The calculation of longitude and latitude from geodesic measurements[*]

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(Originally published: October 1825; translated: August 13, 2009)

1. INTRODUCTION

Consider a geodesic line between two points A and B on the surface of the Earth. Given the position of A, the length of the line and its azimuth at A, we wish to determine the position of B and the azimuth of the line there. This problem occurs so frequently that I undertook to construct tables to simplify the computation. In order to explain the method clearly, I start by deriving the fundamental properties of geodesic lines on a spheroid of revolution. Even though aspects of this derivation may already be well known, the benefit of having the entire development presented together outweighs the cost of repeating it.[1]

2. THE CHARACTERISTIC EQUATION FOR A GEODESIC

Take two points A and B on the surface on a spheroid[2] of revolution joined by some specified curve. Consider two neighboring points on the curve with latitudes ϕ and $\phi + d\phi$ and longitudes relative to A of w and $w + dw$ (measuring east positive). Let the distance between them be ds, the azimuth of line directed toward A be α (measured clockwise from north), the radius of the circle of latitude be r, and the meridional radius of curvature by R; then we find[3]

$$\begin{aligned} ds\cos\alpha &= -R\,d\phi = \frac{dr}{\sin\phi}, \\ ds\sin\alpha &= -r\,dw, \end{aligned} \tag{1}$$

which gives

$$ds = \sqrt{R^2\,d\phi^2 + r^2\,dw^2}.$$

If we write p for $d\phi/dw$ and U for $\sqrt{R^2p^2 + r^2}$, this becomes

$$ds = U\,dw.$$

The distance along the curve between the two points A and B is therefore

$$s = \int U\,dw,$$

where the integration is from A to B. If the curve is the geodesic or *shortest* path, then the relation between ϕ and w must be such that the integral is a minimum. If we perturb this relation so that ϕ is replaced by $\phi + z$ where z is an arbitrary function of w which vanishes at the end points (because these points lie on both curves), then the perturbed length,

$$s' = \int U'\,dw,$$

must be larger than s for all z.

Expanding $U(\phi, p)$ in a Taylor series, we obtain[4]

$$U' = U + \frac{\partial U}{\partial\phi}z + \frac{\partial U}{\partial p}\frac{dz}{dw} + \ldots$$

and therefore we have

$$s' = s + \int\left(\frac{\partial U}{\partial\phi}z + \frac{\partial U}{\partial p}\frac{dz}{dw}\right)dw + \ldots,$$

where we have explicitly included terms only up to first order in z. For s to be a minimum, we require that

$$\int\left(\frac{\partial U}{\partial\phi}z + \frac{\partial U}{\partial p}\frac{dz}{dw}\right)dw + \ldots \geq 0$$

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[*] This is an English translation of *Über die Berechnung der geographischen Längen und Breiten aus geodätischen Vermessungen*, Astronomische Nachrichten **4**(86), 241–254 (1826), doi:10.1002/asna.18260041601. The paper also appears in *Abhandlungen von Friedrich Wilhelm Bessel*, Vol. 3, pp. 5–14 (W. Engelmann, Leipzig, 1876). The translation has been prepared and edited by Charles F. F. Karney ⟨ckarney@sarnoff.com⟩ and Rodney E. Deakin ⟨rod.deakin@rmit.edu.au⟩, with the assistance of Max Hunter and Stephan Brunner. The mathematical notation has been updated to conform to current conventions and, in a few places, the equations have been rearranged for clarity. Several errors have been corrected, a figure has been included, and the tables have been recomputed. A transcription of the original paper with the updated mathematical notation and with the corrections is available at arXiv:0908.1823. A contemporary, but partial, translation into English appeared in Quart. Jour. Roy. Inst. **21**(41), 138–152 (1826).

[1] In Secs. 2–4, Bessel gives a concise summary of the work of several other authors, notably, Clairaut, du Séjour, Legendre, and Oriani. Bessel's contributions, which start in Sec. 5, consist of his methods for expanding the distance and longitude integrals and his compilation of tables to provide a practical method for computing geodesics. Two sentences have been omitted from this translation of the introduction. In one, Bessel refers to two letters he published earlier in the *Astronomische Nachrichten* which do not, however, have a direct bearing on the present work. In the other, he criticizes "du Séjour's method," but without providing details; in any case, such criticism is misplaced because du Séjour had died over 30 years earlier and Bessel does not cite more recent work.

[2] "Spheroid" here is used in the sense of a shape approximating a sphere. Sections 2 and 3 treat the case of a rotationally symmetric earth. In Sec. 4, Bessel specializes to a rotationally symmetric ellipsoid.

[3] The minus signs appear in (1) because α is the back azimuth, pointing to A, while ds advances the geodesic away from A. In this section, Bessel assumes an easterly geodesic so that $ds/dw > 0$. However the final result, Eq. (2), is general.

[4] The notation here employs partial derivatives instead of Bessel's less formal use of differentials.

for all z. Since this must also hold if z is replaced by $-z$ and since we can take z so small that the first order terms are bigger that the sum of all the higher order terms (except if the first order terms vanish), it follows that the condition that s be minimum is

$$\int \left(\frac{\partial U}{\partial \phi} z + \frac{\partial U}{\partial p} \frac{dz}{dw} \right) dw = 0.$$

Integrating the second term by parts to give $z(\partial U/\partial p) - \int z[d(\partial U/\partial p)/dw]\, dw$ and remembering that z vanishes at the end points, we obtain

$$\int z \left\{ \frac{\partial U}{\partial \phi} - \frac{d}{dw} \left(\frac{\partial U}{\partial p} \right) \right\} dw = 0.$$

Since this integral must vanish for *arbitrary* z, we find[5]

$$\frac{\partial U}{\partial \phi} - \frac{d}{dw} \left(\frac{\partial U}{\partial p} \right) = 0$$

or, multiplying by $d\phi/dw = p$,

$$\frac{\partial U}{\partial \phi} \frac{d\phi}{dw} + \frac{\partial U}{\partial p} \frac{dp}{dw} - \frac{dp}{dw} \frac{\partial U}{\partial p} - p \frac{d}{dw} \left(\frac{\partial U}{\partial p} \right) = 0,$$

which on integrating with respect to w becomes[6]

$$U - p \left(\frac{dU}{dp} \right) = \mathrm{const.}$$

Substituting $\sqrt{r^2 + R^2 p^2}$ for U, we obtain[7]

$$\frac{r}{\sqrt{1 + (R^2/r^2) p^2}} = -r \sin \alpha = \mathrm{const.},$$

which is the well known characteristic equation of the geodesic.

If the azimuth of the geodesic at A (in the direction of B) is α' and the distance of A from the rotation axis is r', we have

$$r' \sin(\alpha' + 180^\circ) = r \sin \alpha,$$

or

$$r' \sin \alpha' = -r \sin \alpha. \tag{2}$$

3. THE AUXILIARY SPHERE

Let the maximum distance of the spheroid to the rotation axis be a, so that r and r' are less than or equal to a; we can then write[8]

$$r' = a \cos u', \quad r = a \cos u,$$

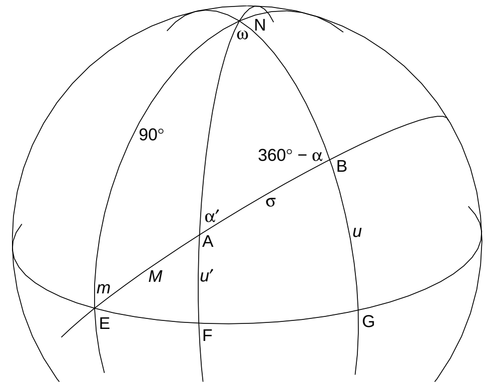

Figure 1 Spherical triangles on the auxiliary sphere. EAB is the geodesic, N is the pole; EFG is the equator; and NE, NAF, and NBG are meridians.

and equation (2) becomes

$$\cos u' \sin \alpha' = -\cos u \sin \alpha. \tag{3}$$

This equation relates two sides of a spherical triangle,[9] $90^\circ - u'$ and $90^\circ - u$, and their opposite angles, $360^\circ - \alpha$ and α'. The third side σ and its opposite angle ω will appear in the following calculations giving elegant expressions for the joint variations of s, u and w. In particular, using the well known differential formulas of spherical trigonometry, we find[10]

$$du = -\cos \alpha \, d\sigma,$$
$$\cos u \, d\omega = -\sin \alpha \, d\sigma.$$

Substituting these in equations (1) and expressing r in terms of u gives

$$\begin{aligned} ds &= a \frac{\sin u}{\sin \phi} d\sigma, \\ dw &= \frac{\sin u}{\sin \phi} d\omega. \end{aligned} \tag{4}$$

4. THE EQUATIONS FOR A GEODESIC ON AN ELLIPSOID

I now assume that the meridian is an ellipse with equatorial semi-axis a, polar semi-axis b, and eccentricity $e = \sqrt{a^2 - b^2}/a$.[11] The equation for an ellipse expressed in terms

[5] This is the Euler-Lagrange equation of the calculus of variations.

[6] This is now known as the Beltrami identity.

[7] A. C. Clairaut gives a geometric derivation of this result in Mém. de l'Acad. Roy. des Sciences de Paris, 1733, 406–416 (1735). The equation also follows from conservation of angular momentum for a mass sliding without friction on a spheroid of revolution.

[8] The quantity u is the *reduced* or *parametric* latitude.

[9] See the triangle ABN on the "auxiliary sphere" in Fig. 1; Equation (3) is the sine rule applied to angles A and B of the triangle.

[10] Here and in the rest of the paper, the differentials give the movement of point B along the geodesic defined with point A and α' held fixed.

[11] In Bessel's time, it was known that the earth could be approximated by an oblate ellipsoid, $a > b$, but the eccentricity had not been determined accurately. Therefore, Bessel computes tables which are applicable to oblate ellipsoids with a range of eccentricities. However, the series expansions that Bessel obtains, (11) and (12), can also to applied to prolate ellipsoids, $a < b$, by allowing $e^2 < 0$.

of cartesian coordinates is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Differentiating this and setting $dy/dx = -\cot\phi$, we obtain

$$\frac{x\sin\phi}{a^2} - \frac{y\cos\phi}{b^2} = 0;$$

eliminating y between these equations then gives

$$x = \frac{a\cos\phi}{\sqrt{1-e^2\sin^2\phi}}.$$

The quantity x is the same as $r = a\cos u$, which gives the relationships between ϕ and u,

$$\begin{aligned}
\cos u &= \frac{\cos\phi}{\sqrt{1-e^2\sin^2\phi}}, &\quad \cos\phi &= \frac{\cos u\sqrt{1-e^2}}{\sqrt{1-e^2\cos^2 u}},\\
\sin u &= \frac{\sin\phi\sqrt{1-e^2}}{\sqrt{1-e^2\sin^2\phi}}, &\quad \sin\phi &= \frac{\sin u}{\sqrt{1-e^2\cos^2 u}},\\
\tan u &= \tan\phi\sqrt{1-e^2}, &\quad \tan\phi &= \frac{\tan u}{\sqrt{1-e^2}},
\end{aligned}$$

and

$$\frac{\sin u}{\sin\phi} = \sqrt{1-e^2\cos^2 u}.$$

Substituting this into (4), we obtain the differential equations for a geodesic on an ellipsoid

$$\begin{aligned}
ds &= a\sqrt{1-e^2\cos^2 u}\,d\sigma,\\
dw &= \sqrt{1-e^2\cos^2 u}\,d\omega.
\end{aligned} \tag{5}$$

5. THE DISTANCE INTEGRAL

To integrate the first of these differential equations, I use the three relations between u', u, α', α and σ,[12]

$$\begin{aligned}
\sin u &= \sin u'\cos\sigma + \cos u'\cos\alpha'\sin\sigma,\\
-\cos u\cos\alpha &= -\sin u'\sin\sigma + \cos u'\cos\alpha'\cos\sigma,\\
-\cos u\sin\alpha &= \cos u'\sin\alpha'.
\end{aligned} \tag{6}$$

It is convenient to write these in terms of the auxiliary angles m and M defined by[13]

$$\begin{aligned}
\sin u' &= \cos m\sin M,\\
\cos u'\cos\alpha' &= \cos m\cos M,\\
\cos u'\sin\alpha' &= \sin m.
\end{aligned} \tag{7}$$

[12] Referring to Fig. 1, consider two central cartesian coordinate systems with the xy plane containing the geodesic EAB, and either A or B lying on the x axis. Equations (6) give the transformation between the coordinates of N in the two systems, $[\sin u', \cos u'\cos\alpha', \cos u'\sin\alpha']$ and $[\sin u, -\cos u\cos\alpha, -\cos u\sin\alpha]$, namely a rotation by σ about the z axis.

[13] The auxiliary angles m and M are an angle and a side of the spherical triangle EAN shown in Fig. 1. Equations (7) are the sine rule on angles E and F of triangle AEF, the cosine rule on angle F of triangle AEF, and the sine rule on angles A and E of triangle ANE.

Equations (6) then become[14]

$$\begin{aligned}
\sin u &= \cos m\sin(M+\sigma),\\
\cos u\cos\alpha &= -\cos m\cos(M+\sigma),\\
\cos u\sin\alpha &= -\sin m.
\end{aligned} \tag{8}$$

This gives

$$\cos^2 u = 1 - \cos^2 m\sin^2(M+\sigma),$$

and the equation for ds becomes

$$ds = a\sqrt{1-e^2}\sqrt{1+k^2\sin^2(M+\sigma)}\,d\sigma, \tag{9}$$

where

$$k = \frac{e\cos m}{\sqrt{1-e^2}}.$$

This differential equation may be integrated in terms of the elliptic integrals introduced by Legendre.[15] Because the tools to compute these special functions are not yet sufficiently versatile,[16] we instead develop a series solution which converges rapidly because e^2 is so small. We readily achieve this by decomposing the term under the square root into two complex factors, namely[17]

$$\begin{aligned}
ds = a\frac{\sqrt{1-e^2}}{1-\epsilon}\,d\sigma\times&\\
\sqrt{1-\epsilon\exp\bigl(2i(M+\sigma)\bigr)}&\sqrt{1-\epsilon\exp\bigl(-2i(M+\sigma)\bigr)},
\end{aligned}$$

where

$$\epsilon = \frac{\sqrt{1+k^2}-1}{\sqrt{1+k^2}+1}, \quad k = \frac{2\sqrt{\epsilon}}{1-\epsilon}.$$

Expanding the two factors in the radicals in infinite series and multiplying the results gives[18]

$$\begin{aligned}
ds = a\frac{\sqrt{1-e^2}}{1-\epsilon}\,d\sigma\bigl[A &- 2B\cos 2(M+\sigma)\\
&- 2C\cos 4(M+\sigma) - 2D\cos 6(M+\sigma) - \ldots\bigr],
\end{aligned}$$

[14] These are analogs of Eqs. (7) with meridian NAF replaced by NBG.

[15] A. M. Legendre, *Exercices du calcul intégral*, Vol. 1 (Courcier, Paris, 1811).

[16] Even though good numerical algorithms for elliptic integrals are available, these usually require linking to an additional library and, for that reason, computations of geodesics are still usually in terms of a series.

[17] The notation has been simplified here compared to Bessel's original formulation in which k and ϵ are expressed in terms of E through $k = \tan E$ and $\epsilon = \tan^2\frac{1}{2}E$. By using ϵ as the expansion parameter and by dividing out the factor $1-\epsilon$, Bessel has ensured that the terms that he is expanding are invariant under the transformation $\epsilon \to -\epsilon$, $M+\sigma \to \pi/2-(M+\sigma)$. This symmetry causes half the terms in the expansions in ϵ to vanish.

[18] The use of complex exponentials facilitates the series expansions by avoiding the need to employ awkward trigonometric identities. If we write $\sqrt{1-x} = 1 - \frac{1}{2}x - \frac{1\cdot1}{2\cdot4}x^2 - \frac{1\cdot1\cdot3}{2\cdot4\cdot6}x^3 - \frac{1\cdot1\cdot3\cdot5}{2\cdot4\cdot6\cdot8}x^4 - \ldots = \sum_j a_j x^j$, then the coefficient of $\cos\bigl(2l(M+\sigma)\bigr)\epsilon^{l+2j}$ is a_j^2 for $l=0$ and $2a_j a_{j+l}$ for $l>0$.

where A, B, C, … are given by

$$A = 1 + \left(\frac{1}{2}\right)^2 \epsilon^2 + \left(\frac{1\cdot 1}{2\cdot 4}\right)^2 \epsilon^4 + \left(\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\right)^2 \epsilon^6 + \dots,$$
$$B = \frac{1}{2}\epsilon - \frac{1\cdot 1}{2\cdot 4}\,\frac{1}{2}\epsilon^3 - \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\,\frac{1\cdot 1}{2\cdot 4}\epsilon^5 - \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}\,\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\epsilon^7 - \dots,$$
$$C = \frac{1\cdot 1}{2\cdot 4}\epsilon^2 - \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\,\frac{1}{2}\epsilon^4 - \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}\,\frac{1\cdot 1}{2\cdot 4}\epsilon^6 - \frac{1\cdot 1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 10}\,\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\epsilon^8 - \dots,$$
$$D = \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\epsilon^3 - \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}\,\frac{1}{2}\epsilon^5 - \frac{1\cdot 1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 10}\,\frac{1\cdot 1}{2\cdot 4}\epsilon^7 - \frac{1\cdot 1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12}\,\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\epsilon^9 - \dots,$$

etc.

Integrating the equation for ds starting at $\sigma = 0$, we obtain

$$s = \frac{b}{1-\epsilon}\bigl[A\sigma - \tfrac{2}{1}B\cos(2M+\sigma)\sin\sigma - \tfrac{2}{2}C\cos(4M+2\sigma)\sin 2\sigma - \tfrac{2}{3}D\cos(6M+3\sigma)\sin 3\sigma - \dots\bigr]. \qquad (10)$$

6. SOLVING THE DISTANCE EQUATION

The series (10) gives the distance s between A and B in terms of u', α', and σ; if, however, s and α' have been measured and u' is known from the latitude at A, then σ is obtained by solving (10). The latitude of B and the azimuth of the geodesic there are found from (8). Equation (10) can be solved either by reverting the series or by successive approximation—the latter way is however the simplest if the tables I have compiled are used.

I write[19]

$$\sigma = \frac{\alpha}{b}s + \beta\cos(2M+\sigma)\sin\sigma + \gamma\cos(4M+2\sigma)\sin 2\sigma + \delta\cos(6M+3\sigma)\sin 3\sigma + \dots, \qquad (11)$$

where

$$\alpha = \frac{648\,000}{\pi}\,\frac{1-\epsilon}{A},$$
$$\beta = \frac{648\,000}{\pi}\,\frac{2B}{A},$$
$$\gamma = \frac{648\,000}{\pi}\,\frac{C}{A},$$
$$\delta = \frac{648\,000}{\pi}\,\frac{2D}{3A},$$

etc.

The tables give the logarithms[20] of α, β, and γ as a function of the argument

$$\log k = \log\frac{e\cos m}{\sqrt{1-e^2}}.$$

By this choice, the variation of $\log\beta$ and $\log\gamma$ are very close to two and four times that of the argument, which simplifies interpolation into the table.[21]

We take $\alpha s/b$ as the first approximation of σ, substitute this into the second term to obtain a second approximation, with which we recalculate the second term and add the third. The convergence of the series is sufficiently fast that, even if the argument is $\bar{1}.1$ (which is only possible if the flattening of the ellipsoid, $1 - b/a$, exceeds $\frac{1}{128}$), the approximation never needs to be carried further in order to keep the errors in σ under $0.001''$. The term involving δ does not exceed $0.0005''$ at this value of the argument.

7. ACCURACY OF THE TABLES

The values of $\log\alpha$ in the table are given to 8 decimal places.[22] An error of half a unit of the last place results in an error of only $0.0005''$ or $0.008\,\mathrm{toise}$ over a distance corresponding to $\sigma = 12°4'$ or 700 000 toises.[23] Similarly, I retain only sufficient digits in the tabulation of $\log\beta$ to ensure that the error in this term is less than $0.0005''$; for this purpose, I use 6 digits at the end of the table and fewer digits for smaller values of the argument. The third term never exceeds $0.17''$, even at the end of the table; therefore I include only 3 decimal places for $\log\gamma$. Thus the errors are $0.001''$ for distances up to 700 000 toises; even if the distance is of the order of a quarter meridian (i.e., $\sigma = 90°$), the error is less than $0.01''$.

8. AN EXAMPLE

In order to illustrate the use of the tables, I consider the results from the great survey by von Müffling.[24] Relative to

[19] The units for σ, α, β, … are arc seconds. Bessel here adopts a conflicting notation for the coefficient α which should not be confused with the azimuth.

[20] In this paper, $\log x$ denotes the common logarithm (base 10) and we use $\operatorname{colog} x = \log(1/x)$. The tables in the original paper contained a number of errors of one unit in the last place. These errors do not, for the most part, affect the results obtained from the tables when rounded to $0.001''$. In addition, there were systematic errors in the tabulated values of $\log\beta$ equivalent to a relative error of order ϵ^2 in β which result in discrepancies from 1 to 17 units in the last place on the final page (the 6-figure portion) of the tables. In calculations involving logarithms, a bar over a numeral indicates that that numeral should be negated, e.g., $\log 0.02 \approx \bar{2}.3 = (-2) + 0.3$. In the original paper, logarithms are written modulo 10, e.g., $\log 0.02 \approx 8.3$. The notation "$(-)$" in these calculations indicates that the quantity whose logarithm is being taken is negative.

[21] The columns headed Δ give the first differences of the immediately preceding columns and aid in interpolating the data. Bessel would have used a table of "proportional parts" to compute the interpolated values.

[22] Working with 8-figure logarithms provides about 2 bits more precision than IEEE single precision floating point numbers.

[23] The toise was a French unit of length. It can be converted to meters by $1\,\mathrm{toise} = 864\,\mathrm{ligne}$, $443.296\,\mathrm{ligne} = 1\,\mathrm{m}$, or $1\,\mathrm{toise} \approx 1.949\,\mathrm{m}$.

[24] F. K. F. von Müffling, Astron. Nachr. **2**(27), 33–38 (1824).

Seeberg (point A), the distance and azimuth to Dunkirk (point B) are[25]

$$\log s = 5.478\,303\,14, \quad \alpha' = 274^\circ\,21'\,3.18''.$$

I assume the latitude of the Observatory at Seeberg to be $\phi' = 50^\circ\,56'\,6.7''$ and the ellipsoid parameters to be $\log b = 6.513\,354\,64$, $\log e = \bar{2}.905\,4355$.[26]

From $\tan u' = \sqrt{1-e^2}\tan\phi'$, we find

$$\begin{aligned}\log\tan\phi' &= 0.090\,626\,65\\ \log\sqrt{1-e^2} &= \bar{1}.998\,590\,60\\ \hline \log\tan u' &= 0.089\,217\,25; \quad u' = 50^\circ\,50'\,39.057''.\end{aligned}$$

Given u' and α', we can compute M, $\cos m$ and $\sin m$ from equations (7):[27]

$$\begin{aligned}\log\sin u' &= \bar{1}.889\,543\,51\\ \log\cos u' &= \bar{1}.800\,326\,27\\ \log\cos\alpha' &= \bar{2}.880\,037\,33\\ \log\sin\alpha' &= \bar{1}.998\,746\,62(-)\\ \hline \log(\cos m\sin M) &= \bar{1}.889\,543\,51\\ \log(\cos m\cos M) &= \bar{2}.680\,363\,60\\ \log\,\sin m &= \bar{1}.799\,072\,89(-)\\ \hline M &= 86^\circ\,27'\,53.949''; \quad 2M = 172^\circ\,55'\,47.9''\\ \log\cos m &= \bar{1}.890\,370\,63 \quad 4M = 345^\circ\,51'\,36''.\end{aligned}$$

The argument in the tables, $\log\bigl((e/\sqrt{1-e^2})\cos m\bigr)$, is

$$\begin{aligned}\log\frac{e}{\sqrt{1-e^2}} &= \bar{2}.906\,845\\ \log\cos m &= \bar{1}.890\,371\\ \hline \text{Argument} &= \bar{2}.797\,216.\end{aligned}$$

Looking up $\log\alpha$ in the tables, and calculating $\alpha s/b$ gives[28]

$$\begin{aligned}\log\alpha &= 5.313\,998\,92\\ \text{colog}\,b &= \bar{7}.486\,645\,36\\ \log s &= 5.478\,303\,14\\ \hline \log\frac{\alpha s}{b} &= 4.278\,947\,42; \quad \frac{\alpha}{b}s = 5^\circ\,16'\,48.481''.\end{aligned}$$

Adopting this as the first approximation to the value of σ, we obtain the second by adding the first term in the series (11),

$$\begin{aligned}\log\beta &= 2.305\,94\\ \log\cos(2M+\sigma) &= \bar{1}.999\,79(-)\\ \log\sin\sigma &= \bar{2}.963\,91\\ \hline & 1.269\,64(-) = -18.61''.\end{aligned}$$

We now update the value of this term with the second approximation of $\sigma = 5^\circ\,16'\,48.5'' - 18.6'' = 5^\circ\,16'\,29.9''$ and so obtain as the third approximation:

$$\begin{aligned}\log\beta &= 2.305\,94\\ \log\cos(2M+\sigma) &= \bar{1}.999\,79(-)\\ \log\sin\sigma &= \bar{2}.963\,48\\ \hline & 1.269\,21(-) = -18.587'',\end{aligned}$$

$$\begin{aligned}\log\gamma &= \bar{2}.394\\ \log\cos(4M+2\sigma) &= \bar{1}.999\\ \log\sin 2\sigma &= \bar{1}.263\\ \hline & \bar{3}.656 = +0.005''.\end{aligned}$$

Gathering the terms in (11) gives $\sigma = 5^\circ\,16'\,48.481'' - 18.587'' + 0.005'' = 5^\circ\,16'\,29.899''$ and so, finally, we determine α, u and ϕ from equations (8),

$$\begin{aligned}M+\sigma &= 91^\circ\,44'\,23.848''\\ \log\sin(M+\sigma) &= \bar{1}.999\,799\,71\\ \log\bigl(-\cos(M+\sigma)\bigr) &= \bar{2}.482\,349\,32\\ \log\cos m &= \bar{1}.890\,370\,63\\ \log(-\sin m) &= \bar{1}.799\,072\,89\\ \hline \log\,\sin u &= \bar{1}.890\,170\,34\\ \log(\cos u\cos\alpha) &= \bar{2}.372\,719\,95\\ \log(\cos u\sin\alpha) &= \bar{1}.799\,072\,89\\ \hline \log\cot\alpha &= \bar{2}.573\,647\,06; \quad \alpha = 87^\circ\,51'\,15.523''\\ \log\cos u &= \bar{1}.799\,377\,50\\ \log\tan u &= 0.090\,792\,84\\ \text{colog}\,\sqrt{1-e^2} &= 0.001\,409\,40\\ \hline \log\tan\phi &= 0.092\,202\,24; \quad \phi = 51^\circ\,2'\,12.719''.\end{aligned}$$

In this example, I carried out the trigonometric calculations to 8 decimals; however the tables of $\log\alpha$, $\log\beta$, and $\log\gamma$ in fact allow α and ϕ to be determined slightly more accurately than this. If only standard 7-figure logarithm tables are available, the last digits in the tabulated values of $\log\alpha$, $\log\beta$, and $\log\gamma$ may be neglected.

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[25] Seeberg: $50^\circ 56'$N $10^\circ 44'$E; Dunkirk: $51^\circ 2'$N $2^\circ 23'$E.

[26] In present-day units, this is $a \approx 6377\,\text{km}$, flattening $f \approx 1/308.6$, $s \approx 586\,\text{km}$. In this example, Bessel uses the toise as the unit of length and the second as the unit of arc.

[27] Bessel solves 3 equations (7) for 2 unknowns M and m. The redundancy serves as a check for the hand calculation and can also improve the accuracy of the calculation, for example, in the case where $\sin m \approx 1$.

[28] It is necessary to use second differences when interpolating in the table for $\log\alpha$. The argument, $\bar{2}.797\,216$, lies $q = 0.7216$ of the way between $\bar{2}.79$ and $\bar{2}.80$. Bessel's central 2nd-order interpolation formula for the last 6 digits of $\log\alpha$ gives $401\,284 + q(-1941) + \frac{1}{4}q(q-1)(1853 - 1004 - 1028) = 399\,892$. For the other table look-ups, linear interpolation using first differences suffices.

9. THE LONGITUDE INTEGRAL

We turn now to the determination of the longitude difference w by integrating (5),

$$dw = \sqrt{1 - e^2 \cos^2 u}\, d\omega.$$

This integral contains two separate constants m and e, which cannot be combined. Thus it not possible to construct tables to allow a rigorous solution of this problem which are valid for arbitrary e.[29] However, we can achieve our goal by sacrificing strict rigor and by making an approximation which results in errors which are inconsequential in our application.

We start by writing

$$dw = d\omega - \bigl(1 - \sqrt{1 - e^2 \cos^2 u}\bigr) d\omega,$$

and substitute in the second term

$$d\omega = \frac{\sin\alpha' \cos u'}{\cos^2 u}\, d\sigma.$$

On integrating, we obtain

$$w = \omega - \sin\alpha' \cos u' \int \frac{1 - \sqrt{1 - e^2 \cos^2 u}}{\cos^2 u}\, d\sigma.$$

Let us write

$$\frac{1 - \sqrt{1 - e^2 \cos^2 u}}{\cos^2 u} = \frac{e^2}{2} (1 + e^2 p \cos^2 u)^q (1 + y);$$

in other words, we set

$$\begin{aligned} 1 + y &= \frac{2(1 - \sqrt{1 - e^2 \cos^2 u})}{e^2 \cos^2 u (1 + e^2 p \cos^2 u)^q} \\ &= \frac{1 + \frac14 e^2 \cos^2 u + \frac18 e^4 \cos^4 u + \frac{5}{64} e^6 \cos^6 u + \ldots}{\left(\begin{aligned} 1 + qpe^2 \cos^2 u + \tfrac{q(q-1)}{1\cdot 2} p^2 e^4 \cos^4 u \\ + \tfrac{q(q-1)(q-2)}{1\cdot 2\cdot 3} p^3 e^6 \cos^6 u + \ldots \end{aligned} \right)}. \end{aligned}$$

The first three terms in the denominator and in the numerator are equal, provided that

$$p = -\tfrac34, \qquad q = -\tfrac13,$$

which gives

$$\begin{aligned} 1 + y &= \frac{1 + \frac14 e^2 \cos^2 u + \frac18 e^4 \cos^4 u + \frac{5}{64} e^6 \cos^6 u + \ldots}{1 + \frac14 e^2 \cos^2 u + \frac18 e^4 \cos^4 u + \frac{7}{96} e^6 \cos^6 u + \ldots} \\ &= 1 + \tfrac{1}{192} e^6 \cos^6 u + \ldots \end{aligned}$$

From this, we see that neglecting y results in an error of order e^8 or an error in w of $\frac{1}{384} e^8 \sigma$. This would not be discernible even in the calculation of long geodesics to 10 decimal places.[30]

Thus, for the present purposes, we may take $y \approx 0$ enabling us to tabulate the integral in a way that is valid for all e.

10. SERIES EXPANSION FOR LONGITUDE

Introducing this approximation, we have

$$\begin{aligned} w &\approx \omega - \frac{e^2}{2} \sin m \int \frac{d\sigma}{\sqrt[3]{1 - \frac34 e^2 \cos^2 u}} \\ &= \omega - \frac{e^2}{2} \sin m \int \frac{d\sigma}{\sqrt[3]{1 - \frac34 e^2 + \frac34 e^2 \cos^2 m \sin^2(M + \sigma)}}. \end{aligned}$$

If we set

$$k' = \frac{\sqrt{\frac34} e \cos m}{\sqrt{1 - \frac34 e^2}},$$

we can express the integral in the second term as

$$\int \frac{d\sigma}{\sqrt[3]{1 - \frac34 e^2}\, \sqrt[3]{1 + k'^2 \sin^2(M + \sigma)}}.$$

Following the same procedure used in expanding the integral for ds in Sec. 5, we introduce ϵ' defined by[31]

$$\epsilon' = \frac{\sqrt{1 + k'^2} - 1}{\sqrt{1 + k'^2} + 1}, \qquad k' = \frac{2\sqrt{\epsilon'}}{1 - \epsilon'},$$

and separate the integrand into two complex factors,

$$\int \frac{\sqrt[3]{(1 - \epsilon')^2 / \bigl(1 - \frac34 e^2\bigr)}\, d\sigma}{\sqrt[3]{1 - \epsilon' \exp\bigl(2i(M + \sigma)\bigr)}\, \sqrt[3]{1 - \epsilon' \exp\bigl(-2i(M + \sigma)\bigr)}}.$$

If we expand these in infinite series, the product becomes[32]

$$\frac{2}{\sqrt[3]{1 - \frac34 e^2}} \int \Bigl(\alpha' + \beta' \cos 2(M + \sigma) + 2\gamma' \cos 4(M + \sigma) + 3\delta' \cos 6(M + \sigma) + \ldots \Bigr) d\sigma,$$

[29] As a practical matter, it would have been impossible for Bessel to provide a complete tabulation of a function of two parameters. He could have tabulated the function for a fixed value of e, which would greatly reduced the utility of his method, especially given the uncertainties in the measurements of e. Instead, Bessel manipulates the expression for dw to move the dependence on the second parameter into a small term that may be neglected.

[30] For a flattening of $\frac{1}{128}$, the error in the longitude difference over a distance equivalent to a quarter meridian, i.e., $10\,000$ km, is less than $0.000\,05''$.

[31] Bessel gives the relationship between k' and ϵ' in terms of E', where $k' = \tan E'$ and $\epsilon' = \tan^2 \frac12 E'$.

[32] There are a series of errors in the original paper leading up to (12). Here we assume that the original Eq. (12) defines α', β', γ', …, which makes this equation analogous to (11), and correct the preceding equations to be consistent.

where[33]

$$\alpha' = \tfrac{1}{2}\sqrt[3]{(1-\epsilon')^2}\left[1 + \left(\frac{1}{3}\right)^2 \epsilon'^2 + \left(\frac{1\cdot 4}{3\cdot 6}\right)^2 \epsilon'^4 + \dots\right],$$

$$\beta' = \tfrac{1}{1}\sqrt[3]{(1-\epsilon')^2}\left[\frac{1}{3}\epsilon' + \frac{1\cdot 4}{3\cdot 6}\,\frac{1}{3}\epsilon'^3 + \frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}\,\frac{1\cdot 4}{3\cdot 6}\epsilon'^5 + \dots\right],$$

$$\gamma' = \tfrac{1}{2}\sqrt[3]{(1-\epsilon')^2}\left[\frac{1\cdot 4}{3\cdot 6}\epsilon'^2 + \frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}\,\frac{1}{3}\epsilon'^4 + \frac{1\cdot 4\cdot 7\cdot 10}{3\cdot 6\cdot 9\cdot 12}\,\frac{1\cdot 4}{3\cdot 6}\epsilon'^6 + \dots\right],$$

$$\delta' = \tfrac{1}{3}\sqrt[3]{(1-\epsilon')^2}\left[\frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}\epsilon'^3 + \frac{1\cdot 4\cdot 7\cdot 10}{3\cdot 6\cdot 9\cdot 12}\,\frac{1}{3}\epsilon'^5 + \frac{1\cdot 4\cdot 7\cdot 10\cdot 13}{3\cdot 6\cdot 9\cdot 12\cdot 15}\,\frac{1\cdot 4}{3\cdot 6}\epsilon'^7 + \dots\right],$$

etc.

Integrating from $\sigma = 0$ then gives

$$w \approx \omega - \frac{e^2 \sin m}{\sqrt[3]{1-\frac{3}{4}e^2}}\Big(\alpha'\sigma + \beta'\cos(2M+\sigma)\sin\sigma + \gamma'\cos(4M+2\sigma)\sin 2\sigma + \delta'\cos(6M+3\sigma)\sin 3\sigma + \dots\Big). \quad (12)$$

11. COMPUTING THE LONGITUDE DIFFERENCE

The first two coefficients of this series are given in the 4th and 5th columns of the tables[34] as functions of the argument

$$\log k' = \log\left(\frac{\sqrt{\frac{3}{4}}e}{\sqrt{1-\frac{3}{4}e^2}}\cos m\right).$$

The convergence is commensurate with the 3 first columns of the tables. We calculate ω using one of the formulas for spherical triangles (Sec. 3), either[35]

$$\sin\omega = \frac{\sin\sigma\sin\alpha'}{\cos u} = \frac{-\sin\sigma\sin\alpha}{\cos u'} = \frac{\sin\sigma\sin m}{\cos u\cos u'},$$

or[36]

$$\tan\tfrac{1}{2}\omega = \frac{\sin\frac{1}{2}(u'-u)}{\cos\frac{1}{2}(u'+u)}\cot\tfrac{1}{2}(\alpha'+\alpha) = \frac{\cos\frac{1}{2}(u'-u)}{\sin\frac{1}{2}(u'+u)}\cot\tfrac{1}{2}(\alpha'-\alpha).$$

[33] See footnote 18 and set $(1-x)^{-1/3} = 1 + \frac{1}{3}x + \frac{1\cdot 4}{3\cdot 6}x^2 + \frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}x^3 + \frac{1\cdot 4\cdot 7\cdot 10}{3\cdot 6\cdot 9\cdot 12}x^4 + \dots$

[34] The value of β' in the tables includes the factor of $648\,000/\pi$ necessary to convert from radians to arc seconds.

[35] The first two relations are the sine rule for angle N of triangle ABN of Fig. 1. The last relation is obtained, for example, by substituting for $\sin\alpha'$ from (7).

[36] These are Napier's analogies for angle N of triangle ABN.

and evaluate w by means of the tables.

I will continue with the example in Sec. 8 and calculate the longitude difference between Dunkirk and Seeberg using this prescription. Solving the spherical triangle for ω gives

$$\begin{aligned} \log\sin\sigma &= \bar{2}.963\,483\,83 \\ \log(-\sin\alpha) &= \bar{1}.999\,695\,39(-) \\ \operatorname{colog}\cos u' &= 0.199\,673\,73 \\ \log\sin\omega &= \bar{1}.162\,852\,95(-); \quad \omega = -8^\circ\,21'\,57.741''. \end{aligned}$$

The argument for the last two columns of the tables is $\log\big((\sqrt{\frac{3}{4}}e/\sqrt{1-\frac{3}{4}e^2})\cos m\big)$, giving

$$\begin{aligned} \log\frac{\sqrt{\frac{3}{4}}e}{\sqrt{1-\frac{3}{4}e^2}} &= \bar{2}.844\,022 \\ \log\cos m &= \bar{1}.890\,371 \\ \text{Argument} &= \bar{2}.734\,393. \end{aligned}$$

Computing the terms in the series (12) gives

$$\begin{aligned} \log\alpha' &= \bar{1}.698\,758 \\ \log(-\sin m) &= \bar{1}.799\,073 \\ \log\frac{e^2}{\sqrt[3]{1-\frac{3}{4}e^2}} &= \bar{3}.811\,575 \\ \log\sigma &= 4.278\,523 \\ & 1.587\,929 = +38.719'', \end{aligned}$$

and

$$\begin{aligned} \log\beta' &= 1.703 \\ \log(-\sin m) &= \bar{1}.799 \\ \log\frac{e^2}{\sqrt[3]{1-\frac{3}{4}e^2}} &= \bar{3}.812 \\ \log\big(\cos(2M+\sigma)\sin\sigma\big) &= \bar{2}.963(-) \\ & \bar{2}.277(-) = -0.019''. \end{aligned}$$

The sum of both terms is $+38.700''$, and adding this to ω, we find the longitude difference,

$$w = -8^\circ\,21'\,19.041''.$$

12. CONCLUSION

This illustration of the use of these tables shows that the accuracy of the calculation is limited not by the neglect of terms of high order in the eccentricity, but by the number of decimal places included. The steps in the calculation are, for the most part, the same as for a spherical earth; in order to account for the earth's ellipticity one needs, in addition, only to solve equation (11) and to evaluate the series (12). Since this approach is sufficiently convenient even for routine use, it is unnecessary to use an approximate method which is valid only for small distances.

(The tables are shown on the following pages.)

TABLES for computing geodesics 1.

Arg	$\log\alpha$	$-\Delta$	$\log\beta$	Δ	$\log\gamma$	Δ	$\log\alpha'$	$-\Delta$	$\log\beta'$	Δ
$\bar{4}.4$	5.314 425 13	1	$\bar{3}.5124$	2000			$\bar{1}.698\,970$	0	$\bar{3}.035$	200
$\bar{4}.5$	5.314 425 12	0	$\bar{3}.7124$	2000			$\bar{1}.698\,970$	0	$\bar{3}.235$	200
$\bar{4}.6$	5.314 425 12	1	$\bar{3}.9124$	2000			$\bar{1}.698\,970$	0	$\bar{3}.435$	200
$\bar{4}.7$	5.314 425 11	2	$\bar{2}.1124$	2000			$\bar{1}.698\,970$	0	$\bar{3}.635$	200
$\bar{4}.8$	5.314 425 09	3	$\bar{2}.3124$	2000			$\bar{1}.698\,970$	0	$\bar{3}.835$	200
$\bar{4}.9$	5.314 425 06	4	$\bar{2}.5124$	2000			$\bar{1}.698\,970$	0	$\bar{2}.035$	200
$\bar{3}.0$	5.314 425 02	6	$\bar{2}.7124$	2000			$\bar{1}.698\,970$	0	$\bar{2}.235$	200
$\bar{3}.1$	5.314 424 96	10	$\bar{2}.9124$	2000			$\bar{1}.698\,970$	0	$\bar{2}.435$	200
$\bar{3}.2$	5.314 424 86	16	$\bar{1}.1124$	2000			$\bar{1}.698\,970$	0	$\bar{2}.635$	200
$\bar{3}.3$	5.314 424 70	25	$\bar{1}.3124$	2000			$\bar{1}.698\,970$	0	$\bar{2}.835$	200
$\bar{3}.4$	5.314 424 45	40	$\bar{1}.5124$	2000			$\bar{1}.698\,970$	1	$\bar{1}.035$	200
$\bar{3}.50$	5.314 424 05	5	$\bar{1}.7124$	200			$\bar{1}.698\,969$	0	$\bar{1}.235$	20
$\bar{3}.51$	5.314 424 00	6	$\bar{1}.7324$	200			$\bar{1}.698\,969$	0	$\bar{1}.255$	20
$\bar{3}.52$	5.314 423 94	5	$\bar{1}.7524$	200			$\bar{1}.698\,969$	0	$\bar{1}.275$	20
$\bar{3}.53$	5.314 423 89	6	$\bar{1}.7724$	200			$\bar{1}.698\,969$	0	$\bar{1}.295$	20
$\bar{3}.54$	5.314 423 83	6	$\bar{1}.7924$	200			$\bar{1}.698\,969$	0	$\bar{1}.315$	20
$\bar{3}.55$	5.314 423 77	7	$\bar{1}.8124$	200			$\bar{1}.698\,969$	0	$\bar{1}.335$	20
$\bar{3}.56$	5.314 423 70	7	$\bar{1}.8324$	200			$\bar{1}.698\,969$	0	$\bar{1}.355$	20
$\bar{3}.57$	5.314 423 63	7	$\bar{1}.8524$	200			$\bar{1}.698\,969$	0	$\bar{1}.375$	20
$\bar{3}.58$	5.314 423 56	7	$\bar{1}.8724$	200			$\bar{1}.698\,969$	0	$\bar{1}.395$	20
$\bar{3}.59$	5.314 423 49	8	$\bar{1}.8924$	200			$\bar{1}.698\,969$	0	$\bar{1}.415$	20
$\bar{3}.60$	5.314 423 41	8	$\bar{1}.9124$	200			$\bar{1}.698\,969$	0	$\bar{1}.435$	20
$\bar{3}.61$	5.314 423 33	8	$\bar{1}.9324$	200			$\bar{1}.698\,969$	0	$\bar{1}.455$	20
$\bar{3}.62$	5.314 423 25	9	$\bar{1}.9524$	200			$\bar{1}.698\,969$	0	$\bar{1}.475$	20
$\bar{3}.63$	5.314 423 16	10	$\bar{1}.9724$	200			$\bar{1}.698\,969$	0	$\bar{1}.495$	20
$\bar{3}.64$	5.314 423 06	9	$\bar{1}.9924$	200			$\bar{1}.698\,969$	0	$\bar{1}.515$	20
$\bar{3}.65$	5.314 422 97	11	0.0124	200			$\bar{1}.698\,969$	1	$\bar{1}.535$	20
$\bar{3}.66$	5.314 422 86	10	0.0324	200			$\bar{1}.698\,968$	0	$\bar{1}.555$	20
$\bar{3}.67$	5.314 422 76	11	0.0524	200			$\bar{1}.698\,968$	0	$\bar{1}.575$	20
$\bar{3}.68$	5.314 422 65	12	0.0724	200			$\bar{1}.698\,968$	0	$\bar{1}.595$	20
$\bar{3}.69$	5.314 422 53	12	0.0924	200			$\bar{1}.698\,968$	0	$\bar{1}.615$	20
$\bar{3}.70$	5.314 422 41	13	0.1124	200			$\bar{1}.698\,968$	0	$\bar{1}.635$	20
$\bar{3}.71$	5.314 422 28	14	0.1324	200			$\bar{1}.698\,968$	0	$\bar{1}.655$	20
$\bar{3}.72$	5.314 422 14	14	0.1524	200			$\bar{1}.698\,968$	0	$\bar{1}.675$	20
$\bar{3}.73$	5.314 422 00	15	0.1724	200			$\bar{1}.698\,968$	0	$\bar{1}.695$	20
$\bar{3}.74$	5.314 421 85	15	0.1924	200			$\bar{1}.698\,968$	0	$\bar{1}.715$	20
$\bar{3}.75$	5.314 421 70	16	0.2124	200			$\bar{1}.698\,968$	0	$\bar{1}.735$	20
$\bar{3}.76$	5.314 421 54	17	0.2324	200			$\bar{1}.698\,968$	1	$\bar{1}.755$	20
$\bar{3}.77$	5.314 421 37	18	0.2524	200			$\bar{1}.698\,967$	0	$\bar{1}.775$	20
$\bar{3}.78$	5.314 421 19	18	0.2724	200			$\bar{1}.698\,967$	0	$\bar{1}.795$	20
$\bar{3}.79$	5.314 421 01	20	0.2924	200			$\bar{1}.698\,967$	0	$\bar{1}.815$	20
$\bar{3}.80$	5.314 420 81	20	0.3124	200			$\bar{1}.698\,967$	0	$\bar{1}.835$	20
$\bar{3}.81$	5.314 420 61	22	0.3324	200			$\bar{1}.698\,967$	0	$\bar{1}.855$	20
$\bar{3}.82$	5.314 420 39	22	0.3524	200			$\bar{1}.698\,967$	0	$\bar{1}.875$	20
$\bar{3}.83$	5.314 420 17	23	0.3724	200			$\bar{1}.698\,967$	0	$\bar{1}.895$	20
$\bar{3}.84$	5.314 419 94	25	0.3924	200			$\bar{1}.698\,967$	1	$\bar{1}.915$	20
$\bar{3}.85$	5.314 419 69	25	0.4124	200			$\bar{1}.698\,966$	0	$\bar{1}.935$	20
$\bar{3}.86$	5.314 419 44	27	0.4324	200			$\bar{1}.698\,966$	0	$\bar{1}.955$	20
$\bar{3}.87$	5.314 419 17	28	0.4524	200			$\bar{1}.698\,966$	0	$\bar{1}.975$	20
$\bar{3}.88$	5.314 418 89	30	0.4724	200			$\bar{1}.698\,966$	0	$\bar{1}.995$	20
$\bar{3}.89$	5.314 418 59	31	0.4924	200			$\bar{1}.698\,966$	1	0.015	20
$\bar{3}.90$	5.314 418 28		0.5124				$\bar{1}.698\,965$		0.035	

TABLES for computing geodesics 2.

Arg	$\log\alpha$	$-\Delta$	$\log\beta$	Δ	$\log\gamma$	Δ	$\log\alpha'$	$-\Delta$	$\log\beta'$	Δ
$\bar{3}.90$	5.314 418 28	32	0.512 35	2000			$\bar{1}.698\,965$	0	0.035	20
$\bar{3}.91$	5.314 417 96	34	0.532 35	2000			$\bar{1}.698\,965$	0	0.055	20
$\bar{3}.92$	5.314 417 62	35	0.552 35	2000			$\bar{1}.698\,965$	0	0.075	20
$\bar{3}.93$	5.314 417 27	37	0.572 35	2000			$\bar{1}.698\,965$	0	0.095	20
$\bar{3}.94$	5.314 416 90	39	0.592 35	2000			$\bar{1}.698\,965$	1	0.115	20
$\bar{3}.95$	5.314 416 51	41	0.612 35	2000			$\bar{1}.698\,964$	0	0.135	20
$\bar{3}.96$	5.314 416 10	42	0.632 35	2000			$\bar{1}.698\,964$	0	0.155	20
$\bar{3}.97$	5.314 415 68	45	0.652 35	2000			$\bar{1}.698\,964$	1	0.175	20
$\bar{3}.98$	5.314 415 23	47	0.672 35	1999			$\bar{1}.698\,963$	0	0.195	20
$\bar{3}.99$	5.314 414 76	48	0.692 34	2000			$\bar{1}.698\,963$	0	0.215	20
$\bar{2}.00$	5.314 414 28	52	0.712 34	2000			$\bar{1}.698\,963$	1	0.235	20
$\bar{2}.01$	5.314 413 76	53	0.732 34	2000			$\bar{1}.698\,962$	0	0.255	20
$\bar{2}.02$	5.314 413 23	56	0.752 34	2000			$\bar{1}.698\,962$	0	0.275	20
$\bar{2}.03$	5.314 412 67	59	0.772 34	2000			$\bar{1}.698\,962$	1	0.295	20
$\bar{2}.04$	5.314 412 08	61	0.792 34	2000			$\bar{1}.698\,961$	0	0.315	20
$\bar{2}.05$	5.314 411 47	65	0.812 34	2000			$\bar{1}.698\,961$	1	0.335	20
$\bar{2}.06$	5.314 410 82	67	0.832 34	2000			$\bar{1}.698\,960$	0	0.355	20
$\bar{2}.07$	5.314 410 15	71	0.852 34	1999			$\bar{1}.698\,960$	0	0.375	20
$\bar{2}.08$	5.314 409 44	74	0.872 33	2000			$\bar{1}.698\,960$	1	0.395	20
$\bar{2}.09$	5.314 408 70	77	0.892 33	2000			$\bar{1}.698\,959$	0	0.415	20
$\bar{2}.10$	5.314 407 93	81	0.912 33	2000			$\bar{1}.698\,959$	1	0.435	20
$\bar{2}.11$	5.314 407 12	85	0.932 33	2000			$\bar{1}.698\,958$	1	0.455	20
$\bar{2}.12$	5.314 406 27	89	0.952 33	2000			$\bar{1}.698\,957$	0	0.475	20
$\bar{2}.13$	5.314 405 38	93	0.972 33	1999			$\bar{1}.698\,957$	1	0.495	20
$\bar{2}.14$	5.314 404 45	98	0.992 32	2000			$\bar{1}.698\,956$	0	0.515	20
$\bar{2}.15$	5.314 403 47	102	1.012 32	2000			$\bar{1}.698\,956$	1	0.535	20
$\bar{2}.16$	5.314 402 45	107	1.032 32	2000			$\bar{1}.698\,955$	1	0.555	20
$\bar{2}.17$	5.314 401 38	112	1.052 32	2000			$\bar{1}.698\,954$	1	0.575	20
$\bar{2}.18$	5.314 400 26	117	1.072 32	1999			$\bar{1}.698\,953$	0	0.595	20
$\bar{2}.19$	5.314 399 09	123	1.092 31	2000			$\bar{1}.698\,953$	1	0.615	20
$\bar{2}.20$	5.314 397 86	128	1.112 31	2000			$\bar{1}.698\,952$	1	0.635	20
$\bar{2}.21$	5.314 396 58	135	1.132 31	2000			$\bar{1}.698\,951$	1	0.655	20
$\bar{2}.22$	5.314 395 23	141	1.152 31	1999			$\bar{1}.698\,950$	1	0.675	20
$\bar{2}.23$	5.314 393 82	147	1.172 30	2000			$\bar{1}.698\,949$	1	0.695	20
$\bar{2}.24$	5.314 392 35	155	1.192 30	2000			$\bar{1}.698\,948$	1	0.715	20
$\bar{2}.25$	5.314 390 80	162	1.212 30	1999	$\bar{4}.207$	40	$\bar{1}.698\,947$	1	0.735	20
$\bar{2}.26$	5.314 389 18	169	1.232 29	2000	$\bar{4}.247$	40	$\bar{1}.698\,946$	1	0.755	20
$\bar{2}.27$	5.314 387 49	177	1.252 29	2000	$\bar{4}.287$	40	$\bar{1}.698\,945$	1	0.775	20
$\bar{2}.28$	5.314 385 72	186	1.272 29	1999	$\bar{4}.327$	40	$\bar{1}.698\,944$	2	0.795	20
$\bar{2}.29$	5.314 383 86	195	1.292 28	2000	$\bar{4}.367$	40	$\bar{1}.698\,942$	1	0.815	20
$\bar{2}.30$	5.314 381 91	203	1.312 28	1999	$\bar{4}.407$	40	$\bar{1}.698\,941$	1	0.835	20
$\bar{2}.31$	5.314 379 88	213	1.332 27	2000	$\bar{4}.447$	40	$\bar{1}.698\,940$	2	0.855	20
$\bar{2}.32$	5.314 377 75	224	1.352 27	2000	$\bar{4}.487$	40	$\bar{1}.698\,938$	1	0.875	20
$\bar{2}.33$	5.314 375 51	234	1.372 27	1999	$\bar{4}.527$	40	$\bar{1}.698\,937$	2	0.895	20
$\bar{2}.34$	5.314 373 17	244	1.392 26	2000	$\bar{4}.567$	40	$\bar{1}.698\,935$	1	0.915	20
$\bar{2}.35$	5.314 370 73	257	1.412 26	1999	$\bar{4}.607$	40	$\bar{1}.698\,934$	2	0.935	20
$\bar{2}.36$	5.314 368 16	268	1.432 25	2000	$\bar{4}.647$	40	$\bar{1}.698\,932$	2	0.955	20
$\bar{2}.37$	5.314 365 48	281	1.452 25	1999	$\bar{4}.687$	40	$\bar{1}.698\,930$	2	0.975	20
$\bar{2}.38$	5.314 362 67	295	1.472 24	1999	$\bar{4}.727$	40	$\bar{1}.698\,928$	2	0.995	20
$\bar{2}.39$	5.314 359 72	308	1.492 23	2000	$\bar{4}.767$	40	$\bar{1}.698\,926$	2	1.015	20
$\bar{2}.40$	5.314 356 64		1.512 23		$\bar{4}.807$		$\bar{1}.698\,924$		1.035	

TABLES for computing geodesics 3.

Arg	$\log\alpha$	$-\Delta$	$\log\beta$	Δ	$\log\gamma$	Δ	$\log\alpha'$	$-\Delta$	$\log\beta'$	Δ
$\bar{2}.40$	5.314 356 64	323	1.512 23	1999	$\bar{4}.807$	40	$\bar{1}.698\,924$	2	1.035	20
$\bar{2}.41$	5.314 353 41	338	1.532 22	1999	$\bar{4}.847$	40	$\bar{1}.698\,922$	2	1.055	20
$\bar{2}.42$	5.314 350 03	353	1.552 21	2000	$\bar{4}.887$	40	$\bar{1}.698\,920$	2	1.075	20
$\bar{2}.43$	5.314 346 50	371	1.572 21	1999	$\bar{4}.927$	40	$\bar{1}.698\,918$	3	1.095	20
$\bar{2}.44$	5.314 342 79	388	1.592 20	1999	$\bar{4}.967$	40	$\bar{1}.698\,915$	2	1.115	20
$\bar{2}.45$	5.314 338 91	406	1.612 19	1999	$\bar{3}.007$	40	$\bar{1}.698\,913$	3	1.135	20
$\bar{2}.46$	5.314 334 85	425	1.632 18	2000	$\bar{3}.047$	40	$\bar{1}.698\,910$	3	1.155	20
$\bar{2}.47$	5.314 330 60	446	1.652 18	1999	$\bar{3}.087$	40	$\bar{1}.698\,907$	3	1.175	20
$\bar{2}.48$	5.314 326 14	466	1.672 17	1999	$\bar{3}.127$	40	$\bar{1}.698\,904$	3	1.195	20
$\bar{2}.49$	5.314 321 48	489	1.692 16	1999	$\bar{3}.167$	40	$\bar{1}.698\,901$	3	1.215	20
$\bar{2}.50$	5.314 316 59	511	1.712 15	1999	$\bar{3}.207$	40	$\bar{1}.698\,898$	4	1.235	20
$\bar{2}.51$	5.314 311 48	535	1.732 14	1999	$\bar{3}.247$	40	$\bar{1}.698\,894$	3	1.255	20
$\bar{2}.52$	5.314 306 13	561	1.752 13	1999	$\bar{3}.287$	40	$\bar{1}.698\,891$	4	1.275	20
$\bar{2}.53$	5.314 300 52	587	1.772 12	1998	$\bar{3}.327$	40	$\bar{1}.698\,887$	4	1.295	20
$\bar{2}.54$	5.314 294 65	615	1.792 10	1999	$\bar{3}.367$	40	$\bar{1}.698\,883$	4	1.315	20
$\bar{2}.55$	5.314 288 50	644	1.812 09	1999	$\bar{3}.407$	40	$\bar{1}.698\,879$	4	1.335	20
$\bar{2}.56$	5.314 282 06	674	1.832 08	1999	$\bar{3}.447$	40	$\bar{1}.698\,875$	5	1.355	20
$\bar{2}.57$	5.314 275 32	705	1.852 07	1998	$\bar{3}.487$	40	$\bar{1}.698\,870$	5	1.375	20
$\bar{2}.58$	5.314 268 27	739	1.872 05	1999	$\bar{3}.527$	40	$\bar{1}.698\,865$	4	1.395	20
$\bar{2}.59$	5.314 260 88	774	1.892 04	1998	$\bar{3}.567$	40	$\bar{1}.698\,861$	6	1.415	20
$\bar{2}.60$	5.314 253 14	810	1.912 02	1998	$\bar{3}.607$	39	$\bar{1}.698\,855$	5	1.435	20
$\bar{2}.61$	5.314 245 04	848	1.932 00	1999	$\bar{3}.646$	40	$\bar{1}.698\,850$	6	1.455	20
$\bar{2}.62$	5.314 236 56	889	1.951 99	1998	$\bar{3}.686$	40	$\bar{1}.698\,844$	6	1.475	20
$\bar{2}.63$	5.314 227 67	930	1.971 97	1998	$\bar{3}.726$	40	$\bar{1}.698\,838$	6	1.495	20
$\bar{2}.64$	5.314 218 37	973	1.991 95	1998	$\bar{3}.766$	40	$\bar{1}.698\,832$	6	1.515	20
$\bar{2}.65$	5.314 208 64	1020	2.011 93	1998	$\bar{3}.806$	40	$\bar{1}.698\,826$	7	1.535	20
$\bar{2}.66$	5.314 198 44	1068	2.031 91	1998	$\bar{3}.846$	40	$\bar{1}.698\,819$	7	1.555	20
$\bar{2}.67$	5.314 187 76	1118	2.051 89	1998	$\bar{3}.886$	40	$\bar{1}.698\,812$	8	1.575	20
$\bar{2}.68$	5.314 176 58	1170	2.071 87	1997	$\bar{3}.926$	40	$\bar{1}.698\,804$	7	1.595	20
$\bar{2}.69$	5.314 164 88	1226	2.091 84	1998	$\bar{3}.966$	40	$\bar{1}.698\,797$	9	1.615	20
$\bar{2}.70$	5.314 152 62	1283	2.111 82	1997	$\bar{2}.006$	40	$\bar{1}.698\,788$	8	1.635	19
$\bar{2}.71$	5.314 139 79	1344	2.131 79	1998	$\bar{2}.046$	40	$\bar{1}.698\,780$	9	1.654	20
$\bar{2}.72$	5.314 126 35	1406	2.151 77	1997	$\bar{2}.086$	40	$\bar{1}.698\,771$	9	1.674	20
$\bar{2}.73$	5.314 112 29	1473	2.171 74	1997	$\bar{2}.126$	40	$\bar{1}.698\,762$	10	1.694	20
$\bar{2}.74$	5.314 097 56	1543	2.191 71	1997	$\bar{2}.166$	40	$\bar{1}.698\,752$	11	1.714	20
$\bar{2}.75$	5.314 082 13	1615	2.211 68	1997	$\bar{2}.206$	40	$\bar{1}.698\,741$	10	1.734	20
$\bar{2}.76$	5.314 065 98	1690	2.231 65	1996	$\bar{2}.246$	40	$\bar{1}.698\,731$	12	1.754	20
$\bar{2}.77$	5.314 049 08	1771	2.251 61	1997	$\bar{2}.286$	40	$\bar{1}.698\,719$	11	1.774	20
$\bar{2}.78$	5.314 031 37	1853	2.271 58	1996	$\bar{2}.326$	40	$\bar{1}.698\,708$	13	1.794	20
$\bar{2}.79$	5.314 012 84	1941	2.291 54	1996	$\bar{2}.366$	39	$\bar{1}.698\,695$	13	1.814	20
$\bar{2}.800$	5.313 993 43	1004	2.311 50	998	$\bar{2}.405$	20	$\bar{1}.698\,682$	6	1.834	10
$\bar{2}.805$	5.313 983 39	1028	2.321 48	998	$\bar{2}.425$	20	$\bar{1}.698\,676$	7	1.844	10
$\bar{2}.810$	5.313 973 11	1051	2.331 46	998	$\bar{2}.445$	20	$\bar{1}.698\,669$	7	1.854	10
$\bar{2}.815$	5.313 962 60	1076	2.341 44	998	$\bar{2}.465$	20	$\bar{1}.698\,662$	7	1.864	10
$\bar{2}.820$	5.313 951 84	1101	2.351 42	998	$\bar{2}.485$	20	$\bar{1}.698\,655$	8	1.874	10
$\bar{2}.825$	5.313 940 83	1127	2.361 40	997	$\bar{2}.505$	20	$\bar{1}.698\,647$	7	1.884	10
$\bar{2}.830$	5.313 929 56	1152	2.371 37	998	$\bar{2}.525$	20	$\bar{1}.698\,640$	8	1.894	10
$\bar{2}.835$	5.313 918 04	1180	2.381 35	998	$\bar{2}.545$	20	$\bar{1}.698\,632$	8	1.904	10
$\bar{2}.840$	5.313 906 24	1207	2.391 33	997	$\bar{2}.565$	20	$\bar{1}.698\,624$	8	1.914	10
$\bar{2}.845$	5.313 894 17	1234	2.401 30	998	$\bar{2}.585$	20	$\bar{1}.698\,616$	8	1.924	10
$\bar{2}.850$	5.313 881 83		2.411 28		$\bar{2}.605$		$\bar{1}.698\,608$		1.934	

TABLES for computing geodesics 4.

Arg	$\log\alpha$	$-\Delta$	$\log\beta$	Δ	$\log\gamma$	Δ	$\log\alpha'$	$-\Delta$	$\log\beta'$	Δ
$\bar{2}.850$	5.313 881 83	1264	2.411 279	9974	$\bar{2}.605$	20	$\bar{1}.698\,608$	8	1.934	10
$\bar{2}.855$	5.313 869 19	1293	2.421 253	9974	$\bar{2}.625$	20	$\bar{1}.698\,600$	9	1.944	10
$\bar{2}.860$	5.313 856 26	1323	2.431 227	9974	$\bar{2}.645$	20	$\bar{1}.698\,591$	9	1.954	10
$\bar{2}.865$	5.313 843 03	1353	2.441 201	9973	$\bar{2}.665$	20	$\bar{1}.698\,582$	9	1.964	10
$\bar{2}.870$	5.313 829 50	1385	2.451 174	9972	$\bar{2}.685$	20	$\bar{1}.698\,573$	9	1.974	10
$\bar{2}.875$	5.313 815 65	1417	2.461 146	9972	$\bar{2}.705$	20	$\bar{1}.698\,564$	10	1.984	10
$\bar{2}.880$	5.313 801 48	1450	2.471 118	9971	$\bar{2}.725$	20	$\bar{1}.698\,554$	9	1.994	10
$\bar{2}.885$	5.313 786 98	1484	2.481 089	9970	$\bar{2}.745$	20	$\bar{1}.698\,545$	10	2.004	10
$\bar{2}.890$	5.313 772 14	1518	2.491 059	9970	$\bar{2}.765$	20	$\bar{1}.698\,535$	10	2.014	9
$\bar{2}.895$	5.313 756 96	1553	2.501 029	9969	$\bar{2}.785$	19	$\bar{1}.698\,525$	11	2.023	10
$\bar{2}.900$	5.313 741 43	1590	2.510 998	9968	$\bar{2}.804$	20	$\bar{1}.698\,514$	10	2.033	10
$\bar{2}.905$	5.313 725 53	1626	2.520 966	9968	$\bar{2}.824$	20	$\bar{1}.698\,504$	11	2.043	10
$\bar{2}.910$	5.313 709 27	1664	2.530 934	9966	$\bar{2}.844$	20	$\bar{1}.698\,493$	11	2.053	10
$\bar{2}.915$	5.313 692 63	1702	2.540 900	9966	$\bar{2}.864$	20	$\bar{1}.698\,482$	11	2.063	10
$\bar{2}.920$	5.313 675 61	1742	2.550 866	9965	$\bar{2}.884$	20	$\bar{1}.698\,471$	12	2.073	10
$\bar{2}.925$	5.313 658 19	1783	2.560 831	9965	$\bar{2}.904$	20	$\bar{1}.698\,459$	12	2.083	10
$\bar{2}.930$	5.313 640 36	1824	2.570 796	9963	$\bar{2}.924$	20	$\bar{1}.698\,447$	12	2.093	10
$\bar{2}.935$	5.313 622 12	1866	2.580 759	9963	$\bar{2}.944$	20	$\bar{1}.698\,435$	12	2.103	10
$\bar{2}.940$	5.313 603 46	1909	2.590 722	9962	$\bar{2}.964$	20	$\bar{1}.698\,423$	13	2.113	10
$\bar{2}.945$	5.313 584 37	1953	2.600 684	9961	$\bar{2}.984$	20	$\bar{1}.698\,410$	13	2.123	10
$\bar{2}.950$	5.313 564 84	1999	2.610 645	9960	$\bar{1}.004$	20	$\bar{1}.698\,397$	13	2.133	10
$\bar{2}.955$	5.313 544 85	2045	2.620 605	9959	$\bar{1}.024$	20	$\bar{1}.698\,384$	14	2.143	10
$\bar{2}.960$	5.313 524 40	2093	2.630 564	9958	$\bar{1}.044$	20	$\bar{1}.698\,370$	14	2.153	10
$\bar{2}.965$	5.313 503 47	2141	2.640 522	9957	$\bar{1}.064$	19	$\bar{1}.698\,356$	14	2.163	10
$\bar{2}.970$	5.313 482 06	2191	2.650 479	9956	$\bar{1}.083$	20	$\bar{1}.698\,342$	15	2.173	10
$\bar{2}.975$	5.313 460 15	2241	2.660 435	9956	$\bar{1}.103$	20	$\bar{1}.698\,327$	15	2.183	10
$\bar{2}.980$	5.313 437 74	2293	2.670 391	9954	$\bar{1}.123$	20	$\bar{1}.698\,312$	15	2.193	10
$\bar{2}.985$	5.313 414 81	2347	2.680 345	9953	$\bar{1}.143$	20	$\bar{1}.698\,297$	16	2.203	9
$\bar{2}.990$	5.313 391 34	2400	2.690 298	9952	$\bar{1}.163$	20	$\bar{1}.698\,281$	15	2.212	10
$\bar{2}.995$	5.313 367 34	2457	2.700 250	9951	$\bar{1}.183$	20	$\bar{1}.698\,266$	17	2.222	10
$\bar{1}.000$	5.313 342 77	2513	2.710 201	9950	$\bar{1}.203$	20	$\bar{1}.698\,249$	17	2.232	10
$\bar{1}.005$	5.313 317 64	2571	2.720 151	9948	$\bar{1}.223$	20	$\bar{1}.698\,232$	17	2.242	10
$\bar{1}.010$	5.313 291 93	2631	2.730 099	9948	$\bar{1}.243$	20	$\bar{1}.698\,215$	17	2.252	10
$\bar{1}.015$	5.313 265 62	2691	2.740 047	9946	$\bar{1}.263$	19	$\bar{1}.698\,198$	18	2.262	10
$\bar{1}.020$	5.313 238 71	2754	2.749 993	9945	$\bar{1}.282$	20	$\bar{1}.698\,180$	18	2.272	10
$\bar{1}.025$	5.313 211 17	2818	2.759 938	9943	$\bar{1}.302$	20	$\bar{1}.698\,162$	19	2.282	10
$\bar{1}.030$	5.313 182 99	2883	2.769 881	9943	$\bar{1}.322$	20	$\bar{1}.698\,143$	19	2.292	10
$\bar{1}.035$	5.313 154 16	2949	2.779 824	9941	$\bar{1}.342$	20	$\bar{1}.698\,124$	20	2.302	10
$\bar{1}.040$	5.313 124 67	3018	2.789 765	9939	$\bar{1}.362$	20	$\bar{1}.698\,104$	20	2.312	10
$\bar{1}.045$	5.313 094 49	3087	2.799 704	9939	$\bar{1}.382$	20	$\bar{1}.698\,084$	20	2.322	10
$\bar{1}.050$	5.313 063 62	3159	2.809 643	9936	$\bar{1}.402$	20	$\bar{1}.698\,064$	21	2.332	10
$\bar{1}.055$	5.313 032 03	3232	2.819 579	9936	$\bar{1}.422$	20	$\bar{1}.698\,043$	22	2.342	9
$\bar{1}.060$	5.312 999 71	3306	2.829 515	9934	$\bar{1}.442$	19	$\bar{1}.698\,021$	22	2.351	10
$\bar{1}.065$	5.312 966 65	3383	2.839 449	9932	$\bar{1}.461$	20	$\bar{1}.697\,999$	22	2.361	10
$\bar{1}.070$	5.312 932 82	3460	2.849 381	9931	$\bar{1}.481$	20	$\bar{1}.697\,977$	23	2.371	10
$\bar{1}.075$	5.312 898 22	3541	2.859 312	9929	$\bar{1}.501$	20	$\bar{1}.697\,954$	24	2.381	10
$\bar{1}.080$	5.312 862 81	3623	2.869 241	9928	$\bar{1}.521$	20	$\bar{1}.697\,930$	24	2.391	10
$\bar{1}.085$	5.312 826 58	3706	2.879 169	9926	$\bar{1}.541$	20	$\bar{1}.697\,906$	25	2.401	10
$\bar{1}.090$	5.312 789 52	3791	2.889 095	9924	$\bar{1}.561$	20	$\bar{1}.697\,881$	25	2.411	10
$\bar{1}.095$	5.312 751 61	3879	2.899 019	9922	$\bar{1}.581$	19	$\bar{1}.697\,856$	26	2.421	10
$\bar{1}.100$	5.312 712 82		2.908 941		$\bar{1}.600$		$\bar{1}.697\,830$		2.431	