3. THE SOLUTION OF TRANSFORMATION PARAMETERS

Coordinate transformations, as used in practice, are models describing the assumed mathematical relationships between points in two rectangular coordinate systems; in these notes, the u,v and the x,y systems. To determine the parameters of any transformation, coordinates of points common to both systems must be known. These points are known as "control points" or "common points". The number of common points required for the solution of transformation parameters depends on the number of parameters in the transformation. In 2D transformations, each common point gives rise to two equations, thus p common points will give $n = 2p$ equations. Therefore, if the four parameters of a *Linear Conformal* transformation are to be determined, then a minimum of two common points are required to solve for the parameters. For *Affine* transformations, (six parameters) and *3rd-order Polynomial* transformations (up to twenty parameters), minimums of three and ten common points respectively are required for a solution of the parameters.

It is good measurement practice to determine coordinate transformation parameters by using more than the minimum number of common points. This introduces redundant equations into the solution for the parameters and the theory of *least squares* is employed to calculate the best estimates. Parameters calculated in this manner are usually more reliable and the least squares process allows precision estimation of the parameters as well as an assessment (via residuals) of how well the transformation model fits the common points. By using least squares, several types of transformations can be "tested" on the common points to assess their suitability.

3.1. The Solution Process

The solution for the parameters for any transformation involves the following steps

(i) Choose the appropriate transformation model assumed to link the x,y and u,v coordinate systems.
(ii) Select the common points ensuring that there are sufficient to allow a redundant set of equations.
(iii) Select the appropriate least squares adjustment model to be used to estimate the parameters.
(iv) Select the appropriate weight matrix for the model.
(v) Solve for the parameters $\mathbf{x}$ and residuals $\mathbf{v}$.

(vi) Assess the suitability of the model by analysis of the parameters and residuals.

3.2. Solution of 2D Linear Conformal Transformation Parameters

The 2D Linear Conformal transformation, consisting of rotation, scaling and translation is set out in Sections 1.3 and 1.6. The transformation model for the $k = 1, 2, 3, \ldots, p$ common points is given by equations (1.11)

$$\begin{bmatrix} x_k \\ y_k \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} u_k \\ v_k \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} \tag{3.1}$$

Elements of the solution process defined above is set in detail below.

3.2.1. Mathematical Models

Two least squares models can be constructed to solve for the parameters of a 2D Linear Conformal transformation. Which one is selected will depend on how the *u,v* and *x,y* coordinates are to be treated. The two choices of models are.

(a) Only the *x,y* coordinates are treated as observations with residuals v_x and v_y. This leads to an adjustment model of the form $\mathbf{v} + \mathbf{Bx} = \mathbf{f}$. The solution for the parameters is direct (requiring no iteration) and relatively simple; it is by far the most popular model. This technique of simply adding residuals to account for the inconsistency in the model is similar to that used in statistical regression problems. This adjustment model suffers in comparison to model (b) due to its inability to properly treat both sets of coordinates as observations. That is, one cannot properly assign (or construct) covariance matrices for both the *x,y* and *u,v* coordinates (the observations) of the common points. Hence, solutions from this model may not be the best estimates. On the other hand, in many applications, nothing is known of the precision of the coordinates and this adjustment model may be the most appropriate.

(b) Both the *x,y* and *u,v* coordinates are treated as observations with residuals v_x, v_y, v_u and v_v. This leads to an adjustment model of the form $\mathbf{Av} + \mathbf{B}\delta\mathbf{x} = \mathbf{f}^0$ requiring an iterative solution process. This model has an advantage over model (a) as the precision of the coordinates (the observations) can be properly taken into account in the solution.

Model (a) $\mathbf{v} + \mathbf{Bx} = \mathbf{f}$

Equations (3.1) can be expressed in the form of *observation equations* where v_{x_k} and v_{y_k} are small unknown corrections or residuals simply added to the equations to account for the assumed inconsistency in the model. We could think of these residuals as consisting of two parts; one part associated with the *u,v* system and the other associated with the transformed *x,y* system; the subscripts *x* and *y* attached to the residuals simply reflect the fact that they have been added to the "transformed" side of the model.

$$\begin{bmatrix} x_k \\ y_k \end{bmatrix} + \begin{bmatrix} v_{x_k} \\ v_{y_k} \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} u_k \\ v_k \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} \tag{3.2}$$

Re-arranging (3.2) so that all the "unknowns" are on to the left of the equals sign and the observations are to the right gives

$$\begin{aligned} v_{x_k} - a\,u_k - b\,v_k - t_x &= -x_k \\ v_{y_k} - a\,v_k + b\,u_k - t_y &= -y_k \end{aligned} \tag{3.3}$$

For p common points and $u = 4$ unknown parameters, the partitioned matrix representation of the $n = 2p$ equations (3.3) is

$$\begin{bmatrix} v_{x_1} \\ v_{x_2} \\ v_{x_3} \\ \vdots \\ v_{x_p} \\ \hdashline v_{y_1} \\ v_{y_2} \\ v_{y_3} \\ \vdots \\ v_{y_p} \end{bmatrix} + \begin{bmatrix} -u_1 & -v_1 & -1 & 0 \\ -u_2 & -v_2 & -1 & 0 \\ -u_3 & -v_3 & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -u_p & -v_p & -1 & 0 \\ \hdashline -v_1 & u_1 & 0 & -1 \\ -v_2 & u_2 & 0 & -1 \\ -v_3 & u_3 & 0 & -1 \\ \vdots & \vdots & \vdots & \vdots \\ -v_p & u_p & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ t_x \\ t_y \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 \\ -x_3 \\ \vdots \\ -x_p \\ \hdashline -y_1 \\ -y_2 \\ -y_3 \\ \vdots \\ -y_p \end{bmatrix} \tag{3.4}$$

These equations are represented by the matrix equation

$$\mathbf{v} + \mathbf{Bx} = \mathbf{f} \tag{3.5}$$

where

- $\mathbf{v}$ is an $(n,1)$ column vector of residuals
- $\mathbf{B}$ is an (n,u) matrix of coefficients

$\mathbf{x}$ is a $(u,1)$ vector of unknown parameters

$\mathbf{f}$ is an $(n,1)$ column vector of numeric terms (coordinates)

The equations for the solution of parameters $\mathbf{x}$ and residuals $\mathbf{v}$ is set out in section 2.5.3 noting that in this case $\mathbf{x}$ and $\mathbf{f}$ have replaced $\delta\mathbf{x}$ and $\mathbf{f}^0$ respectively. The general form of the normal equations $\left(\mathbf{B}^T\mathbf{W}\mathbf{B}\right)\mathbf{x} = \mathbf{B}^T\mathbf{W}\mathbf{f}$ (or $\mathbf{N}\mathbf{x} = \mathbf{t}$) assuming that $\mathbf{W} = \mathbf{Q} = \mathbf{I}$ are

$$\left[\begin{array}{cc|cc} \sum_{k=1}^{p}\left(u_k^2+v_k^2\right) & 0 & \sum_{k=1}^{p} u_k & \sum_{k=1}^{p} v_k \\ & \sum_{k=1}^{p}\left(u_k^2+v_k^2\right) & \sum_{k=1}^{p} v_k & -\sum_{k=1}^{p} u_k \\ \hline & & n & 0 \\ \text{symmetric} & & & n \end{array}\right] \begin{bmatrix} a \\ b \\ t_x \\ t_y \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{p}\left(u_k x_k + v_k y_k\right) \\ \sum_{k=1}^{p}\left(v_k x_k - u_k y_k\right) \\ \sum_{k=1}^{p} x_k \\ \sum_{k=1}^{p} y_k \end{bmatrix} \tag{3.6}$$

Centroidal coordinates

Computational savings can be made by reducing coordinates to a *centroid*. For the p common points, the coordinates of the centroid in the x,y system are

$$x_c = \frac{x_1 + x_2 + x_3 + \cdots + x_p}{p} = \frac{\sum_{k=1}^{p} x_k}{p}$$

$$y_c = \frac{y_1 + y_2 + y_3 + \cdots + y_p}{p} = \frac{\sum_{k=1}^{p} y_k}{p}$$

and the *centroidal* coordinates of these same p points are then

$$\begin{array}{ll} \bar{x}_1 = x_1 - x_c & \bar{y}_1 = y_1 - y_c \\ \bar{x}_2 = x_2 - x_c & \bar{y}_2 = y_2 - y_c \\ \bar{x}_3 = x_3 - x_c & \bar{y}_3 = y_3 - y_c \\ \vdots & \vdots \\ \bar{x}_p = x_p - x_c & \bar{y}_3 = y_3 - y_c \end{array}$$

Similar relationships can be written for centroidal coordinates in the u,v system. A useful property of the centroidal coordinates of the p common points is that their sums equal zero, ie,

$$\sum_{k=1}^{p} \bar{x}_k = 0 \quad \sum_{k=1}^{p} \bar{y}_k = 0 \quad \sum_{k=1}^{p} \bar{u}_k = 0 \quad \sum_{k=1}^{p} \bar{v}_k = 0$$

Thus, replacing *x,y* and *u,v* coordinates with their centroidal counterparts $\bar{x}, \bar{y}$ and $\bar{u}, \bar{v}$ reduces the transformation (3.1) to

$$\begin{bmatrix} \bar{x}_k \\ \bar{y}_k \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} \bar{u}_k \\ \bar{v}_k \end{bmatrix} \tag{3.7}$$

It should be noted here that translations t_x and t_y are both zero when centroidal coordinates are used indicating that the centroids x_c, y_c and u_c, v_c are the same point.

For p common points and $u = 2$ unknown parameters, the partitioned matrix representation of the $n = 2p$ equations resulting from the centroidal model (3.7) is

$$\begin{bmatrix} v_{x_1} \\ v_{x_2} \\ v_{x_3} \\ \vdots \\ v_{x_p} \\ \hdashline v_{y_1} \\ v_{y_2} \\ v_{y_3} \\ \vdots \\ v_{y_p} \end{bmatrix} + \begin{bmatrix} -\bar{u}_1 & -\bar{v}_1 \\ -\bar{u}_2 & -\bar{v}_2 \\ -\bar{u}_3 & -\bar{v}_3 \\ \vdots & \vdots \\ -\bar{u}_p & -\bar{v}_p \\ \hdashline -\bar{v}_1 & \bar{u}_1 \\ -\bar{v}_2 & \bar{u}_2 \\ -\bar{v}_3 & \bar{u}_3 \\ \vdots & \vdots \\ -\bar{v}_p & \bar{u}_p \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -\bar{x}_1 \\ -\bar{x}_2 \\ -\bar{x}_3 \\ \vdots \\ -\bar{x}_p \\ \hdashline -\bar{y}_1 \\ -\bar{y}_2 \\ -\bar{y}_3 \\ \vdots \\ -\bar{y}_p \end{bmatrix} \tag{3.8}$$

These equations are represented by the matrix equation (3.5). With $\mathbf{W} = \mathbf{Q} = \mathbf{I}$, the normal equations have the following simple form containing only three different numbers

$$\begin{bmatrix} \sum_{k=1}^{n} \left(\bar{u}_k^2 + \bar{v}_k^2\right) & 0 \\ 0 & \sum_{k=1}^{n} \left(\bar{u}_k^2 + \bar{v}_k^2\right) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{p} \left(\bar{u}_k \bar{x}_k + \bar{v}_k \bar{y}_k\right) \\ \sum_{k=1}^{p} \left(\bar{v}_k \bar{x}_k - \bar{u}_k \bar{y}_k\right) \end{bmatrix} \tag{3.9}$$

The solutions for the parameters *a* and *b* are

$$a = \frac{\sum_{k=1}^{p} \left(\bar{u}_k \bar{x}_k + \bar{v}_k \bar{y}_k\right)}{\sum_{k=1}^{p} \left(\bar{u}_k^2 + \bar{v}_k^2\right)} \tag{3.10}$$

$$b = \frac{\sum_{k=1}^{p}\left(\overline{v}_k\overline{x}_k - \overline{u}_k\overline{y}_k\right)}{\sum_{k=1}^{p}\left(\overline{u}_k^2 + \overline{v}_k^2\right)} \tag{3.11}$$

The translations t_x and t_y are obtained by re-arranging (3.1) as

$$\begin{bmatrix} t_x \\ t_y \end{bmatrix} = \begin{bmatrix} x_c \\ y_c \end{bmatrix} - \begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} u_c \\ v_c \end{bmatrix}$$

or

$$\begin{aligned} t_x &= x_c - au_c - bv_c \\ t_y &= y_c + bu_c - av_c \end{aligned} \tag{3.12}$$

After calculation of the parameters, $\mathbf{x} = \begin{bmatrix} a & b & t_x & t_y \end{bmatrix}^T$ the residuals are calculated using (3.8).

Model (b) $\mathbf{Av} + \mathbf{B}\delta\mathbf{x} = \mathbf{f}^0$

Equations (3.1) can be written in functional form

$$\begin{aligned} f_1 &= au_k + bv_k + t_x - x_k = 0 \\ f_2 &= -bu_k + av_k + t_y - y_k = 0 \end{aligned} \tag{3.13}$$

where the vector of parameters is $\mathbf{x}^T = \begin{bmatrix} a & b & t_x & t_y \end{bmatrix}$ and the vector of observations is $\mathbf{l} = \begin{bmatrix} x_k & y_k & u_k & v_k \end{bmatrix}^T$ with cofactor matrices as estimates of precision. Applying the principles of section 2.1 gives a set of equations of the form $\mathbf{Av} + \mathbf{B}\delta\mathbf{x} = \mathbf{f}^0$ where, for a single point, the coefficient matrices $\mathbf{A}$ and $\mathbf{B}$ are

$$\mathbf{A} = \left.\frac{\partial F}{\partial \hat{\mathbf{l}}}\right|_{l,x^0} = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{bmatrix} = \begin{bmatrix} -1 & 0 & a^0 & b^0 \\ 0 & -1 & -b^0 & a^0 \end{bmatrix}$$

$$\mathbf{B} = \left.\frac{\partial F}{\partial \hat{\mathbf{x}}}\right|_{l,x^0} = \begin{bmatrix} \frac{\partial f_1}{\partial a} & \frac{\partial f_1}{\partial b} & \frac{\partial f_1}{\partial t_x} & \frac{\partial f_1}{\partial t_y} \\ \frac{\partial f_2}{\partial a} & \frac{\partial f_2}{\partial b} & \frac{\partial f_2}{\partial t_x} & \frac{\partial f_2}{\partial t_y} \end{bmatrix} = \begin{bmatrix} u_k & v_k & 1 & 0 \\ v_k & -u_k & 0 & 1 \end{bmatrix}$$

The vector of numeric terms $\mathbf{f}^0$ is

$$\mathbf{f}^0 = -F\left(\mathbf{l},\mathbf{x}^0\right) = \begin{bmatrix} -a^0u_k - b^0v_k - t_x^0 + x_k \\ b^0u_k - a^0v_k - t_y^0 + y_k \end{bmatrix}$$

Note that in the matrices above $\mathbf{x}^0 = \begin{bmatrix} a^0 & b^0 & t_x^0 & t_y^0 \end{bmatrix}^T$ is a vector of approximate values of the parameters and the matrices **A**, **B** and the vector $\mathbf{f}^0$ are evaluated with these approximate values. If they are unknown, then they are set to zero for the first iteration.

Centroidal coordinates

Computational savings can be made by using gentroidal coordinates. The number of parameters is reduced to two, since translations t_x and t_y are eliminated. Equations (3.1) can be written in functional form using centroidal coordinates

$$\begin{aligned} f_1 &= \quad a\bar{u}_k + b\bar{v}_k - \bar{x}_k = 0 \\ f_2 &= -b\bar{u}_k + a\bar{v}_k - \bar{y}_k = 0 \end{aligned} \tag{3.14}$$

where the vector of parameters is $\mathbf{x} = \begin{bmatrix} a & b \end{bmatrix}^T$ and the vector of observations is $\mathbf{l} = \begin{bmatrix} \bar{x}_k & \bar{y}_k & \bar{u}_k & \bar{v}_k \end{bmatrix}^T$ with cofactor matrices as estimates of precision. Applying the principles of section 2.1 gives a set of equations of the form $\mathbf{Av} + \mathbf{B}\delta\mathbf{x} = \mathbf{f}^0$ where, for a single point, the coefficient matrices and vector of numeric terms become

$$\mathbf{A} = \left.\frac{\partial F}{\partial \hat{\mathbf{l}}}\right|_{l,x^0} = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{bmatrix} = \begin{bmatrix} -1 & 0 & a^0 & b^0 \\ 0 & -1 & -b^0 & a^0 \end{bmatrix}$$

$$\mathbf{B} = \left.\frac{\partial F}{\partial \hat{\mathbf{x}}}\right|_{l,x^0} = \begin{bmatrix} \frac{\partial f_1}{\partial a} & \frac{\partial f_1}{\partial b} \\ \frac{\partial f_2}{\partial a} & \frac{\partial f_2}{\partial b} \end{bmatrix} = \begin{bmatrix} \bar{u}_k & \bar{v}_k \\ \bar{v}_k & -\bar{u}_k \end{bmatrix}$$

The vector of numeric terms $\mathbf{f}^0$ is

$$\mathbf{f}^0 = -F\left(\mathbf{l},\mathbf{x}^0\right) = \begin{bmatrix} -a^0\bar{u}_k - b^0\bar{v}_k + \bar{x}_k \\ b^0\bar{u}_k - a^0\bar{v}_k + \bar{y}_k \end{bmatrix}$$

For a single point, the matrix equation $\mathbf{Av} + \mathbf{B}\delta\mathbf{x} = \mathbf{f}^0$ has the following form

$$\begin{bmatrix} -1 & 0 & a^0 & b^0 \\ 0 & -1 & -b^0 & a^0 \end{bmatrix} \begin{bmatrix} v_{x_k} \\ v_{y_k} \\ v_{u_k} \\ v_{v_k} \end{bmatrix} + \begin{bmatrix} \overline{u}_k & \overline{v}_k \\ \overline{v}_k & -\overline{u}_k \end{bmatrix} \begin{bmatrix} \delta a \\ \delta b \end{bmatrix} = \begin{bmatrix} -a^0\overline{u}_k - b^0\overline{v}_k + \overline{x}_k \\ b^0\overline{u}_k - a^0\overline{v}_k + \overline{y}_k \end{bmatrix}$$

For the $p = 4$ common points, the partitioned matrix representation of the equation $\mathbf{Av} + \mathbf{B}\delta\mathbf{x} = \mathbf{f}^0$ resulting from the centroidal model given by the functional equations (3.14) is given as

$$\left[\begin{array}{cccc|cccc|cccc|cccc} -1 & 0 & a^0 & b^0 & & & & & & & & & & & & \\ 0 & -1 & -b^0 & a^0 & & & & & & & & & & & & \\ \hline & & & & -1 & 0 & a^0 & b^0 & & & & & & & & \\ & & & & 0 & -1 & -b^0 & a^0 & & & & & & & & \\ \hline & & & & & & & & -1 & 0 & a^0 & b^0 & & & & \\ & & & & & & & & 0 & -1 & -b^0 & a^0 & & & & \\ \hline & & & & & & & & & & & & -1 & 0 & a^0 & b^0 \\ & & & & & & & & & & & & 0 & -1 & -b^0 & a^0 \end{array}\right] \left[\begin{array}{c} v_{x_1} \\ v_{y_1} \\ v_{u_1} \\ v_{v_1} \\ \hline v_{x_2} \\ v_{y_2} \\ v_{u_2} \\ v_{v_2} \\ \hline v_{x_3} \\ v_{y_3} \\ v_{u_3} \\ v_{v_3} \\ \hline v_{x_4} \\ v_{y_4} \\ v_{u_4} \\ v_{v_4} \end{array}\right] + \left[\begin{array}{cc} \overline{u}_1 & \overline{v}_1 \\ \overline{v}_1 & -\overline{u}_1 \\ \hline \overline{u}_2 & \overline{v}_2 \\ \overline{v}_2 & -\overline{u}_2 \\ \hline \overline{u}_3 & \overline{v}_3 \\ \overline{v}_3 & -\overline{u}_3 \\ \hline \overline{u}_4 & \overline{v}_4 \\ \overline{v}_4 & -\overline{u}_4 \end{array}\right] \begin{bmatrix} \delta a \\ \delta b \end{bmatrix} = \left[\begin{array}{r} -a^0\overline{u}_1 - b^0\overline{v}_1 + \overline{x}_1 \\ b^0\overline{u}_1 - a^0\overline{v}_1 + \overline{y}_1 \\ \hline -a^0\overline{u}_2 - b^0\overline{v}_2 + \overline{x}_2 \\ b^0\overline{u}_2 - a^0\overline{v}_2 + \overline{y}_2 \\ \hline -a^0\overline{u}_3 - b^0\overline{v}_3 + \overline{x}_3 \\ b^0\overline{u}_3 - a^0\overline{v}_3 + \overline{y}_3 \\ \hline -a^0\overline{u}_4 - b^0\overline{v}_4 + \overline{x}_4 \\ b^0\overline{u}_4 - a^0\overline{v}_4 + \overline{y}_4 \end{array}\right]$$