# COORDINATE TRANSFORMATIONS IN SURVEYING AND MAPPING

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Coordinate transformations are used in surveying and mapping to transform coordinates in one "system" to coordinates in another system, and take many forms. For example

- Map projections are transformations of geographical coordinates, latitude φ and longitude λ on a sphere or ellipsoid, to rectangular (or Cartesian) coordinates on a plane.
- Polar–Rectangular conversions where coordinates of points in polar coordinates, say bearings and distances, are converted to rectangular coordinates.
- Two-Dimensional (2D) transformations where the coordinates of points in one rectangular system (*x*, *y*) are transformed into coordinates in another rectangular system (*X*, *Y*).
- Three-Dimensional (3D) transformations where coordinates of points in one right-handed rectangular system (x,y,z) are transformed into another rectangular system (X,Y,Z).

3D transformations also include transformations from geographical coordinates ( $\phi$ ,  $\lambda$ ) on a reference surface (sphere or ellipsoid), to rectangular coordinates (*X*, *Y*,*Z*) whose origin is at the centre of the reference surface, or to a local rectangular system (*E*,*N*,*U*) whose origin is a point on the reference surface.

The effect of a transformation on a group of points defining a 2D polygon or 3D object varies from simple changes of location and orientation (without any change in shape or size), to uniform scale change (no change in shape), and, finally, to changes in shape and size of different degrees of non-linearity (Mikhail 1976).

In general we consider points in space as being connected to the origin O of a 3D right-handed rectangular coordinate system X, Y, Z. Such a system can be visualised as the corner of a room where the intersection of two walls and the floor provide three reference lines OX, OY and OZ, known as the *X*-, *Y*- and *Z*-axes that are (usually) at right angles to one another. The *X*-*Z* and *Y*-*Z* planes are the walls and the *X*-*Y* plane is the floor.



The three mutually perpendicular axes *X*, *Y* and *Z* are related by the *right-hand rule* as follows:

If the thumb, the forefinger and the second finger of the right hand are placed mutually at right angles then the thumb points in the Z-direction, the forefinger points in the X-direction and the second finger points in the Y-direction.

The axes X, Y and Z (in the cyclic order XYZ) are a *right-handed* system (or dextral system) since a rotation from X towards Y advances a right-handed screw in the direction of Z. Similarly, a rotation from Y towards Z advances a right-handed screw in the direction of X and so on. The diagram on the left shows the *right-hand screw rule* for the positive directions of rotations and axes of a right-handed rectangular coordinate system. These rotations are considered positive anticlockwise when looking along the axis towards the origin; the positive sense of rotation being determined by the *right-hand-grip* 

*rule* where an imaginary right hand grips the axis with the thumb pointing in the positive direction of the axis and the natural curl of the fingers indicating the positive direction of rotation. In the following pages, transformations in two-dimensional (2D) space are discussed: in such cases points are considered to have only *X*, *Y* coordinates, i.e., they lie in the *X*-*Y* plane with a *Z*-value = 0.

# 1. TRANSFORMATIONS IN TWO-DIMENSIONAL (2D) SPACE

In 2D transformations all points lie in a plane. In these notes it is assumed that 2D transformations are transformations from one rectangular coordinate system (u,v) to another rectangular system (x,y). In addition, unless stated otherwise, a rotation is an angle considered to be positive in an anticlockwise direction as determined by the *right-hand-grip rule*. This is consistent with mathematics, where angles are measured positive anticlockwise from the *x*-axis and also in applications in Photogrammetry and Remote Sensing.

#### 1.1. Transformations involving Rotation only

u,v coordinates are transformed to x,y coordinates by considering a rotation of the u,v coordinate axes through a positive anticlockwise angle  $\theta$ . The transformation equations can be expressed in the following way

$$\begin{aligned} x &= u\cos\theta + v\sin\theta\\ y &= -u\sin\theta + v\cos\theta \end{aligned} \tag{1.1}$$

or in matrix notation

| x               | $\int \cos\theta$     | $\sin\theta \left[ u \right]$  | (1.2) |
|-----------------|-----------------------|--------------------------------|-------|
| $y \rfloor^{-}$ | $\lfloor -\sin\theta$ | $\cos\theta \rfloor v \rfloor$ | (1.2) |

As an example consider the polygon *ABCD* whose *u*, *v* coordinates are rotated by a positive anticlockwise angle  $\theta = 30^\circ$ . Figure 1 shows the initial location of the polygon in the *u*, *v* system and Figure 2 shows its transformed (rotated) location in the *x*, *y* system.



Figure 1 Polygon ABCD with u, v coordinates in metres

y

D

211.603

51.036

Figure 2 Rotated polygon *ABCD* with *x*,*y* coordinates in metres

x

Comparing Figures 1 and 2 it appears that the size and shape of the polygon *ABCD* has not changed but its orientation with respect to the coordinate axes has. This can be verified by considering the dimensions (bearings and distances) of the polygon *ABCD* derived from the two coordinate sets.

| Line | Bearing  | Distance | Ι | Line | Bearing           | Distance |
|------|----------|----------|---|------|-------------------|----------|
| AB   | 30° 00′  | 200.000  | - | AB   | 60° 00′           | 200.000  |
| BC   | 120° 00′ | 100.000  |   | BC   | $150^{\circ}00'$  | 100.000  |
| CD   | 210° 00′ | 257.735  | ( | CD   | $240^{\circ}00'$  | 257.735  |
| DA   | 330° 00′ | 115.470  |   | DA   | $0^{\circ} \ 00'$ | 115.470  |
|      |          |          |   |      |                   |          |

Polygon dimensions in the *u*,*v* system

A

D

Polygon dimensions in the *x*,*y* system

This example demonstrates that a rotation of the coordinate axes causes an apparent rotation, in an opposite direction, of any polygon defined within the coordinate system. The size and shape of the polygon does not change.

Equation (1.1) and its matrix equivalent (1.2) can be obtained by considering Figure 3.



Figure 3 x,y coordinates of P as functions of u, v coordinates and rotation  $\theta$ 

#### 1.1.1. Rotation matrices

Equation (1.2) can be expressed as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{R} \begin{bmatrix} u \\ v \end{bmatrix}$$
(1.3)

where  $\mathbf{R} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$  is known as a rotation matrix. Rotation matrices are orthogonal, i.e.,

the sum of squares of the elements of any row or column is equal to unity and an orthogonal matrix has the unique property that its inverse is equal to its transpose, i.e.,  $\mathbf{R}^{-1} = \mathbf{R}^{T}$ . This useful property allows us to write the transformation from *x*, *y* coordinates to *u*, *v* coordinates as follows.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{R} \begin{bmatrix} u \\ v \end{bmatrix}$$
$$\mathbf{R}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{R}^{-1} \mathbf{R} \begin{bmatrix} u \\ v \end{bmatrix}$$
$$\mathbf{R}^{T} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{I} \begin{bmatrix} u \\ v \end{bmatrix}$$

and rearranging gives

$$\begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{R}^{T} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
(1.4)

We could write (1.4) as

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{R}^* \begin{bmatrix} x \\ y \end{bmatrix}$$

which in words means: the *x*,*y* coordinates are transformed (rotated) to *u*,*v* coordinates. Equation (1.3) on the other hand means: the *u*,*v* coordinates are transformed (rotated) to *x*,*y* coordinates and it is interesting to note that **R** and **R**<sup>\*</sup> are in fact the same rotation matrix except in the former,  $\theta$  is positive anticlockwise and in the latter  $\theta$  is positive clockwise. Note that  $\sin(-\theta) = -\sin\theta$  and  $\cos(-\theta) = \cos\theta$ .

#### 1.1.2. Orthogonal Matrices

Orthogonal matrices are extremely useful since their inverse is equal to their transpose. Rotation matrices **R** are orthogonal, hence  $\mathbf{R}^{-1} = \mathbf{R}^{T}$ . A proof of this can be found in Allan (1997) and is repeated here.

Consider the effect of a rotation on the coordinates  $\mathbf{x}$  of a point P, expressed as

$$\mathbf{X} = \mathbf{R}\mathbf{x}$$

X is the transformed (or rotated) coordinates and R is the rotation matrix. Multiplying both sides of the equation by the inverse of R gives

$$\mathbf{R}^{-1}\mathbf{X} = \mathbf{R}^{-1}\mathbf{R}\mathbf{x}$$

but from matrix algebra  $\mathbf{R}^{-1}\mathbf{R} = \mathbf{I}$  and  $\mathbf{I}\mathbf{x} = \mathbf{x}$  so

$$\mathbf{R}^{-1}\mathbf{X} = \mathbf{x}$$
$$\mathbf{x} = \mathbf{R}^{-1}\mathbf{X}$$

The length (actually squared length) of the line from the origin to the original position of point *P* is given by  $\mathbf{x}^T \mathbf{x}$  and the length from the origin to the new (rotated) position is given by  $\mathbf{X}^T \mathbf{X}$ . This length does not change due to rotation, i.e., it is invariant under rotation. Hence

$$\mathbf{x}^{T}\mathbf{x} = \mathbf{X}^{T}\mathbf{X}$$

 $\mathbf{X} = \mathbf{R}\mathbf{x}$ 

but

or

so

$$\mathbf{x}^T \mathbf{x} = (\mathbf{R}\mathbf{x})^T \mathbf{R}\mathbf{x}$$
$$= \mathbf{x}^T \mathbf{R}^T \mathbf{R}\mathbf{x}$$

For this result to be possible

|     | $\mathbf{R}^T \mathbf{R} = \mathbf{I}$   |
|-----|--|
| but | $\mathbf{R}^{-1}\mathbf{R} = \mathbf{I}$ |

Therefore

 $\mathbf{R}^T = \mathbf{R}^{-1}$ 

Thus the *inverse* of a rotation matrix is equal to its *transpose*.

### 1.1.3. Rotation of Axes versus Rotation of Object

In these notes it is assumed that a rotation angle is a positive anticlockwise angle as determined by the *right-hand-grip rule* and that "apparent" rotations of objects (polygons) are caused by a rotation of the coordinate axes. This is not the only way that an object can be rotated.



Consider Figure 4 where *P* with coordinates *x*, *y* moves to *P'* with coordinates x', y' by a positive anticlockwise rotation  $\phi$ . The coordinates of *P'* are

$$x' = d\cos(\theta + \phi) = d(\cos\theta\cos\phi - \sin\theta\sin\phi)$$
  

$$y' = d\sin(\theta + \phi) = d(\sin\theta\cos\phi + \cos\theta\sin\phi)$$
(1.5)



The coordinates of *P* are  $x = d \cos \theta$  and  $y = d \sin \theta$  which can be substituted into (1.5) to give

$$\begin{aligned} x' &= x\cos\phi - y\sin\phi\\ y' &= y\cos\phi + x\sin\phi \end{aligned} \quad \text{or in matrix form} \quad \begin{bmatrix} x'\\ y' \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi\\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x\\ y \end{bmatrix} = \overline{\mathbf{R}} \begin{bmatrix} x\\ y \end{bmatrix} \tag{1.6}$$

Where  $\overline{\mathbf{R}}$  is a rotation matrix and the rotation angle  $\phi$  is a "right-handed" rotation. Inspection of equations (1.3) and (1.6) shows that  $\overline{\mathbf{R}}$  is <u>not the same form as  $\mathbf{R}$ </u>, in fact it is identical in form to  $\mathbf{R}^{T}$ .

The rotation matrix  $\mathbf{R}$  causes an apparent rotation of the object by rotation of the coordinate axes whilst the rotation matrix  $\mathbf{\bar{R}}$  rotates the object itself. Both  $\mathbf{R}$  and  $\mathbf{\bar{R}}$  are "right-hand" rotation matrices (one is the transpose of the other) and there is often confusion amongst users of transformation software in defining the type of rotation and the positive direction of rotation. You must be very careful in defining rotation, i.e., you must state what is being rotated, either axes or object and what is the positive direction of rotation. In these notes it is always assumed that the coordinate axes are being rotated and the rotations are always positive anticlockwise as defined by the right-hand-grip rule.

#### **1.2.** Transformations involving Rotation $\theta$ and a Scale change *s*

u,v coordinates are transformed to x,y coordinates by considering a rotation of the u,v coordinate axes through a positive anticlockwise angle  $\theta$  and a scaling of the u,v coordinates by a factor s. The transformation equations can be expressed in the following way

$$x = (s\cos\theta)u + (s\sin\theta)v$$
  

$$y = -(s\sin\theta)u + (s\cos\theta)v$$
(1.7)

or in matrix notation

$$\begin{bmatrix} x \\ y \end{bmatrix} = s \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$
(1.8)

Often, the coefficients of u and v in (1.7) are written as  $a = s \cos \theta$  and  $b = s \sin \theta$  giving

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$
(1.9)

and the scale factor s and the rotation angle  $\theta$  are given by

$$s = \sqrt{a^2 + b^2}$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$
(1.10)

As an example consider the polygon *ABCD* whose *u*,*v* coordinates are rotated by a positive anticlockwise angle  $\theta = 30^{\circ}$  and scaled by a factor s = 0.6. Figure 1 shows the initial location of the polygon in the *u*,*v* system and Figure 5 shows its transformed (rotated and scaled) location in the *x*,*y* system.



Figure 5 Rotated and scaled polygon ABCD with x, y coordinates in metres

Comparing Figures 1 and 5 it appears that the shape of the polygon *ABCD* has not changed but its size and orientation with respect to the coordinate axes has. This can be verified by considering the dimensions (bearings and distances) and area of the polygon *ABCD* derived from the two coordinate sets.

| Line  | Bearing    | Distance       | Line | Bearing     | Distance |
|-------|------------|----------------|------|-------------|----------|
| AB    | 30° 00′    | 200.000        | AB   | 60° 00′     | 120.000  |
| BC    | 120° 00′   | 100.000        | BC   | 150° 00′    | 60.000   |
| CD    | 210° 00′   | 257.735        | CD   | 240° 00′    | 154.641  |
| DA    | 330° 00′   | 115.470        | DA   | 0° 00′      | 69.282   |
| Area= | =22,886.75 | m <sup>2</sup> | Area | =8,239.23 n | $n^2$    |

Polygon dimensions in the *u*,*v* system

Polygon dimensions in the *x*,*y* system

Inspection of the two sets of dimensions reveals that bearings have been rotated by an angle  $\theta = 30^{\circ}$  and distances scaled by a factor s = 0.6. Note that the shape of the polygon is unchanged but the area of the transformed figure has been reduced by a factor of  $s^2$ .

# **1.3.** Transformations involving Rotation $\theta$ , Scale change *s* and Translations $t_x$ , $t_y$

u, v coordinates are first transformed to x', y' coordinates by considering a rotation of the u, vcoordinate axes through a positive anticlockwise angle  $\theta$  and a scaling of the u, v coordinates by a factor *s*. The x', y' coordinates are then transformed into x, y coordinates by the addition of translations  $t_x$  and  $t_y$ . The transformation equations can be expressed in the following way

$$x = (s\cos\theta)u + (s\sin\theta)v + t_x$$
  

$$y = -(s\sin\theta)u + (s\cos\theta)v + t_y$$
(1.11)

or in matrix notation

|    | $\begin{bmatrix} x \\ y \end{bmatrix} = s \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$ | (1.12) |
|----|--|--------|
| or | $\begin{bmatrix} x \\ y \end{bmatrix} = s \mathbf{R} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$  |        |

Similarly to before writing  $a = s \cos \theta$  and  $b = s \sin \theta$  gives

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$
(1.13)

This transformation is referred to by several names

- (i) Four-parameter transformation, the four parameters being  $a, b, t_x, t_y$ ,
- (ii) 2D Linear Conformal transformation,
- (iii) Similarity transformation and
- (iv) Helmert's transformation, after the German geodesist F.R. Helmert (1843-1917).

The transformation equations may be derived by considering Figure 6. The x', y' coordinates are obtained by rotating and scaling the u, v coordinates and then the x, y coordinates obtained by adding the translations  $t_x$  and  $t_y$ . Note that  $t_x$  and  $t_y$  may be negative.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = s \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$



Figure 6. Schematic diagram of rotated and translated axes

As an example of a 2D Linear Conformal transformation, consider the polygon *ABCD* whose u, v coordinates are rotated by a positive anticlockwise angle  $\theta = 30^{\circ}$ , scaled by a factor s = 0.6 and translated by  $t_x = 50.000$  m and  $t_y = 150.000$  m Figure 1 shows the initial location of the polygon in the u, v system and Figure 7 shows its transformed (rotated, scaled and translated) location in the x, y system.



Figure 7 Rotated, scaled and translated polygon ABCD with x,y coordinates in metres

Comparing Figures 1 and 7 it appears that the shape of the polygon *ABCD* has not changed but its area and orientation with respect to the coordinate axes has. This can be verified by considering the dimensions (bearings and distances) and area of the polygon *ABCD* derived from the two coordinate sets.

| Line                            | Bearing  | Distance | Line  | Bearing     | Distance       |
|---------------------------------|----------|----------|-------|-------------|----------------|
| AB                              | 30° 00′  | 200.000  | AB    | 60° 00′     | 120.000        |
| BC                              | 120° 00′ | 100.000  | BC    | 150° 00′    | 60.000         |
| CD                              | 210° 00′ | 257.735  | CD    | 240° 00′    | 154.641        |
| DA                              | 330° 00′ | 115.470  | DA    | 0° 00′      | 69.282         |
| $Area = 22,886.75 \mathrm{m}^2$ |          | $m^2$    | Area= | =8,239.23 n | n <sup>2</sup> |

Polygon dimensions in the *u*,*v* system

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Polygon dimensions in the x,y system
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Inspection of the two sets of dimensions reveals that bearings and distances of the polygon in the u,v system have been has been rotated by an angle  $\theta = 30^{\circ}$  and scaled by a factor s = 0.6. Note that the shape of the polygon is unchanged but the area of the transformed figure has been reduced by a factor of  $s^2$ . Comparison with the previous transformation demonstrates that translation has no effect on the area and shape of a polygon.

#### 1.4. Affine Transformations

u, v coordinates are transformed to x, y coordinates by the following equations containing six parameters; four coefficients a, b, d and e and two translations c and f. Affine transformations are often called *6-parameter transformations*. The transformation equations are

$$\begin{aligned} x &= au + bv + c \\ y &= du + ev + f \end{aligned}$$
 (1.14)

or in matrix notation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}$$
(1.15)

The parameters *a*, *b*, *d* and *e* are scalar quantities that can, if desired, be linked to scale factors in certain directions, rotation of axes and a "skew angle" (see the next section of these notes). Note that the parameters *a* and *b* are different the *a* and *b* of the 4-parameter transformation of the previous section. The parameters *c* and *f* are translations and are identical to  $t_x$  and  $t_y$  of the four-parameter transformation.

Affine transformations deform the shape of polygons, thus altering areas, but parallel lines are preserved in the transformation. As an example of an Affine transformation, consider the polygon *ABCD* shown in Figure 1. The *u*,*v* coordinates are transformed to *x*,*y* coordinates using equation (1.15) with a = 1.20, b = -0.50, c = 0, d = 0.25, e = 0.90 and f = 0

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1.20 & -0.50 \\ 0.25 & 0.90 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



Figure 8 Affine transformation of polygon *ABCD* with *x*,*y* coordinates in metres

Comparing Figures 1 and 8 it appears that the shape, area and orientation of the polygon *ABCD* has changed. This can be verified by considering the dimensions (bearings and distances) and area of the polygon *ABCD* derived from the two coordinate sets.

| Line                          | Bearing  | Distance | Line                     | Bearing      | Distance |
|-------------------------------|----------|----------|--------------------------|--------------|----------|
| AB                            | 30° 00′  | 200.000  | AB                       | 10° 27′ 40″  | 183.942  |
| BC                            | 120° 00′ | 100.000  | BC                       | 100° 15′ 57″ | 131.020  |
| CD                            | 210° 00′ | 257.735  | CD                       | 190° 27′ 40″ | 237.041  |
| DA                            | 330° 00′ | 115.470  | DA                       | 302° 21′ 16″ | 141.203  |
| $Area=22,886.75 \mathrm{m}^2$ |          | Area     | $=27,578.52\mathrm{m}^2$ |              |          |

Polygon dimensions in the *u*,*v* system

Polygon dimensions in the *x*,*y* system

Some properties of Affine transformations can be deduced from comparisons of dimensions.

- 1. Calculating scale factors  $s = \frac{dist \text{ in } x, y \text{ plane}}{dist \text{ in } u, v \text{ plane}}$  for the sides of *ABCD* gives  $s_{AB} = 0.919710$ ,  $s_{BC} = 1.310200$ ,  $s_{CD} = 0.919708$  and  $s_{DA} = 1.222854$  from which we may conclude that scale factor is not constant in an Affine transformation.
- 2. Calculating angular differences (*u*,*v*-angle *x*,*y*-angle) at each corner of *ABCD* gives  $A: +8^{\circ} 06' 24'', B: -0^{\circ} 11' 43'', C: +0^{\circ} 11' 43'' and D: -8^{\circ} 06' 24''$  from which we may conclude that the transformation <u>does not preserve angular relationships</u>. This is to be expected, since the scale is not constant. Also, it should be noted that <u>parallel lines in the</u> <u>original polygon are still parallel in the transformed polygon</u>. This property can be deduced from (1.14) considering coordinate differences  $\Delta x = x_j - x_k$ ,  $\Delta y = y_j - y_k$ ,  $\Delta u = u_j - u_k$  and  $\Delta v = v_j - v_k$  giving

$$\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix}$$
(1.16)

Hence parallel lines, *AB* and *CD* in our example, which have  $\Delta u$  and  $\Delta v$  coordinate differences in proportion to the lengths of the lines, will be transformed to another set of coordinate differences  $\Delta x$  and  $\Delta y$  in exactly the same proportion.

Thus, parallel lines are preserved in the transformation. This property also holds for the previous transformations of sections 1.1, 1.2 and 1.3.

In this example, the translations c and f are zero. As was demonstrated in the previous transformation (section 1.3), size and shape are not changed by translation. Identical results to those above will be obtained for the same values of a, b, d, e and any values of the translations c and f.

#### 1.5. Geometric Interpretation of the Parameters of a 2D Affine Transformation

The Affine transformation given by equations (1.14) or (1.15) is

| x = a u + b v + c | · ,· ,.               | $\begin{bmatrix} x \end{bmatrix} \begin{bmatrix} a \end{bmatrix}$     | $b \left[ u \right] \left[ c \right]$     |
|-------------------|-----------------------|---|---|
| y = d u + e v + f | of in matrix notation | $\begin{bmatrix} y \end{bmatrix}^{=} \begin{bmatrix} d \end{bmatrix}$ | $e \left[ v \right]^{+} \left[ f \right]$ |

It is usual (as was stated at the beginning of these notes) to consider the *x*,*y* and the *u*,*v* coordinate systems as both being rectangular or orthogonal systems, i.e., the *x* and *y* axes perpendicular to each other and the *u* and *v* axes are perpendicular to each other. But this is not always a convenient way to account for apparent distortions in the shape of the same polygon (or object) in two different coordinate systems. The following derivation of equations for scale factors  $s_u$  and  $s_v$  in the directions of the *u*- and *v*-axes and rotation and "skew" angles  $\theta$  and  $\alpha$  is due to Methley (1986) and is similar to a derivation by Wolf & Dewitt (2000). It allows a geometric interpretation of the parameters *a*, *b*, *d* and *e* of the Affine transformation.

Consider the case where it is assumed that the *u*,*v* system is a <u>non-orthogonal</u> system, i.e., the *u* and *v* axes are not perpendicular, and the *x*,*y* system is orthogonal. In addition, it is assumed that the scale factor *s* is not constant in every direction but has values  $s_u$  and  $s_v$  in the directions of the *u*-and *v*-axes, i.e., the scale factor  $s_u$  is constant along lines parallel to the *u*-axis and  $s_v$  is constant along lines parallel to the *v*-axis. The transformation from *u*,*v* to *x*,*y* coordinates can be considered as three separate transformations.







In Figure 9 the non-orthogonal u,v coordinates are transformed into an orthogonal u',v' system scaled by  $s_u$  and  $s_v$ .  $\alpha$  is the skew angle. Note that the u,v coordinates are distances measured along lines of constant u or v, not distances perpendicular to the coordinate axes.

$$u' = s_u(u) + s_v(v \sin \alpha)$$
  

$$v' = s_v(v \cos \alpha)$$
(1.17)

or in matrix notation

$$\begin{bmatrix} u'\\v' \end{bmatrix} = \begin{bmatrix} s_u & s_v \sin \alpha\\ 0 & s_v \cos \alpha \end{bmatrix} \begin{bmatrix} u\\v \end{bmatrix}$$
(1.18)

**1.5.2.** 2nd Transform:  $u', v' \rightarrow u'', v''$ 



(rotation)

Figure 10

The u',v' system is rotated (positive anticlockwise) by an angle  $\theta$  to the u'',v'' system.

$$\begin{bmatrix} u''\\v'' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta\\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u'\\v' \end{bmatrix}$$
(1.19)

**1.5.3.** 3rd Transform  $u'', v'' \rightarrow x, y$  (translation)



Figure 11

The u'', v'' system is parallel to the x, y system and offset by the translations  $T_x$  and  $T_y$ 

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u'' \\ v'' \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \end{bmatrix}$$
(1.20)

Combining equations (1.18), (1.19) and (1.20) and replacing  $T_x$  and  $T_y$  with *c* and *f* respectively gives

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} s_u & s_v \sin\alpha \\ 0 & s_v \cos\alpha \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}$$
$$= \begin{bmatrix} s_u \cos\theta & s_v \cos\theta \sin\alpha + s_v \sin\theta \cos\alpha \\ -s_u \sin\theta & -s_v \sin\theta \sin\alpha + s_v \cos\theta \cos\alpha \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}$$
(1.21)

Equating the elements of the coefficient matrices of equations (1.15) and (1.21) gives

$$a = s_u \cos\theta \tag{1.22}$$

$$b = s_{v} (\cos\theta \sin\alpha + \sin\theta \cos\alpha)$$
  
=  $s_{v} \sin(\theta + \alpha)$  (1.23)

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Scales:

$$d = -s_u \sin\theta \tag{1.24}$$

$$e = s_{v} \left( \cos \theta \cos \alpha - \sin \theta \sin \alpha \right)$$
$$= s_{v} \cos(\theta + \alpha)$$
(1.25)

From equations (1.22) to (1.25)  $s_u$ ,  $s_v$ ,  $\theta$  and  $\alpha$  can be obtained from

$$s_u = \sqrt{a^2 + d^2} \tag{1.26}$$

$$s_v = \sqrt{b^2 + e^2}$$
 (1.27)

Rotation angle 
$$\theta$$
:  $\tan \theta = \frac{-d}{a}$  (1.28)

Skew angle 
$$\alpha$$
:  $\tan(\theta + \alpha) = \frac{b}{e}$  (1.29)

Alternatively, after computing  $s_u$  and  $s_v$  from equations (1.26) and (1.27)

Rotation angle 
$$\theta$$
:  $\cos \theta = \frac{a}{s_u}$  (1.30)

Skew angle 
$$\alpha$$
:  $\cos(\theta + \alpha) = \frac{e}{s_v}$  (1.31)

#### 1.5.4. Another Geometric interpretation of the Affine Transformation

If we consider the original u, v system to be orthogonal, i.e., the skew angle  $\alpha = 0$  then equation (1.21) becomes, since  $\sin \alpha = 0$ ,  $\cos \alpha = 1$ 

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s_u \cos\theta & s_v \sin\theta \\ -s_u \sin\theta & s_v \cos\theta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}$$
(1.32)

Equating the elements of the coefficient matrices of equations (1.15) and (1.32) gives

$$a = s_u \cos \theta$$
  

$$b = s_v \sin \theta$$
  

$$d = -s_u \sin \theta$$
  

$$e = s_v \cos \theta$$
  
(1.33)

Scales:

From equations (1.33)  $s_{\mu}$ ,  $s_{\nu}$ , and  $\theta$  can be obtained from

$$s_u = \sqrt{a^2 + d^2} \tag{1.34}$$

$$s_v = \sqrt{b^2 + e^2}$$
 (1.35)

Rotation angle 
$$\theta$$
:  $\tan \theta = \frac{b}{a}$  (1.36)

# 1.6. 2D Polynomial Transformation

A polynomial function of a single variable P(x) is defined as

$$P(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n = \sum_{k=0}^n c_k x^k$$

and a polynomial function of two variables P(x, y) is defined by an equation of the form

$$P(x, y) = \sum_{m=0}^{p} \sum_{n=0}^{q} c_{m,n} x^{m} y^{n}$$

Thus a polynomial transformation of *u*,*v* coordinates to *x*,*y* coordinates can be expressed as

$$x = P(u, v) = \sum_{m=0}^{p} \sum_{n=0}^{q} c_{m,n} u^{m} v^{n}$$
  
=  $c_{00} u^{0} v^{0} + c_{01} u^{0} v^{1} + c_{02} u^{0} v^{2} + c_{03} u^{0} v^{3} + \cdots$   
+  $c_{10} u^{1} v^{0} + c_{11} u^{1} v^{1} + c_{12} u^{1} v^{2} + c_{13} u^{1} v^{3} + \cdots$   
+  $c_{20} u^{2} v^{0} + c_{21} u^{2} v^{1} + c_{22} u^{2} v^{2} + c_{23} u^{2} v^{3} + \cdots$   
+  $\cdots$ 

$$y = P(u, v) = \sum_{m=0}^{p} \sum_{n=0}^{q} d_{m,n} u^{m} v^{n}$$
  
=  $d_{00} u^{0} v^{0} + d_{01} u^{0} v^{1} + d_{02} u^{0} v^{2} + d_{03} u^{0} v^{3} + \cdots$   
+  $d_{10} u^{1} v^{0} + d_{11} u^{1} v^{1} + d_{12} u^{1} v^{2} + d_{13} u^{1} v^{3} + \cdots$   
+  $d_{20} u^{2} v^{0} + d_{21} u^{2} v^{1} + d_{22} u^{2} v^{2} + d_{23} u^{2} v^{3} + \cdots$   
+  $\cdots$ 

Noting that  $u^0 = v^0 = 1$ , simplifying the coefficients and arranging in ascending orders (i.e., 1st order terms contain *u* or *v*, 2nd order terms contain  $u^2$  or  $v^2$  or uv, etc.) a polynomial transformation can be expressed as

$$x = c_0 + c_1 u + c_2 v + c_3 u v + c_4 u^2 + c_5 v^2 + c_6 u^2 v + c_7 u v^2 + c_8 u^3 + c_9 v^3 + \cdots$$
  

$$y = d_0 + d_1 u + d_2 v + d_3 u v + d_4 u^2 + d_5 v^2 + d_6 u^2 v + d_7 u v^2 + d_8 u^3 + d_9 v^3 + \cdots$$
(1.37)

In general, polynomial transformations deform the size and shape of polygons.

Ignoring second- and higher-order terms in (1.37) gives a "first-order" polynomial transformation, which is in fact, the previously described Affine transformation (or 6-parameter transformation) of section 1.4.

$$\begin{aligned} x &= c_0 + c_1 u + c_2 v \\ y &= d_0 + d_1 u + d_2 v \end{aligned}$$
 (1.38)

where  $c_0$  and  $d_0$  are translations.

If  $c_1 = d_2$  and  $c_2 = -d_1$  then the first-order polynomial transformation becomes a 2D Linear Conformal transformation (or 4-parameter transformation) described in section 1.3.

$$x = c_0 + c_1 u - c_2 v y = d_0 + c_2 u + c_1 v$$
 (1.39)

As an example of a 2D Polynomial transformation, consider the polygon *ABCD* shown in Figure 1. The u,v coordinates are transformed to x,y coordinates using a 2nd order 2D Polynomial transformation

$$x = c_0 + c_1 u + c_2 v + c_3 u v + c_4 u^2 + c_5 v^2$$
  

$$y = d_0 + d_1 u + d_2 v + d_3 u v + d_4 u^2 + d_5 v^2$$
(1.40)

|                                 | k | $c_k$   | $d_{k}$ |
|---------------------------------|---|---------|---------|
|                                 | 0 | 64.250  | -50.500 |
|                                 | 1 | +1.2000 | -0.2500 |
| with the following coefficients | 2 | +0.2000 | +1.2000 |
|                                 | 3 | -0.0013 | -0.0002 |
|                                 | 4 | -0.0008 | +0.0011 |
|                                 | 5 | +0.0001 | -0.0002 |



Figure 12 Polynomial transformation of polygon ABCD with x, y coordinates in metres

It is important to note that polynomial transformations change straight lines in the u,v system to curved lines in the x,y system. For example in the polygon ABCD, consider the 200 metre straight line AB in the u,v system broken up into 1 metre segments, ie. AB is now defined by 201 u,v coordinate pairs and there is a linear relationship between them. Under a polynomial transformation each u,v coordinate pair is transformed to an x,y pair with a non-linear relationship between each pair. Thus the 200 line segments defining a straight line AB in the u,v system will be transformed into 200 chords of a curved line joining AB in the x,y system. In the limit, the chords become infinitesimal line elements ds of a complex curve between AB.

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#### 1.7. 2D Linear Conformal Transformations

(The following section is taken from Deakin, 1998 with some modifications to the notation.)

C.F. Gauss showed that the necessary and sufficient condition for a conformal transformation from the ellipsoid to the map plane is given by the complex expression (Lauf 1983)

$$x + iy = f(\chi + i\omega) \tag{1.41}$$

where the function  $f(\chi + i\omega)$  is analytic, containing isometric parameters  $\chi$  (isometric latitude)

and  $\omega$  (longitude). *i* is the imaginary number ( $i^2 = -1$ ).

[It should be noted here that isometric means: *of equal measure*, and on the surface of the ellipsoid (or sphere) latitude and longitude are not equal measures of length. This is obvious if we consider a point near the pole where similar distances along a meridian and a parallel of latitude will correspond to vastly different angular values of latitude and longitude. Hence in conformal map projections, isometric latitude is determined to ensure that angular changes correspond to linear changes.]

A necessary condition for an analytic function is that it must satisfy the Cauchy-Riemann equations

$$\frac{\partial x}{\partial \chi} = \frac{\partial y}{\partial \omega}$$
 and  $\frac{\partial x}{\partial \omega} = -\frac{\partial y}{\partial \chi}$  (1.42)

Using this theorem, a conformal transformation from one plane rectangular coordinate system u,v (isometric parameters) to another plane rectangular system x,y (also isometric parameters) is given by the complex expression

$$x + iy = f(u + iv) \tag{1.43}$$

A function f(u + iv) which satisfies the Cauchy-Riemann equations, is a complex polynomial, hence (1.43) can be given as

$$x + iy = \sum_{k=0}^{n} (a_k + ib_k) (u + iv)^k$$
(1.44)

Equation (1.44) can be expanded to the first power (k = 1) giving

$$x + iy = (a_0 + ib_0)(u + iv)^0 + (a_1 + ib_1)(u + iv)^1$$
  
=  $a_0 + ib_0 + a_1u + a_1iv + ib_1u + i^2b_1v$ 

Equating real and imaginary parts (remembering that  $i^2 = -1$ ) gives

or in matrix notation with translations  $a_0$  and  $b_0$  between the coordinate axes

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}$$
(1.46)

These equations are of similar form to (1.13) of section 1.3 "Transformations involving Rotation, Scale and Translations" and properly describe a *2D Linear Conformal transformation*. Note that the elements of the leading diagonal of the coefficient matrix (a rotation matrix multiplied by a scale factor) are identical and the off-diagonal elements the same magnitude but opposite sign.

Equations (1.45) are essentially the same equations as in Jordan/Eggert/Kneissal (1963, pp. 70-73) in the section headed "Das Helmertsche Verfahren (Helmertsche Transformation)" (Helmert's Transformation) although there is no reference to the original source. It is probable that F.R. Helmert developed this conformal transformation in his masterpiece on geodesy, *Die mathematischen und physikalischen Theorem der höheren Geodäsie*, (The mathematics and physical theorems of higher geodesy) on which he worked from 1877 and published in two parts: vol. 1, *Die mathematischen Theorem* (1880) and vol. 2, *Die physikalischen Theorem* (1884) [DSB 1972]. This probably accounts for the common usage of the term Helmert transformation when describing a 2D Linear Conformal transformation.

The partial derivatives of (1.45) are

$$\frac{\partial x}{\partial u} = a_1, \quad \frac{\partial x}{\partial v} = -b_1, \quad \frac{\partial y}{\partial u} = b_1 \text{ and } \quad \frac{\partial y}{\partial v} = a_1$$

which satisfy the Cauchy-Riemann equations

$$\frac{\partial x}{\partial u} = \frac{\partial y}{\partial v}$$
 and  $\frac{\partial x}{\partial v} = -\frac{\partial y}{\partial u}$ 

so verifying that the transformation is conformal.

#### 1.8. 2D Polynomial Conformal Transformation

Using Gauss' theorem of conformal mapping and a suitable complex polynomial, expansions beyond the first power give rise to conformal polynomial transformations. As in the previous section, let the conformal transformation be given by the complex expression

$$x + iy = \sum_{k=0}^{n} (A_k + iB_k) (u + iv)^k$$
(1.47)

And expanding to the third power (k = 3) gives

$$x + iy = (A_0 + iB_0)(u + iv)^0 + (A_1 + iB_1)(u + iv)^1 + (A_2 + iB_2)(u + iv)^2 + (A_3 + iB_3)(u + iv)^3$$

With  $i^2 = -1$ ,  $(u + iv)^2 = u^2 + 2iuv - v^2$  and  $(u + iv)^3 = u^3 + 3iu^2v - 3uv^2 - iu^3$ 

$$\begin{aligned} x + iy &= A_0 + iB_0 \\ &+ A_1 u + iA_1 v + iB_1 u - B_1 v \\ &+ A_2 u^2 + 2iA_2 uv - A_2 v^2 + iB_2 u^2 - 2B_2 uv - iB_2 v^2 \\ &+ A_3 u^3 + 3iA_3 u^2 v - 3A_3 uv^2 - iA_3 v^3 + iB_3 u^3 - 3B_3 u^2 v - 3iB_3 uv^2 + B_3 v^3 \end{aligned}$$

Equating real and imaginary parts gives the 3rd-order 2D Conformal Polynomial transformation

$$x = A_0 + A_1 u - B_1 v + A_2 (u^2 - v^2) - B_2 (2uv) + A_3 (u^3 - 3uv^2) - B_3 (3u^2v - v^3)$$
  

$$y = B_0 + B_1 u + A_1 v + B_2 (u^2 - v^2) + A_2 (2uv) + B_3 (u^3 - 3uv^2) + A_3 (3u^2v - v^3)$$
(1.48)

The partial derivatives of (1.48) are

$$\frac{\partial x}{\partial u} = A_1 + A_2(2u) - B_2(2v) + A_3(3u^2 - 3v^2) - B_3(6uv)$$
  
$$\frac{\partial x}{\partial v} = -B_1 - A_2(2v) - B_2(2u) - A_3(6uv) - B_3(3u^2 - 3v^2)$$
  
$$\frac{\partial y}{\partial u} = B_1 + B_2(2u) + A_2(2v) + B_3(3u^2 - 3v^2) + A_3(6uv)$$
  
$$\frac{\partial y}{\partial v} = A_1 - B_2(2v) + A_2(2u) - B_3(6uv) + A_3(3u^2 - 3v^2)$$

which satisfy the Cauchy-Riemann equations

$$\frac{\partial x}{\partial u} = \frac{\partial y}{\partial v}$$
 and  $\frac{\partial x}{\partial v} = -\frac{\partial y}{\partial u}$ 

so verifying that the transformation is conformal.

The 2nd-order 2D Polynomial Conformal transformation can be obtained from (1.48) as

$$x = A_0 + A_1 u - B_1 v + A_2 (u^2 - v^2) - B_2 (2uv)$$
  

$$y = B_0 + B_1 u + A_1 v + B_2 (u^2 - v^2) + A_2 (2uv)$$
(1.49)

These equations also satisfy the Cauchy-Riemann equations. 2D Polynomial Conformal transformations preserve shapes of infinitesimally small regions but <u>not</u> finite regions.

As an example of a 2D Polynomial Conformal transformation consider the polygon *ABCD* shown in Figure 1. The u,v coordinates are transformed to x,y coordinates using a 2nd-order 2D Polynomial Conformal transformation (see equations (1.49)) with the following coefficients



Figure 13 Polynomial Conformal transformation of polygon ABCD with x, y coordinates in metres

Comparing Figures 1 and 13, it appears that the shape, area and orientation of the polygon *ABCD* has changed. This can be verified by considering the dimensions (bearings and distances) and area of the polygon *ABCD* derived from the two coordinate sets.

| Line                 | Bearing  | Distance | Line Bearin            | g Distance       |
|----------------------|----------|----------|------------------------|------------------|
| AB                   | 30° 00′  | 200.000  | AB 354° 43' 2          | 22" 200.233      |
| BC                   | 120° 00′ | 100.000  | <i>BC</i> 86° 06'1     | 9″ 122.429       |
| CD                   | 210° 00′ | 257.735  | <i>CD</i> 186° 23′ 2   | 25" 266.274      |
| DA                   | 330° 00′ | 115.470  | $DA = 307^{\circ} 31'$ | 56" 93.433       |
| Area=22,886.75 $m^2$ |          | $m^2$    | Area=22,900.3          | $1 \mathrm{m}^2$ |

Polygon dimensions in the *u*,*v* system

Polygon dimensions in the *x*,*y* system

Some properties of Polynomial Conformal transformations can be deduced from comparisons of dimensions.

1. Calculating scale factors  $s = \frac{dist \text{ in } x, y \text{ plane}}{dist \text{ in } u, v \text{ plane}}$  for the sides of *ABCD* gives  $s_{AB} = 1.001165$ ,  $s_{BC} = 1.224290$ ,  $s_{CD} = 1.033131$  and  $s_{DA} = 0.809154$  from which we may conclude that <u>scale</u> factor is not constant for lines of finite length in a Polynomial Conformal transformation.

2. Calculating angular differences (*u*,*v*-angle – *x*,*y*-angle) at each corner of *ABCD* gives *A*: -12° 48'34", *B*: +1° 22'57", *C*: +10° 17'06" and *D*: +1° 08'31" from which we may conclude that the transformation <u>does not preserve angular relationships between lines of finite length</u>. This is to be expected, since the scale is not constant.

Conformal transformations have the useful property that the scale factor is constant in any direction around a point, thus angles are preserved and shape is retained. However, in general this property only applies to infinitesimally small regions around a point. As can be seen by inspection of the Conformal Polynomial transformation above, angles have been deformed and scale is not constant for lines of finite length. However, as the lengths of lines become differentially small then scale will be preserved in any direction around a point.

#### 1.9. 2D Equal Area Transformations

For various reasons, it may be desirable to effect a transformation that preserves area relationships. That is, regions in the u,v plane having certain area ratios are transformed into regions in the x,y plane having the same area ratios. Such transformations are known as *equal-area* transformations and they may be derived by considering some elementary theory of map projections (Deakin 1994).

For a transformation from the sphere  $\phi, \lambda$  to the projection plane *X*, *Y* where  $X = f_1(\phi, \lambda)$  and  $Y = f_2(\phi, \lambda)$ , Lauf (1983), gives the Gaussian Fundamental Quantities *E*, *F* and *G* for the projection plane as

$$E = \left(\frac{\partial X}{\partial \phi}\right)^2 + \left(\frac{\partial Y}{\partial \phi}\right)^2$$
$$F = \frac{\partial X}{\partial \phi}\frac{\partial X}{\partial \lambda} + \frac{\partial Y}{\partial \phi}\frac{\partial Y}{\partial \lambda}$$
$$G = \left(\frac{\partial X}{\partial \lambda}\right)^2 + \left(\frac{\partial Y}{\partial \lambda}\right)^2$$

The elemental area dA on the projection plane is given by  $dA = J d\phi d\lambda$  where

$$J = \sqrt{EG - F^2} = \left| \frac{\partial X}{\partial \phi} \frac{\partial Y}{\partial \lambda} - \frac{\partial Y}{\partial \phi} \frac{\partial X}{\partial \lambda} \right|$$

Using similar differential relationships; if a transformation is made <u>in the plane</u> between the *u*,*v* system and the *x*,*y* system such that  $x = f_3(u, v)$  and  $y = f_4(u, v)$  where an element of area in the *u*,*v* plane is  $da = du \, dv$ ; then the corresponding element of area in the *x*,*y* plane is  $dA = J \, du \, dv$  where  $J = \left| \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} \right|$ . Now for an equal-area transformation *da* must equal *dA*, which leads to

the equal-area condition in the plane

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = 1$$
(1.50)

There are many equal-area transformations in the plane, which satisfy equation (1.50). One such set of transformations may be derived as follows.

Let 
$$x = f(u)$$
, i.e., x is a function of u only, then  $\frac{\partial x}{\partial u} = f'(u)$  and  $\frac{\partial x}{\partial v} = 0$ .

Equation (27) becomes  $f'(u)\frac{\partial y}{\partial v} = 1$  and solving y by integration gives

 $y = \int \frac{1}{f'(u)} dv = \frac{v}{f'(u)}$ . Similar reasoning can be used to derive an expression for x when

y = g(v). These equal-area transformations in the plane can be summarised as

$$x = f(u), \quad y = \frac{v}{f'(u)}$$
 (1.51)

$$x = \frac{u}{g'(v)}, \quad y = g(v)$$
 (1.52)

Equal-area transformations can be effected by selecting an appropriate function f(u), differentiating the function to obtain f'(u) and then using equations (1.51). Alternatively, an appropriate function g(v) can be selected and equations (1.52) used.

As an example, consider the following polynomial transformation where the function f(u) is selected as a 4h-order polynomial and equations (1.51) used.

$$x = f(u) = A_0 + A_1 u + A_2 u^2 + A_3 u^3 + A_4 u^4$$
$$y = \frac{v}{f'(u)} = \frac{v}{A_1 + 2A_2 u + 3A_3 u^2 + 4A_4 u^3}$$

These equations satisfy the equal-area condition (1.50), which is a differential relationship developed by equating area elements da and dA, but <u>do not</u> transform finite polygons of area A in the u,v system to polygons of the same area in the x,y system.

To achieve an equal-area transformation of a polygon we need to consider three elementary transformations, which preserve area (1) rotation, (2) translation and (3) compression-expansion (Dyer and Snyder 1989). The first two have been shown to preserve area (see the preceding sections). The third is intuitive since we may expand the coordinates in one direction and compress them in another by the same ratio without affecting areas of polygons.

The equations for an equal-area transformation can be developed by considering (i) u, v coordinates transformed to x', y' coordinates by a rotation about the origin

$$x' = u\cos\theta - v\sin\theta$$
  

$$y' = u\sin\theta + v\cos\theta$$
(1.53)

And then (ii) x', y' coordinates transformed to x, y coordinates by compression-expansion and translation

$$x = A_0 + A_1 x'$$
  

$$y = B_0 + \frac{y'}{A_1}$$
(1.54)

Equations (1.53) and (1.54), which both satisfy the equal-area condition (1.50), may be combined in the following way

$$x = A_0 + A_1 \left( a \, u - v \sqrt{1 - a^2} \right)$$
  

$$y = B_0 + \frac{1}{A_1} \left( u \sqrt{1 - a^2} + a \, v \right)$$
(1.55)

As an example of an Equal-Area transformation, consider the polygon *ABCD* shown in Figure 1. The *u*,*v* coordinates are transformed to *x*,*y* coordinates using equations (1.55) with the following coefficients: a = 0.8,  $A_0 = 221.000$ ,  $B_0 = -81.818$  and  $A_1 = 1.10$ 



Figure 14 Equal-Area transformation of polygon ABCD with x,y coordinates in metres

Comparing Figures 1 and 14, it appears that the shape, area and orientation of the polygon *ABCD* has changed but its area remains the same. This can be verified by considering the dimensions (bearings and distances) and area of the polygon *ABCD* derived from the two coordinate sets.

| Line                   | Bearing  | Distance | Line                     | Bearing      | Distance |
|------------------------|----------|----------|--------------------------|--------------|----------|
| AB                     | 30° 00′  | 200.000  | AB                       | 351° 42′ 21″ | 182.421  |
| BC                     | 120° 00′ | 100.000  | BC                       | 84° 18′ 50″  | 109.749  |
| CD                     | 210° 00′ | 257.735  | CD                       | 171° 42′ 20″ | 235.081  |
| DA                     | 330° 00′ | 115.470  | DA                       | 289° 26' 38″ | 123.872  |
| Area=22,886.75 $m^2$ A |          | Area     | $=22,886.62\mathrm{m}^2$ |              |          |

Polygon dimensions in the *u*,*v* system

Polygon dimensions in the *x*,*y* system

Some properties of this Equal-Area transformation can be deduced from comparisons of dimensions.

- 1. The area remains unchanged by the transformation.
- 2. Calculating scale factors  $s = \frac{dist \text{ in } x, y \text{ plane}}{dist \text{ in } u, v \text{ plane}}$  for the sides of *ABCD* gives  $s_{AB} = 0.912105$ ,  $s_{BC} = 1.097490$ ,  $s_{CD} = 0.912104$  and  $s_{DA} = 1.072763$  from which we may conclude that <u>scale</u> <u>factor is not constant</u>. Note though, that parallel lines have the same scale factor.
- Calculating angular differences (*u*,*v*-angle *x*,*y*-angle) at each corner of *ABCD* gives
  A: +2°15′43″, B: +2°36′29″, C: -2°36′30″ and D: -2°15′42″ from which we may conclude that the transformation does not preserve angular relationships. This is to be expected, since the scale is not constant. Note though, that parallel lines in the *u*,*v* system are still parallel in the *x*,*y* system, as in the Affine transformation.

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