Appendix B

This appendix contains a worked example of a *Least Squares* traverse adjustment by the method of *Variation of Coordinates* with added *constraint equations.*

Figure 1 below shows the observed bearings and distances of a traverse network connecting points *A*, *B*, *C*, *D*, *E* and *F*. The line *AB* 110º 15' 20" is the datum for bearings which have been observed on both "faces" of a theodolite and mean observed bearings used as backbearings. bearing miscloses are indicated by observations recorded on both sides of lines *CD*, *CF* and *AF*.

<u>These are the original observations and only these observations or measurements directly derived from them are used in the adjustment.</u>

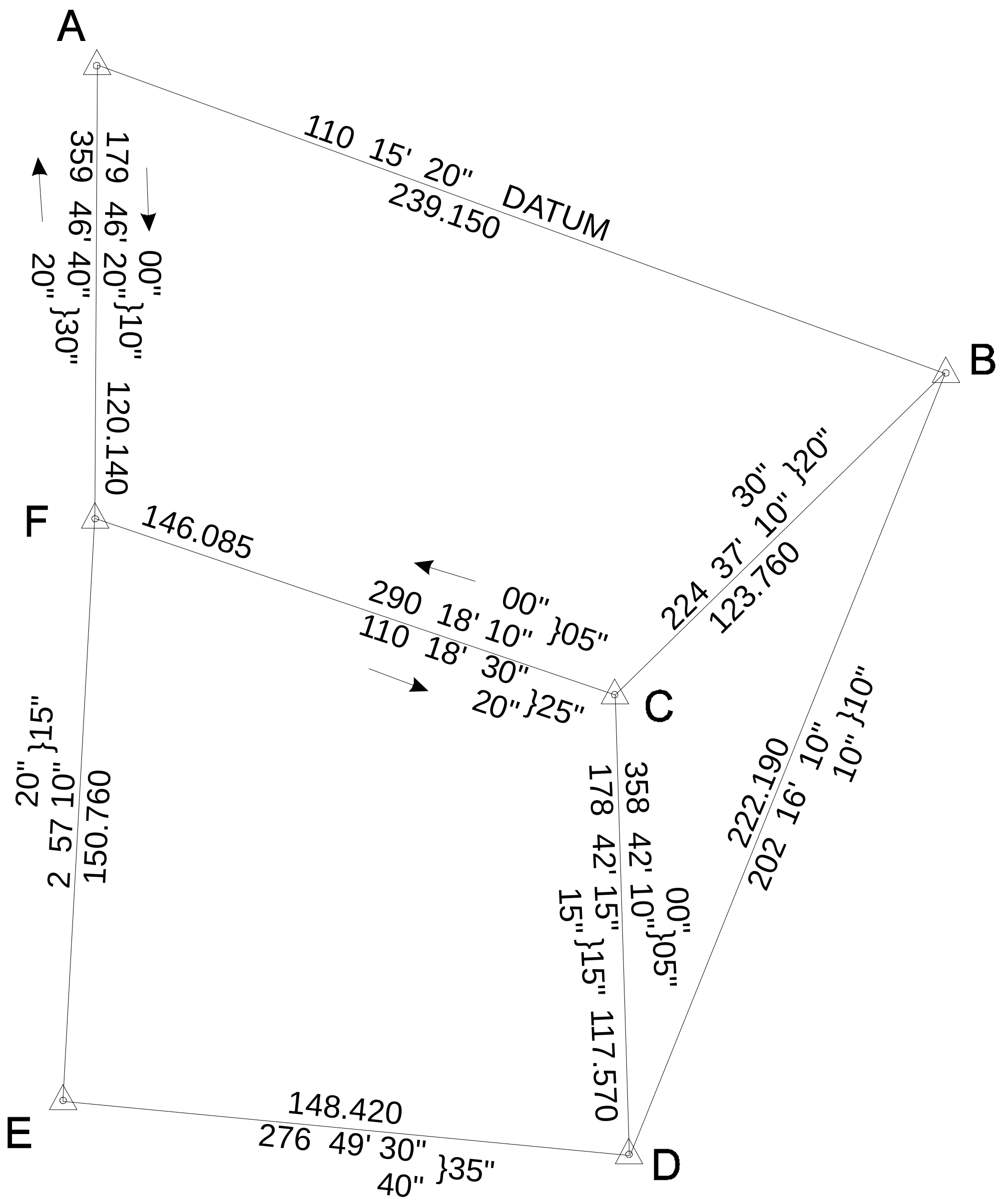

Figure 1. Traverse diagram and field observations

Figure 2 below shows the measurements adopted for the purposes of assessing the accuracy of the survey. Mean bearings of the lines *CD*, *CF* and *AF* have been deduced from the observations and closures taken out on the three connected figures *ABCF*, *BCD* and *CDEF*.

Linear miscloses and accuracies (linear misclose divided by traverse length) are shown on the diagram.

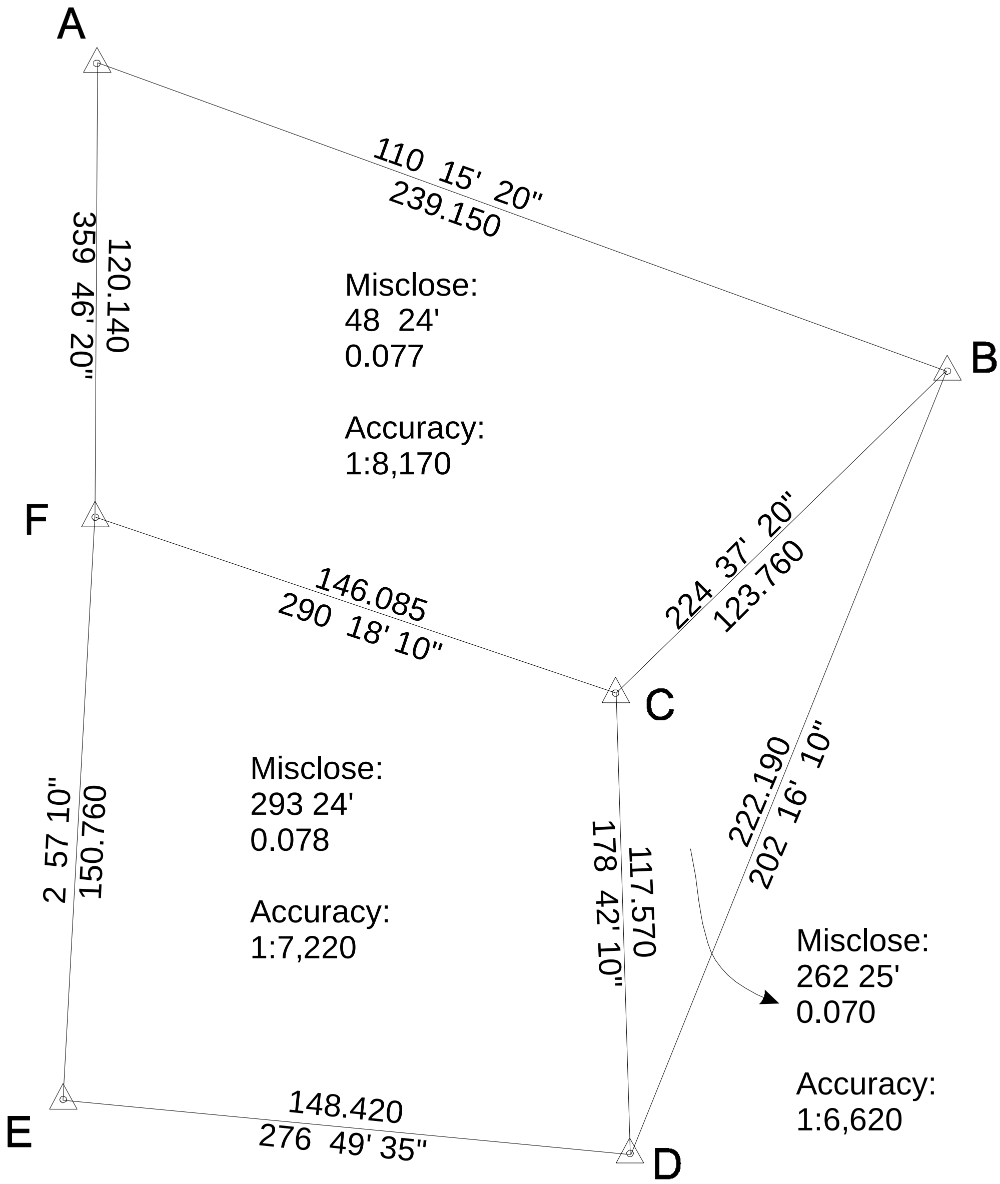

Figure 2. Traverse miscloses and accuracies

Whilst none of the miscloses are excessive, they do present problems in detecting errors in calculations based on these measurements since combinations of the misclose will be transferred to derived figures.

A *Least Squares* adjustment will provide a mathematicall consistent set of adjusted measurements for the traverse network.

The adjustment process is set out in a number of well-defined steps as follows:

STEP 1: DETERMINATION OF *CONSTRAINTS*

For the purpose of the exercise the following *constraints* are applied:

1:	*A* is a fixed station		
2:	the bearing	*AB*	is to be fixed at 110º 15' 20"
3:	the bearing	*DE*	is to be fixed at 276º 49' 35"
4:	the distance	*CF*	is to be fixed at 146.050 m
5:	the angle	*CFE*	is to be fixed at 72º 39' 00"

The minimal constraints necessary for the adjustment by *Variation of Coordinates* are one fixed point and one fixed bearing, or two fixed points.

Fixing the bearing *DE,* the distance *CF* and the angle *CFE* are additional constraints over and above these minimums and will induce distortions into the network. The sum of the squares of the residuals, $\mathbf{v}^T\mathbf{W}\mathbf{v}$ will be a minimum for this configuration of constraints, but it will be larger than $\mathbf{v}^T\mathbf{W}\mathbf{v}$ for the network with minimal constraints.

In many surveys, there may be no points with previously determined coordinate values that can be regarded as fixed points for the purposes of an adjustment. Adopting two points as fixed as minimal constraints on the network will result in a larger value of $\mathbf{v}^T\mathbf{W}\mathbf{v}$ than if one point and one bearing are held fixed.

A traverse adjustment, using the technique outlined in this paper, with only one fixed point and one fixed bearing may be regarded as a *free network adjustment* whose sum of squares of residuals $\mathbf{v}^T\mathbf{W}\mathbf{v}$ is the smallest possible value.

STEP 2: DETERMINATION OF APPROXIMATE COORDINATES AND COMPUTED BEARINGS AND DISTANCES

Figure 3 below shows approximate coordinates derived from the field measurements based on fixed coordinate values for A. Computed bearings and distances ϕ' and s' are calculated from these approximate coordinates.

Station A is "fixed" and the other stations are regarded as "floating".

The adjustment process will determine corrections to be applied to the approximate coordinates of the floating stations.

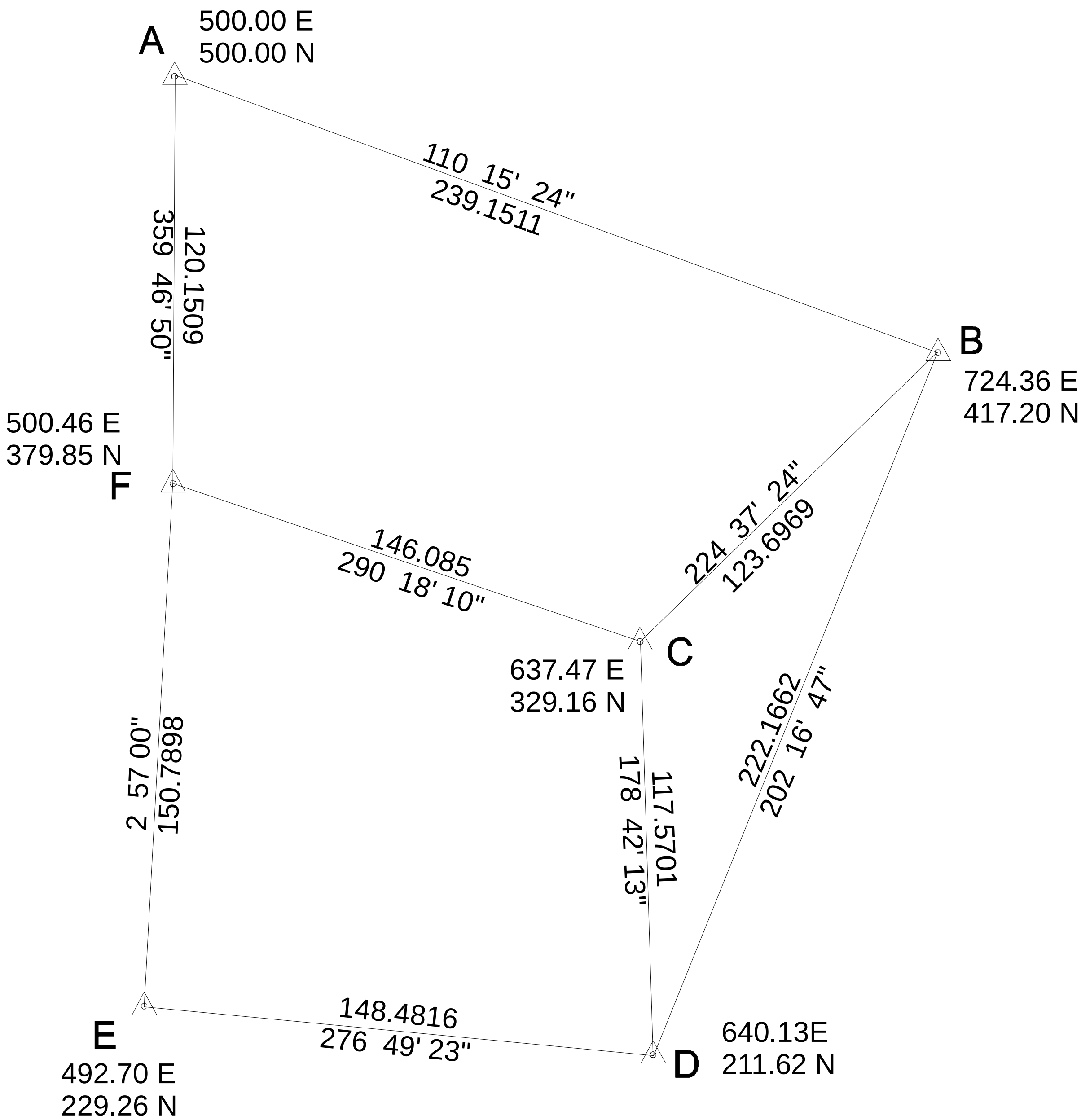

Figure 3. Approximate coordinates and computed bearings and distances

STEP 3: SCHEME DIAGRAM AND OBSERVATIONS

Figure 4 below is a scheme diagram and the numbered arrows on the lines indicate the observed *directions* (1 to 16) and measured *distances* (17 to 24).

A *direction* can be regarded as a reading of the un-oriented theodolite circle wheras a *bearing* assumes the theodolite is oriented to North.

Only "observed" directions and measured distances are used in the adjustment and if bearings have been observed on the traverse lines then directions must be deduced from these observations.

Directions and distances are tabulated below, directions followed by distances. Directions are derived from the mean observed bearings from Figure 1 with 0º 00' 00" as the direction of the backsight and successive directions for the foresights in clockwise order. Standard deviations are apriori estimates of 10" for directions and 0.010 m for measured distances.

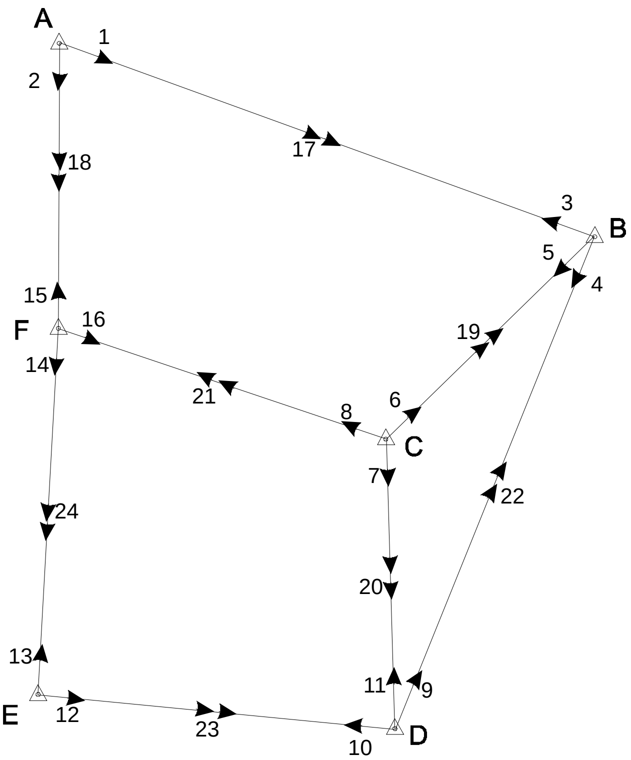

Figure 4. Scheme diagram

No	Station At	Station To	Observation	St. Dev
1	A	B	0º 00' 00"	10"
2	A	F	69º 30' 50"	10"
3	B	A	0º 00' 00"	10"
4	B	D	272º 00' 50"	10"
5	B	C	294º 22' 00"	10"
6	C	B	0º 00' 00"	10"
7	C	D	134º 04' 55"	10"
8	C	F	245º 40' 45"	10"
9	D	B	0º 00' 00"	10"
10	D	E	254º 33' 25"	10"
11	D	C	336º 25' 55"	10"
12	E	D	0º 00' 00"	10"
13	E	F	266º 07' 40"	10"
14	F	E	0º 00' 00"	10"
15	F	A	176º 49' 15"	10"
16	F	C	287º 21' 10"	10"
17	A	B	239.150 m	0.010 m
18	A	F	120.140 m	0.010 m
19	C	B	123.760 m	0.010 m
20	C	D	117.570 m	0.010 m
21	C	F	146.085 m	0.010 m
22	D	B	222.190 m	0.010 m
23	E	D	148.420 m	0.010 m
24	E	F	150.760 m	0.010 m

Table of measurements

STEP 5: DETERMINE ADJUSTMENT "CONSTANTS"

Number of *Residual Equations n*

Inspection of Figure 4 and the Table of measurements shows 24 measurements in the adjustment. Each measurement will yield a *residual equation.*

$$n = 24$$

Number of *Constraint Equations c*

Constraining the bearings *AB* and *DE,* the distance *CF* and the angle *CFE* to predetermined values mean 4 *constraint equations* will be added to the *normal equations.*

$$c = 4$$

Number of *Unknowns u*

Inspection of the observation equation (1) shows that there is an orientation constant associated with every set of directions at a particular traverse station. These are unknowns in the adjustment.

Every floating station in the network has two unknown coordinates, which are also, unknowns in the network.

$$u = 6 + (5 \times 2) = 16$$

$$\textit{Degree of Freedom } r = n - u + c = 24 - 16 + 4 = 12$$

STEP 6: FORM MATRIX **B** AND VECTOR **f** OF RESIDUAL EQUATIONS

Tabulation of the coefficients and numerical terms of the 24 residual equations will lead to the matrix **B** and the vector **f**

Residual equations for directions and distances are given by equations (3) and (4).

For the purpose of this example, the corrections to the approximate coordinates and orientation constants are in centimetres (cm) and seconds of arc (sec) respectively which means that;

> In equation (3), residuals and numerical terms for the direction residual equation are in seconds of arc and the direction coefficients *a* and *b* are in sec/cm and
>
> $$a = \frac{-\sin\phi'}{s'} \times \frac{\rho''}{100} \qquad b = \frac{\cos\phi'}{s'} \times \frac{\rho''}{100}$$
>
> where ρ'' is the number of seconds in 1 radian
>
> In equation (4), residuals and numerical terms for the distance residual equation are in centimetres and the distance coefficients *c* and *d* are given by
>
> $$c = \cos\phi' \qquad d = \sin\phi'$$

Four variations of equations (3) and (4) are possible depending upon whether

1:	P_i	floating	and	P_k	floating
2:	P_i	floating	and	P_k	fixed
3:	P_i	fixed	and	P_k	floating
4:	P_i	fixed	and	P_k	fixed

The 24 residual equations are tabulated below showing the coefficient matrix $\mathbf{B}$ and the vector numerical terms $\mathbf{f}$.

Components $\mathbf{B}_{11}$, $\mathbf{B}_{12}$, $\mathbf{B}_{22}$, $\mathbf{f}_1$ and $\mathbf{f}_2$ as per equation (6) are outlined.

STEP 7: FORM OF NORMAL EQUATION COEFFICIENT MATRIX **N** AND VECTOR OF NUMERICAL TERMS **t**

The *normal equations* are formed according to equations (7) where;

$$\mathbf{N} = \mathbf{B}^T \mathbf{W} \mathbf{B} \quad \text{and} \quad \mathbf{t} = \mathbf{B}^T \mathbf{W} \mathbf{f}$$

Since all the observations can be assumed as independent, the estimate of the variance matrix $\mathbf{Q}$ and its inverse $\mathbf{W}$ are diagonal and the matrix operations can be simplified in the following manner.

Divide each row of the coefficient matrix **B** and the vector of numeric terms **f** by the standard deviation for that particular row (or equation). This operation forms a matrix $\overline{\mathbf{B}}$ of order *n* by $(u+1)$ where the elements of the additional column are the "new" numerical terms.

The elements of the upper-triangular portion of a symmetric matrix $\overline{\mathbf{N}}$ of order $(u+1)$ are formed according to the summation

$$\overline{\mathrm{n}}_{\mathrm{ij}} = \sum_{k=1}^{k=n} \left(\overline{\mathrm{b}}_{ki} \, \overline{\mathrm{b}}_{kj} \right) \qquad i = 1 \text{ to } u+1, \; j = 1 \text{ to } u+1$$

$\overline{\mathbf{N}}$ can be partitioned as

$$\overline{\mathbf{N}} = \left[\begin{array}{c|c} \overline{\mathbf{N}}_{11} & \overline{\mathbf{N}}_{12} \\ \hline \overline{\mathbf{N}}_{21} & \overline{\mathbf{N}}_{22} \end{array} \right]$$

where

$\overline{\mathbf{N}}_{11}$	is the *u* by *u* coefficient matrix **N**
$\overline{\mathbf{N}}_{12}$	is the *u* by 1 vector of numeric terms **t**
$\overline{\mathbf{N}}_{21} = \overline{\mathbf{N}}_{12}$	
$\overline{\mathbf{N}}_{22} = \mathbf{f}^T \mathbf{W} \mathbf{f}$	is the number used in the calculation of the variance factor σ_0^2

The upper-triangular portion of the symmetric coefficient matrix **N** and the vector of numeric terms **t** (both multiplied by -1) and the coefficient matrix **C** and the numerical terms **g** of the 4 constraint equations are shown on the following page.

STEP 8: SOLUTION OF EQUATIONS

Inversion of the previous matrix and multiplication by the numerical terms gives the solutions according to equation (13) as;

$$\begin{aligned}
\Delta N_B &= +0.606701 \text{ cm} & \Delta Z_A &= +0.416271'' & k_1 &= +0.668691 \\
\Delta E_B &= -0.421938 \text{ cm} & \Delta Z_B &= +9.689480'' & k_2 &= +0.369904 \\
\Delta N_C &= -0.839612 \text{ cm} & \Delta Z_C &= -2.746694'' & k_3 &= -0.538359 \\
\Delta E_C &= -1.488356 \text{ cm} & \Delta Z_D &= -19.636998'' & k_4 &= +0.245726 \\
\Delta N_D &= -1.244428 \text{ cm} & \Delta Z_E &= +8.641835'' & & \\
\Delta E_D &= -1.645333 \text{ cm} & \Delta Z_F &= +3.237281'' & & \\
\Delta N_E &= -0.717378 \text{ cm} & & & & \\
\Delta E_E &= +1.332055 \text{ cm} & & & & \\
\Delta N_F &= -2.290640 \text{ cm} & & & & \\
\Delta E_F &= +1.845148 \text{ cm} & & & &
\end{aligned}$$

The corrections are added to the approximate coordinates (shown in Figure 3) and to the approximate orientation constants (shown at the bottom of table of measurements) to give;

$$\begin{aligned}
N_B &= 417.205 \text{ m} & N_C &= 329.152 \text{ m} & N_D &= 211.608 \text{ m} \\
E_B &= 724.356 \text{ m} & E_C &= 637.455 \text{ m} & E_D &= 640.114 \text{ m}
\end{aligned}$$

$$\begin{aligned}
N_E &= 229.253 \text{ m} & N_F &= 379.827 \text{ m} \\
E_E &= 492.713 \text{ m} & E_F &= 500.478 \text{ m}
\end{aligned}$$

$$\begin{aligned}
Z_A &= 110^0\, 15'\, 24.4'' & Z_B &= 290^0\, 15'\, 33.7'' & Z_C &= 44^0\, 37'\, 21.3'' \\
Z_D &= 22^0\, 16'\, 27.4'' & Z_E &= 96^0\, 49'\, 31.6'' & Z_F &= 182^0\, 57'\, 03.2''
\end{aligned}$$

STEP 9: CALCULATION OF ADJUSTED BEARINGS AND DISTANCES

A diagram showing the adjusted bearings and distances is shown below

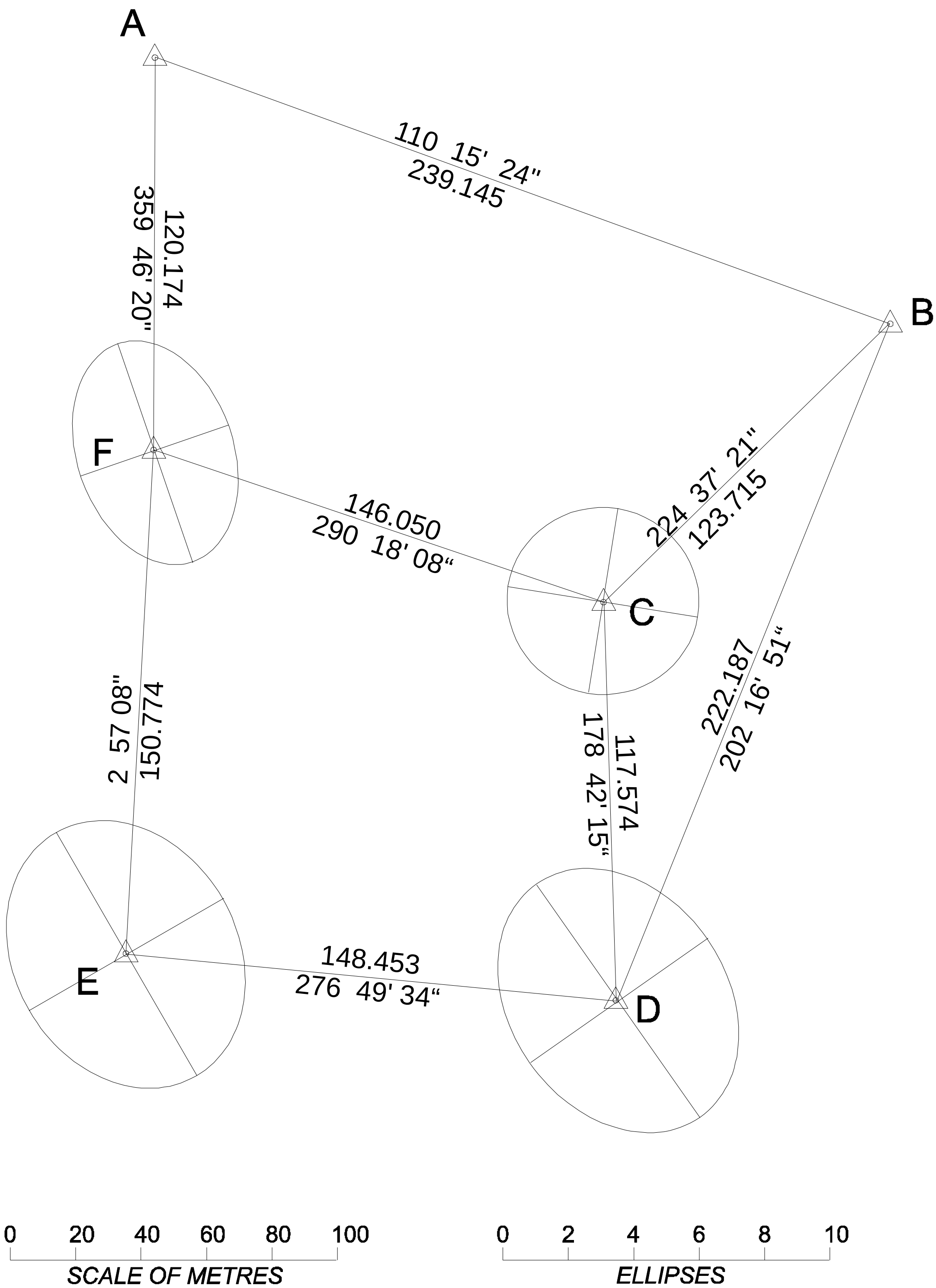

Figure 5. Diagram of adjusted traverse network

The constraints applied to the network;

1: *A* is a fixed station
2: the bearing *AB* is to be fixed at 110º 15' 20"
3: the bearing *DE* is to be fixed at 276º 49' 35"
4: the distance *CF* is to be fixed at 146.050 m
5: the angle *CFE* is to be fixed at 72º 39' 00"

have been satisfied. The difference of 1" in the bearing of the line *DE* is due to rounding errors.

STEP 10: CALCULATION OF RESIDUALS

The residuals, computed from equation (5) $\mathbf{v} = \mathbf{B}\mathbf{x} - \mathbf{f}$ are;

Directions

$$v_1 = -4.1'' \quad v_2 = +4.1'' \quad v_3 = -13.3'' \quad v_4 = +27.3''$$
$$v_5 = -14.0'' \quad v_6 = -1.5'' \quad v_7 = -0.3'' \quad v_8 = +1.9''$$
$$v_9 = +23.6'' \quad v_{10} = -17.0'' \quad v_{11} = -6.4'' \quad v_{12} = +3.7''$$
$$v_{13} = -3.4'' \quad v_{14} = +4.9'' \quad v_{15} = +0.2'' \quad v_{16} = -5.1''$$

Distances

$$v_{17} = -0.5 \text{ cm} \quad v_{18} = +3.4 \text{ cm} \quad v_{19} = -4.5 \text{ cm} \quad v_{20} = -0.4 \text{ cm}$$
$$v_{21} = -3.5 \text{ cm} \quad v_{22} = -0.2 \text{ cm} \quad v_{23} = +3.3 \text{ cm} \quad v_{24} = +1.4 \text{ cm}$$

STEP 11: COMPREHENSIVE CHECKS

As a comprehensive check on the adjustment, the residuals are substituted into the original *Observation Equations* (1) and (2) and compared with the computed values.

STEP 12: PRECISION ESTIMATION

$$\text{Variance Factor} \quad \sigma_0^2 = \frac{\mathbf{v}^T\mathbf{W}\mathbf{v}}{r} = \frac{78.56}{12} = 6.55$$

The numerator $\mathbf{v}^T\mathbf{W}\mathbf{v}$ has been calculated by dividing each residual by the standard deviation for that observation, squaring and adding. The denominator is the degrees of freedom in the network.

Mikhail (1976, pp. 285-88) shows that an unbiased estimate of the variance factor is given by equation (16) as;

$$\sigma_0^2 = \frac{\mathbf{v}^T\mathbf{W}\mathbf{v}}{r} = \frac{\mathbf{f}^T\mathbf{W}\mathbf{f} - \mathbf{x}^T\mathbf{t}}{r}$$

and is based on the assumption that the expected mean of the residuals is zero, or no bias in the observations. The additional constraints placed on the adjustment make it <u>unlikely</u> that $\sigma_0^2 = 6.55$ (as computed above) is <u>an unbiased estimate of the variance factor</u>.

Equation (15) gives the relationship between the "true" measurement variances and the *apriori* estimates, and using the value $\sigma_0^2 = 6.55$ indicates that better estimates of variances may have been

$\sigma_\alpha = \pm 26''$ (standard deviation of observed direction)

$\sigma_l = \pm 0.026\ \text{m}$ (standard deviation of observed distance)

rather than the estimates of $\sigma_\alpha = \pm 10''$ and $\sigma_l = \pm 0.010\ \text{m}$ that were used.

In light of the aforementioned and some experience in adjustment of observations, a surveyor may choose to ignore the computed variance factor and assume that the apriori variance estimates are in fact "true" variances. This implies that $\sigma_0^2 = 1$.

For the purpose of this example traverse network, which is entirely fictitious, the computed variance factor $\sigma_0^2 = 6.55$ will be used. In other circumstances, this relatively large value may require further investigation.

PRECISION OF TRAVERSE BEARINGS AND DISTANCES

Because of the added constraints in the network, the variance matrix of the adjusted quantities is given by equation (18). That part of the matrix inverse $-\alpha$ that relates to the coordinates is shown below. [See equations (13) and (14)]

Station *B*		Station C		Station *D*		Station *E*		Station *F*	
ΔN_B	ΔE_B	ΔN_C	ΔE_C	ΔN_D	ΔE_D	ΔN_E	ΔE_E	ΔN_F	ΔE_F
0.050	-0.134	0.050	-0.068	0.053	-0.068	0.055	-0.079	0.064	-0.063
	0.363	-0.136	0.186	-0.144	0.184	-0.148	0.214	-0.174	0.172
		0.352	0.002	0.252	-0.039	0.244	0.024	0.317	-0.011
			0.340	-0.072	0.382	-0.080	0.448	-0.141	0.287
				0.544	-0.142	0.539	-0.098	0.303	-0.054
					0.648	-0.129	0.537	-0.243	0.307
	SYMMETRIC					0.537	-0.114	0.306	-0.057
							0.670	-0.269	0.339
								0.479	-0.081
									0.261

Only precisions of unconstrained bearings and distances are determined in a traverse network adjustment.

As an example, the precision of the adjusted bearing and distance of the line *CD* will be determined using equations (21a) and (21b). The relevant portions of the variance matrix are highlighted above and the direction/distance coefficients are obtained from the coefficient matrix **B**. Note that C = i and $D = k$

$$
\begin{array}{lll}
a_{ik} = -0.397 \text{ sec / cm} & \sigma_{N_i}^2 = 0.352 \text{ cm}^2 & \sigma_{E_i}^2 = 0.340 \text{ cm}^2 \\
b_{ik} = -17.539 \text{ sec / cm} & \sigma_{N_k}^2 = 0.544 \text{ cm}^2 & \sigma_{E_k}^2 = 0.648 \text{ cm}^2 \\
 & \sigma_{N_i N_k} = 0.252 \text{ cm}^2 & \sigma_{E_i E_k} = 0.382 \text{ cm}^2 \\
c_{ik} = -1.000 & \sigma_{N_i E_i} = 0.002 \text{ cm}^2 & \sigma_{N_i E_k} = -0.039 \text{ cm}^2 \\
d_{ik} = 0.023 & \sigma_{N_k E_k} = -0.142 \text{ cm}^2 & \sigma_{N_k E_i} = -0.072 \text{ cm}^2
\end{array}
$$

and since the variance factor is common to all elements

$$
\begin{array}{lll}
\sigma_{\phi_{ik}}^2 = 6.55 \times 68.56 \text{ sec}^2 & \text{and} & \sigma_{\phi_{ik}} = \pm 21'' \\
\sigma_{s_{ik}}^2 = 6.55 \times 0.393 \text{ cm}^2 & \text{and} & \sigma_{s_{ik}} = \pm 0.016 \text{m}
\end{array}
$$

which are the standard deviations of the adjusted bearing and distance of the line *CD.*

ERROR ELLIPSES

Error ellipses can be computed for stations *C, D, E* and *F* in the network. No error ellipse can be computed for *B* since it is constrained to lie on a particular bearing from the fixed station *A*.

Equations (22) and (23) give the necessary formulae for calculating the parameters of individual error ellipses, a = length of semi-major axis, b = length of semi-minor axis and θ = bearing of major axis.

Point C $\quad \sigma_N^2 = 0.352 \text{ cm}^2 \quad \sigma_E^2 = 0.340 \text{ cm}^2 \quad \sigma_{EN} = 0.002 \text{ cm}^2$

$$a = 0.594 \text{ cm} \qquad b = 0.583 \text{ cm} \qquad \tan 2\theta = \frac{0.004}{0.012}$$

$$2\theta = 18^{\circ}\, 26'\, 06''$$

$$\theta = 9^{\circ}\, 13'$$

Point D $\quad \sigma_N^2 = 0.544 \text{ cm}^2 \quad \sigma_E^2 = 0.648 \text{ cm}^2 \quad \sigma_{EN} = -0.142 \text{ cm}^2$

$$a = 0.864 \text{ cm} \qquad b = 0.667 \text{ cm} \qquad \tan 2\theta = \frac{-0.284}{-0.104}$$

$$2\theta = 249^{\circ}\, 53'\, 15''$$

$$\theta = 124^{\circ}\, 57'$$

Point E $\quad \sigma_N^2 = 0.537 \text{ cm}^2 \quad \sigma_E^2 = 0.670 \text{ cm}^2 \quad \sigma_{EN} = -0.114 \text{ cm}^2$

$$a = 0.858 \text{ cm} \qquad b = 0.687 \text{ cm} \qquad \tan 2\theta = \frac{-0.228}{-0.133}$$

$$2\theta = 239^{\circ}\, 44'\, 37''$$

$$\theta = 119^{\circ}\, 52'$$

Point F $\quad \sigma_N^2 = 0.479 \text{ cm}^2 \quad \sigma_E^2 = 0.261 \text{ cm}^2 \quad \sigma_{EN} = -0.081 \text{ cm}^2$

$$a = 0.711 \text{ cm} \qquad b = 0.484 \text{ cm} \qquad \tan 2\theta = \frac{-0.162}{0.218}$$

$$2\theta = 323^{\circ}\, 23'\, 00''$$

$$\theta = 161^{\circ}\, 42'$$

Error ellipses are shown on Figure 5.

SUMMARY

Several points need to be made about the adjustment example.

1. The variance factor is quite large which may be due to the following;

   (a) there may be undetected blunders in the observations,
   (b) the apriori estimates of the standard deviations are incorrect,
   (c) the constraints are tending to distort the adjustment.

2. The solution shown is a "first iteration" based on the approximate coordinates of Figure 3. It is unlikely that a further iteration based on the new coordinates will produce corrections greater than 1 mm to these coordinates.

3. If only one point in the network is held fixed, and no constraints are applied, the matrix $\mathbf{N}$ is singular and solution of the system of normal equations is impossible. Adding an appropriate constraint will ensure that the combined system of normal and constraint equations is not singular and enable the equations to be solved. Since the matrix $\mathbf{N}$ is formed from $\mathbf{B}$, an appropriate constraint equation

cannot be a linear combination or repetition of any residual equations, such as an angle or distance constraint. An appropriate constraint is a bearing.

This is illustrated by pinning the network diagram to the wall through one traverse point. Constraining a distance or angle will not stop the diagram from rotating about the fixed point and making the coordinates of the other traverse points indeterminate.

4. In this example, the distance and angle constraints are linear combinations of some residual equations. It is possible to remove these equations from matrix **B**, form **N** and solve the system with the constraints. This may lead to different results. Additional investigation is required in this area.