5. PROPAGATION OF VARIANCES APPLIED TO LEAST SQUARES
ADJUSTMENT OF INDIRECT OBSERVATIONS

A most important outcome of a least squares adjustment is that estimates of the precisions of
the quantities sought, the elements of $x$, the unknowns or the parameters, are easily obtained
from the matrix equations of the solution. Application of the Law of Propagation of
Variances demonstrates that $N^{-1}$, the inverse of the normal equation coefficient matrix is
equal to the cofactor matrix $Q_{xx}$ that contains estimates of the variances and covariances of
the elements of $x$. In addition, estimates of the precisions of the residuals and adjusted
observations may be obtained. This most useful outcome enables a statistical analysis of the
results of a least squares adjustment and provides the practitioner with a degree of confidence
in the results.

5.1. Cofactor matrices for adjustment of indirect observations

The observation equations for adjustment of indirect observations is given by

$$\mathbf{v} + Bx = f$$  \hspace{1cm} (5.1)

$f$ is an $(n,1)$ vector of numeric terms derived from the $(n,1)$ vector of observations $l$ and the
$(n,1)$ vector of constants $d$ as

$$f = d - l$$  \hspace{1cm} (5.2)

Associated with the vector of observations $l$ is a variance-covariance matrix $\Sigma_{ll}$ as well as a
cofactor matrix $Q_{ll}$ and a weight matrix $W_{ll} = Q_{ll}^{-1}$. Remember that in most practical
applications of least squares, the matrix $\Sigma_{ll}$ is unknown, but estimated \textit{a priori} by $Q_{ll}$ that
contains estimates of the variances and covariances and $\Sigma_{ll} = \sigma_0^2 Q_{ll}$ where $\sigma_0^2$ is the
reference variance or variance factor.

Note: In the derivations that follow, the subscript “$ll$” is dropped from $Q_{ll}$ and $W_{ll}$.

If equation (5.2) is written as

$$\mathbf{f} = (-\mathbf{I})l + \mathbf{d}$$  \hspace{1cm} (5.3)
then (5.3) is in a form suitable for employing the Law of Propagation of Variances developed in Chapter 3; i.e., if \( y = Ax + b \) and \( y \) and \( x \) are random variables linearly related and \( b \) is a vector of constants then \( Q_{yy} = A Q_{xx} A^T \). Hence, the cofactor matrix of the numeric terms \( f \) is

\[
Q_{ff} = (-1)Q(-1)^T = Q
\]

Thus the cofactor matrix of \( f \) is also the \textit{a priori} cofactor matrix of the observations \( l \).

The solution "steps" in the least squares adjustment of indirect observations are set out Chapter 2 and restated as

\[
N = B^T W B
\]

\[
t = B^T W f
\]

\[
x = N^{-1} t
\]

\[
v = f - B x
\]

\[
\hat{l} = l + v
\]

To apply the Law of Propagation of Variances, these equations may be re-arranged in the form \( y = Ax + b \) where the terms in parenthesis ( ) constitute the \( A \) matrix.

\[
t = (B^T W) f \tag{5.4}
\]

\[
x = (N^{-1}) t \tag{5.5}
\]

\[
v = f - B x = f - B N^{-1} t = f - B N^{-1} B^T W f = (I - B N^{-1} B^T W) f \tag{5.6}
\]

\[
\hat{l} = l + v = l + f - B x = d - B x = (-B)x + d \tag{5.7}
\]

Applying the Law of Propagation of Variances to equations (5.4) to (5.7) gives the following cofactor matrices
\[ Q_{tt} = (B^TW)Q_{ff}(B^TW)^T = N \] (5.8)

\[ Q_{xx} = (N^{-1})Q_{tt}(N^{-1})^T = N^{-1} \] (5.9)

\[ Q_{vv} = (I - BN^{-1}B^TW)Q_{ff}(I - BN^{-1}B^TW)^T \]
\[ = Q - BN^{-1}B^T \] (5.10)

\[ Q_{ff} = (-B)Q(-B)^T \]
\[ = BN^{-1}B^T \]
\[ = Q - Q_{vv} \] (5.11)

Variance-covariance matrices for \( t, x, v \) and \( \hat{l} \) are obtained by multiplying the cofactor matrix by the variance factor \( \sigma_0^2 \).

### 5.2. Calculation of the quadratic form \( v^TWv \)

The \textit{a priori} estimate of the variance factor may be computed from

\[ \hat{\sigma}_0^2 = \frac{v^TWv}{r} \] (5.12)

where \( v^TWv \) is the quadratic form, and

\( r = n - u \) is the degrees of freedom where \( n \) is the number of observations and \( u \) is the number of unknown parameters. \( r \) is also known as the number of redundancies.

A derivation of equation (5.12) is given below. The quadratic form \( v^TWv \) may be computed in the following manner.

Remembering, for the method of indirect observations, the following matrix equations

\[ N = B^TWB \]
\[ t = B^Wf \]
\[ x = N^{-1}t \]
\[ v = f - Bx \]

then
\[ v^T W v = (f - Bx)^T W (f - Bx) \]
\[ = (f^T - x^T B^T) W (f - Bx) \]
\[ = (f^T W - x^T B^T W) (f - Bx) \]
\[ = f^T W f - f^T W B x - x^T B^T W f + x^T B^T W B x \]
\[ = f^T W f - 2 f^T W B x + x^T B^T W B x \]
\[ = f^T W f - 2 f^T x + x^T N x \]
\[ = f^T W f - 2 x^T t + x^T t \]

and

\[ v^T W v = f^T W f - x^T t \]
(5.13)

5.3. Calculation of the Estimate of the Variance Factor \( \hat{\sigma}_0^2 \)

The variance-covariance matrices of residuals \( \Sigma_v \), adjusted observations \( \Sigma_{ii} \) and computed parameters \( \Sigma_{xx} \) are calculated from the general relationship

\[ \Sigma = \sigma^2 \Omega \]
(5.14)

Cofactor matrices \( Q_{vv} \), \( Q_{xx} \) and \( Q_{ii} \) are computed from equations (5.9) to (5.11) and so it remains to determine an estimate of the variance factor \( \hat{\sigma}_0^2 \).

The development of a matrix expression for computing \( \hat{\sigma}_0^2 \) is set out below and follows Mikhail (1976, pp.285-288). Some preliminary relationships will be useful.

1. If \( A \) is an \((n,n)\) square matrix, the sum of its diagonal elements is a scalar quantity called the trace of \( A \) and denoted by \( tr(A) \). The following relationships are useful

\[ tr(A + B) = tr(A) + tr(B) \quad \text{for} \quad A \text{ and } B \text{ of same order} \]  
(5.15)

\[ tr(A^T) = tr(A) \]  
(5.16)

and for the quadratic form \( x^T A x \) where \( A \) is symmetric

\[ x^T A x = tr(xx^T A) \]  
(5.17)
2. The variance-covariance matrix $\Sigma_{xx}$ given by equation (3.21) can be expressed in the following manner, remembering that $\mathbf{x}$ is a vector of random variables and $\mathbf{m}_x$ is a vector of means.

$$\Sigma_{xx} = E\{(x - \mathbf{m}_x)(x - \mathbf{m}_x)^T\}$$

$$= E\{(x - \mathbf{m}_x)(x^T - \mathbf{m}_x^T)\}$$

$$= E\{xx^T - x\mathbf{m}_x^T - \mathbf{m}_x x^T + \mathbf{m}_x \mathbf{m}_x^T\}$$

$$= E\{xx^T\} - E\{x\mathbf{m}_x^T\} - E\{\mathbf{m}_x x^T\} + E\{\mathbf{m}_x \mathbf{m}_x^T\}$$

$$= E\{xx^T\} - E\{x\}\mathbf{m}_x^T - \mathbf{m}_x E\{x^T\} + \mathbf{m}_x \mathbf{m}_x^T$$

Now from equation (3.18) $\mathbf{m}_x = E\{x\}$ hence

$$\Sigma_{xx} = E\{xx^T\} - \mathbf{m}_x \mathbf{m}_x^T - \mathbf{m}_x \mathbf{m}_x^T + \mathbf{m}_x \mathbf{m}_x^T$$

$$= E\{xx^T\} - \mathbf{m}_x \mathbf{m}_x^T$$

(5.18)

or

$$E\{xx^T\} = \Sigma_{xx} + \mathbf{m}_x \mathbf{m}_x^T$$

(5.19)

3. The expected value of the residuals is zero, i.e.,

$$E\{\mathbf{v}\} = \mathbf{m}_v = 0$$

(5.20)

4. By definition (see Chapter 2) the weight matrix $\mathbf{W}$, the cofactor matrix $\mathbf{Q}$ and the variance-covariance matrix $\Sigma$ are related by

$$\mathbf{W} = \mathbf{Q}^{-1} = \sigma_0^2 \Sigma^{-1}$$

(5.21)

Now, for the least squares adjustment of indirect observations the following relationships are recalled

$$\mathbf{v} + \mathbf{Bx} = \mathbf{f}, \quad \mathbf{N} = \mathbf{B}^T \mathbf{WB}, \quad \mathbf{t} = \mathbf{B}^T \mathbf{Wf}$$

$$\mathbf{Q}_{ff} = \mathbf{Q}, \quad \mathbf{Q}_{it} = \mathbf{N}, \quad \mathbf{Q}_{xx} = \mathbf{N}^{-1}$$

Bearing in mind equation (5.21), the following relationships may be introduced

$$\Sigma^{-1} = \frac{1}{\sigma_0^2} \mathbf{W}, \quad \mathbf{M} = \mathbf{B}^T \Sigma^{-1} \mathbf{B}$$

and from these follow
\[ \Sigma_g = \Sigma, \quad \Sigma_n = \sigma_0^4 \mathbf{M}, \quad \Sigma_{xx} = \mathbf{M}^{-1} \]

In addition, the expectation of the vector \( \mathbf{f} \) is the mean \( \mathbf{m}_f \) and so we may write

\[
\mathbf{m}_f = E\{\mathbf{f}\} = E\{\mathbf{v} + \mathbf{Bx}\} = E\{\mathbf{v}\} + \mathbf{B}E\{\mathbf{x}\}
\]

Now since \( E\{\mathbf{x}\} = \mathbf{m}_x \) and \( E\{\mathbf{v}\} = \mathbf{0} \)

\[
\mathbf{m}_f = \mathbf{Bm}_x \tag{5.22}
\]

Now the quadratic form

\[
\mathbf{v}^T \mathbf{Wv} = \sigma_0^2 \left( \mathbf{v}^T \Sigma^{-1} \mathbf{v} \right) \tag{5.23}
\]

and from equation (5.13)

\[
\mathbf{v}^T \mathbf{Wv} = \mathbf{f}^T \mathbf{Wf} - \mathbf{x}^T \mathbf{t} = \mathbf{f}^T \mathbf{Wf} - \mathbf{x}^T \mathbf{Nx}
\]

Using the relationships above

\[
\mathbf{v}^T \Sigma^{-1} \mathbf{v} = \mathbf{f}^T \Sigma^{-1} \mathbf{f} - \mathbf{x}^T \mathbf{Mx}
\]

Now the expected value of this quadratic form is

\[
E\{\mathbf{v}^T \Sigma^{-1} \mathbf{v}\} = E\{\mathbf{f}^T \Sigma^{-1} \mathbf{f} - \mathbf{x}^T \mathbf{Mx}\} = E\{\mathbf{f}^T \Sigma^{-1} \mathbf{f}\} - E\{\mathbf{x}^T \mathbf{Mx}\}
\]

Recognising that the terms on the right-hand-side are both quadratic forms, equation (5.17) can be used to give

\[
E\{\mathbf{v}^T \Sigma^{-1} \mathbf{v}\} = tr\left(\left[\mathbf{ff}^T \Sigma^{-1}\right]\right) - E\left\{tr\left(\mathbf{xx}^T \mathbf{M}\right)\right\} = tr\left(E\{\mathbf{ff}^T \Sigma^{-1}\}\right) - tr\left(E\{\mathbf{xx}^T \mathbf{M}\}\right) = tr\left(E\{\mathbf{ff}^T \Sigma^{-1}\}\right) - tr\left(E\{\mathbf{xx}^T \Sigma^{-1}\}\right) = tr\left(E\{\mathbf{ff}^T \Sigma^{-1}\}\right) - tr\left(E\{\mathbf{xx}^T \Sigma^{-1}\}\right)
\]

Now using equation (5.19)

\[
E\{\mathbf{v}^T \Sigma^{-1} \mathbf{v}\} = tr\left(\left[\Sigma_{gg} + \mathbf{m}_f \mathbf{m}_f^T \right] \Sigma^{-1}\right) - tr\left(\left[\Sigma_{xx} + \mathbf{m}_f \mathbf{m}_f^T \right] \mathbf{M}\right) = tr\left(\mathbf{I}_{nn} \mathbf{m}_f \mathbf{m}_f^T \Sigma^{-1}\right) - tr\left(\mathbf{I}_{nn} \mathbf{m}_f \mathbf{m}_f^T \mathbf{M}\right) = tr\left(\mathbf{I}_{nn} - \mathbf{I}_{uu}\right) - tr\left(\mathbf{m}_f \mathbf{m}_f^T \Sigma^{-1} + \mathbf{m}_f \mathbf{m}_f^T \mathbf{M}\right) = (n - u) - m_f^T \Sigma^{-1} \mathbf{m}_f + m_f^T \mathbf{Mm}_f
\]
From equation (5.22) \( \mathbf{m}_f = \mathbf{Bm}_r \) hence using the rule for matrix transpose

\[
\mathbf{m}_f^T = (\mathbf{Bm}_r)^T = \mathbf{m}_r^T \mathbf{B}^T,
\]

then

\[
E\{\mathbf{v}^T \mathbf{\Sigma}^{-1} \mathbf{v}\} = (n - u) - \mathbf{m}_i^T \mathbf{B}^T \mathbf{\Sigma}^{-1} \mathbf{B} \mathbf{m}_i + \mathbf{m}_i^T \mathbf{M}_i \mathbf{m}_i
\]

\[
= (n - u) - \mathbf{m}_i^T \mathbf{M}_i + \mathbf{m}_i^T \mathbf{M}_i
\]

\[
= (n - u)
\]

Thus according to equation (5.23) and the expression above

\[
E\{\mathbf{v}^T \mathbf{W} \mathbf{v}\} = \sigma_0^2 E\{\mathbf{v}^T \mathbf{\Sigma}^{-1} \mathbf{v}\}
\]

\[
= \sigma_0^2 (n - u)
\]

from which follows

\[
\sigma_0^2 = \frac{E\{\mathbf{v}^T \mathbf{W} \mathbf{v}\}}{n - u}
\]

Consequently, an unbiased estimate of the variance factor \( \hat{\sigma}_0^2 \) can be computed from

\[
\hat{\sigma}_0^2 = \frac{\mathbf{v}^T \mathbf{W} \mathbf{v}}{n - u} = \frac{\mathbf{v}^T \mathbf{W} \mathbf{v}}{r} \quad (5.24)
\]

\( r = n - u \) is the number of redundancies in the adjustment and is known as the degrees of freedom.

Using equation (5.13) an unbiased estimate of the variance factor \( \hat{\sigma}_0^2 \) can be computed from

\[
\hat{\sigma}_0^2 = \frac{\mathbf{f}^T \mathbf{W} \mathbf{f} - \mathbf{x}^T \mathbf{t}}{r} \quad (5.25)
\]