7. LINEARIZATION USING TAYLOR'S THEOREM AND THE DERIVATION OF SOME COMMON SURVEYING OBSERVATION EQUATIONS

In many surveying "problems" the solution depends upon selection of a *mathematical model* suitable to the problem, and using this, together with the *observations* (or measurements) obtain a solution.

For example, a surveyor is required to determine the location (the coordinates) of a point. From this "unknown" point, they can see three known points (i.e., points of known coordinates). Understanding geometric principles, the surveyor measures the directions to these three known points with a theodolite, determines the two angles α and β between the three lines and "solves the problem". In surveying parlance, this technique of solution of position is known as a resection; the mathematical model is based on geometric principles and the observations are the directions, from which the necessary angles are obtained for a solution.

Choosing a resection, as an example of a "surveying problem" is appropriate, since it demonstrates the case of determining quantities (the coordinates of the unknown point) from indirect measurements. That is, the surveyor's measurements of directions are indirect measurements of coordinate differences between the unknown point and the known points.

In many surveying problems, the observations exceed the necessary number required for a unique solution. Again, using a resection as an example, consider the case where the surveyor (at an unknown point) measures the directions to four known points. There are now multiple solutions for the resection point, since the four directions give rise to three angles, exceeding the minimum geometric requirements for a unique solution. That is, there is a *redundancy* in the mathematical model. In this case of the resection, and other surveying problems where there are redundant measurements, the method of *least squares* can be employed to obtain the *best estimate* of the "unknowns".

Least squares (as a method of determining best estimates), depends upon the formation of sets of *observation equations* and their solution. The normal techniques of solution of systems of equations require that the sets of observation equations must be *linear*, i.e., "unknowns" linearly related to measurements. This is not always the case. For example, in a resection, the measurements, directions α_{ik} from the unknown point P_i to known points P_k, are non-linear functions of the coordinate differences (the unknowns).

The observation equations for observed directions in a mathematical model of a resection have the general form

$$\alpha_{ik} + v_{ik} + z_i = \tan^{-1}\left(\frac{E_k - E_i}{N_k - N_i}\right) \tag{7.1}$$

α_{ik}	are the observed directions from the resection point P_i to the known points P_k,
v_{ik}	are the residuals (small corrections) associated with observed directions,
z_i	is an orientation "constant"; the bearing of the Reference Object (RO) for the set of observed directions,
E_k, N_k	are the east and north coordinates of the known points, and
E_i, N_i	are the east and north coordinates of the resection point.

Clearly, in this case, the measurements α_{ik} are non-linear functions of the unknowns E_i, N_i and any system of equations in the form of (7.1) would be *non-linear* and could not be solved by normal means. Consequently, whenever the equations in a mathematical model are non-linear functions linking the measurements with the unknowns, some method of linearization must be employed to obtain sets of *linear equations.*

The most common method of linearization is by using *Taylor's* theorem to represent the function as a power series consisting of zero order terms, 1st order terms, 2nd order terms and higher order terms. By choosing suitable approximations, second and higher-order terms can be neglected, yielding a linear approximation to the function. This linear approximation of the mathematical model can be used to form sets of linear equations, which can be solved by normal means.

7.1. Taylor's Theorem

This theorem, due to the English mathematician Brook Taylor (1685–1731) enables the value of a real function $f(x)$ near a point $x = a$ to be estimated from the values $f(a)$ and the derivatives of $f(x)$ evaluated at $x = a$. Taylor's theorem also provides an estimate of the error made in a polynomial approximation to a function. The Scottish mathematician Colin Maclaurin (1698–1746) developed a special case of Taylor's theorem, which was named in his honour, where the function $f(x)$ is expanded about the origin $x = a = 0$. The citations below, from the Encyclopaedia Britannica give some historical information about Taylor and Maclaurin.

> **Taylor**, Brook (b. Aug. 18, 1685, Edmonton, Middlesex, Eng.– d. Dec. 29, 1731, London), British mathematician noted for his contributions to the development of calculus.
> In 1708 **Taylor** produced a solution to the problem of the centre of oscillation. The solution went unpublished until 1714, when his claim to priority was disputed by the noted Swiss mathematician Johann Bernoulli. **Taylor**'s *Methodus incrementorum directa et inversa* (1715; "Direct and Indirect Methods of Incrementation") added to higher mathematics a new branch now called the calculus of finite differences. Using this new development, he was the first to express mathematically the movement of a vibrating string on the basis of mechanical principles. *Methodus* also contained the celebrated formula known as **Taylor**'s theorem, the importance of which remained unrecognized until 1772. At that time the French mathematician Joseph-Louis Lagrange realized its importance and proclaimed it the basic principle of differential calculus.
> A gifted artist, **Taylor** set forth in *Linear Perspective* (1715) the basic principles of perspective. This work and his *New Principles of Linear Perspective* contained the first general treatment of the principle of vanishing points. **Taylor** was elected a fellow of the Royal Society of London in 1712 and in that same year sat on the committee for adjudicating Sir Isaac Newton's and Gottfried Wilhelm Leibniz's conflicting claims of priority in the invention of calculus.

> **Maclaurin**, Colin (b. February 1698, Kilmodan, Argyllshire, Scot.–d. June 14, 1746, Edinburgh), Scottish mathematician who developed and extended Sir Isaac Newton's work in calculus, geometry, and gravitation. A child prodigy, he entered the University of Glasgow at age 11. At the age of 19, he was elected professor of mathematics at Marischal College, Aberdeen, and two years later he became a fellow of the Royal Society of London. At this time he became acquainted with Newton. In his most important work, *Geometrica Organica; Sive Descriptio Linearum Curvarum Universalis* (1720; "Organic Geometry, with the Description of the Universal Linear Curves"), **Maclaurin** developed several theorems similar to some in Newton's *Principia,* introduced the method of generating conics (the circle, ellipse, hyperbola, and parabola) that bears his name, and showed that certain types of curves (of the third and fourth degree) can be described by the intersection of two movable angles. On the recommendation of Newton, he was made professor of mathematics at the University of Edinburgh in 1725. In 1740 he shared, with the mathematicians Leonhard Euler and Daniel Bernoulli, the prize offered by the Académie des Sciences for an essay on tides. His *Treatisw of Fluxions* (1742) was written in reply to criticisms by George Berkeley of England that Newton's calculus was based on faulty reasoning. In this essay he showed

that stable figures for a homogeneous rotating fluid mass are the ellipsoids of revolution, later known as **Maclaurin**'s ellipsoids. He also gave in his *Fluxions*, for the first time, the correct theory for distinguishing between maxima and minima in general and pointed out the importance of the distinction in the theory of the multiple points of curves. The **Maclaurin** series, a special case of the Taylor series, was named in his honour. In 1745, when Jacobites (supporters of the Stuart king James II and his descendants) were marching on Edinburgh, **Maclaurin** took a prominent part in preparing trenches and barricades for the city's defense. As soon as the rebel army captured Edinburgh, **Maclaurin** fled to England until it was safe to return. The ordeal of his escape ruined his health, and he died at age 48. **Maclaurin**'s *Account of Sir Isaac Newton's Philosophical Discoveries* was published posthumously, as was his *Treatise of Algebra* (1748). "De Linearum Geometricarum Proprietatibus Generalibus tractatus" ("A Tract on the General Properties of Geometrical Lines"), noted for its elegant geometric demonstrations, was appended to his *Algebra*.

Taylor's theorem may be expressed in the following form

$$\begin{aligned} f(x) &= f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots \\ &\quad + \frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + R_n \end{aligned} \tag{7.2}$$

where R_n is the remainder after n terms and $\lim\limits_{n\to\infty} R_n = 0$ for $f(x)$ about $x = a$

$f'(a),\ f''(a),\ \cdots$ etc are derivatives of the function $f(x)$ evaluated at $x = a$.

Taylor's theorem can also be expressed as power series

$$f(x) = \sum_{k=0}^{n} f^{(k)}(a) \frac{(x-a)^k}{k!} \tag{7.3}$$

where $f^{(k)}(a)$ denotes the kth derivative of the function $f(x)$ evaluated at $x = a$ and $f^{(0)}(a)$ is the function $f(x)$ evaluated at $x = a$, and $0! = 1$.

Other forms of Taylor's theorem may be obtained by a change of notation, for example: let $x = a + h$, then $f(x) = f(a+h)$ and $x - a = h$. Substitution into equation (7.2) gives

$$\begin{aligned} f(x) = f(a+h) &= f(a) + h\, f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \cdots \\ &\quad + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + R_n \end{aligned} \tag{7.4}$$

This may be a more convenient form of Taylor's theorem for a particular application.

Inspection of equations (7.2), (7.3) and (7.4) show that Taylor's theorem can be used to expand a non-linear function (about a point) into a linear series. Expansions of this form, also called **Taylor's series**, are a convergent power series approximating $f(x)$.

Taylor's series for functions of two variables

Say $\phi = f(x,y)$ then the Taylor series expansion of the function ϕ about $x=a$ and $y=b$ is

$$\begin{aligned}\phi = f(a,b) &+ (x-a)\frac{\partial f}{\partial x} + (y-b)\frac{\partial f}{\partial y} \\ &+ \frac{1}{2!}\left\{(x-a)^2\frac{\partial^2 f}{\partial x^2} + (y-b)^2\frac{\partial^2 f}{\partial y^2} + (x-a)(y-b)\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\right\} + \cdots \qquad (7.5)\end{aligned}$$

where $f(a,b)$ is the function ϕ evaluated at $x=a$ and $y=b$

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2},$ etc are partial derivatives of the function ϕ evaluated at $x=a$ and $y=b$.

Taylor's series for functions of three variables

Say $\phi = f(x,y,z)$ then the Taylor series expansion of the function ϕ about $x=a$, $y=b$ and $z=c$

$$\begin{aligned}\phi = f(a,b,c) &+ (x-a)\frac{\partial f}{\partial x} + (y-b)\frac{\partial f}{\partial y} + (z-c)\frac{\partial f}{\partial z} \\ &+ \frac{1}{2!}\left\{(x-a)^2\frac{\partial^2 f}{\partial x^2} + (y-b)^2\frac{\partial^2 f}{\partial y^2} + (z-c)^2\frac{\partial^2 f}{\partial z^2}\right. \\ &\left. + (x-a)(y-b)\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + (x-a)(z-c)\frac{\partial f}{\partial x}\frac{\partial f}{\partial z} + (y-b)(z-c)\frac{\partial f}{\partial y}\frac{\partial f}{\partial z}\right\} + \cdots \qquad (7.6)\end{aligned}$$

where $f(a,b,c)$ is the function ϕ evaluated at $x=a$, $y=b$ and $z=c$

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \frac{\partial^2 f}{\partial x^2},$ etc are partial derivatives evaluated at $x=a$, $y=b$ and $z=c$.

Extensions to four or more variables follow a similar pattern. Equations (7.5) and (7.6) show only terms up to the 2nd order; no remainder terms are shown.

7.2. Linear Approximations to Functions using Taylor's Theorem

In the Taylor expansions of functions shown above, suppose that the variables $x,\ y,\ z,\cdots$ etc are expressed as $x = x^0 + \Delta x$, $y = y^0 + \Delta y$, $z = z^0 + \Delta z \cdots$ etc where $x^0,\ y^0,\ z^0,\cdots$ etc are approximate values and $\Delta x,\ \Delta y,\ \Delta z,\cdots$ etc are small corrections.

The Taylor series expansion of a single variable can be expressed as

$$\begin{aligned}\phi = f(x) &= f\left(x^0\right) + \left(x - x^0\right)\frac{df}{dx} + \left(x - x^0\right)^2\frac{d^2 f}{dx^2} + \left(x - x^0\right)^3\frac{d^3 f}{dx^3} + \cdots \\ &= f\left(x^0\right) + \Delta x\frac{df}{dx} + \left(\Delta x\right)^2\frac{d^2 f}{dx^2} + \left(\Delta x\right)^3\frac{d^3 f}{dx^3} + \cdots \\ &= f\left(x^0\right) + \Delta x\frac{df}{dx} + \text{ higher order terms}\end{aligned}$$

where the derivatives $\frac{df}{dx},\ \frac{d^2 f}{dx^2},\ \frac{d^3 f}{dx^3},\cdots$ etc are evaluated at the approximation x^0. If the correction Δx is small, then $\left(\Delta x\right)^2,\ \left(\Delta x\right)^3,\cdots$ etc will be exceedingly small and the higher order terms may be neglected, giving the following linear approximation

For $\phi = f(x)$
$$\phi = f(x) \simeq f(x^0) + \Delta x\frac{df}{dx} \tag{7.7}$$

Using similar reasoning, linear approximations can be written for functions of two and three variables.

For $\phi = f(x, y)$
$$\phi = f(x, y) \simeq f\left(x^0, y^0\right) + \Delta x\frac{\partial f}{\partial x} + \Delta y\frac{\partial f}{\partial y} \tag{7.8}$$

For $\phi = f(x, y, z)$,
$$\phi = f(x, y, z) \simeq f\left(x^0, y^0, z^0\right) + \Delta x\frac{\partial f}{\partial x} + \Delta y\frac{\partial f}{\partial y} + \Delta z\frac{\partial f}{\partial z} \tag{7.9}$$

Similar linear approximations can be written for functions of four or more variables. In equations (7.7), (7.8) and (7.9) the derivatives are evaluated at the approximations $x^0,\ y^0,\ z^0$.

Generalizing this linear form gives, for $\phi = f\left(x_1, x_2, x_3, \cdots x_n\right)$

$$\phi = f(x_1, x_2, x_3, \cdots x_n) \simeq f\left(x_1^0, x_2^0, x_3^0, \cdots x_n^0\right) + \Delta x_1 \frac{\partial f}{\partial x_1} + \Delta x_2 \frac{\partial f}{\partial x_2} + \Delta x_3 \frac{\partial f}{\partial x_3} + \cdots + \Delta x_n \frac{\partial f}{\partial x_n} \tag{7.10}$$

This equation can be written in matrix form

$$\phi = f\left(\mathbf{x}\right) \simeq f\left(\mathbf{x}^0\right) + \mathbf{j}\Delta\mathbf{x} \tag{7.11}$$

where **x** is a vector of variables, $\mathbf{x}^0$ a vector of approximate values of the variables, **j** is a row vector of partial derivatives and $\Delta\mathbf{x}$ is a column vector of corrections.

Suppose this generalized form, equation (7.11), is extended to the general case of *m* variables $y_1,\ y_2,\ y_3,\ \cdots\ y_m$ and each variable y_k is a function of a set of variables $x_1,\ x_2,\ x_3,\ \cdots\ x_n$ i.e.,

$$\begin{aligned}
y_1 &= f_1\left(x_1, x_2, x_3, \cdots x_n\right) \\
y_2 &= f_2\left(x_1, x_2, x_3, \cdots x_n\right) \\
&\vdots \\
y_m &= f_m\left(x_1, x_2, x_3, \cdots x_n\right)
\end{aligned}$$

Expressing each variable y_k in a linearized form gives

$$\begin{aligned}
y_1 &= y_1^0 + \frac{\partial y_1}{\partial x_1}\Delta x_1 + \frac{\partial y_1}{\partial x_2}\Delta x_2 + \cdots + \frac{\partial y_1}{\partial x_n}\Delta x_n \\
y_2 &= y_2^0 + \frac{\partial y_2}{\partial x_1}\Delta x_1 + \frac{\partial y_2}{\partial x_2}\Delta x_2 + \cdots + \frac{\partial y_2}{\partial x_n}\Delta x_n \\
&\vdots \\
y_m &= y_m^0 + \frac{\partial y_m}{\partial x_1}\Delta x_1 + \frac{\partial y_m}{\partial x_2}\Delta x_2 + \cdots + \frac{\partial y_m}{\partial x_n}\Delta x_n
\end{aligned} \tag{7.12}$$

Equations (7.12) can be expressed in matrix notation as

$$\mathbf{y} = \mathbf{y}^0 + \mathbf{J}\,\Delta\mathbf{x} \tag{7.13}$$

where

y is an (*m*,1) vector of (unknown) function values, $\mathbf{y} = \begin{bmatrix} y_1 & y_2 & \cdots & y_m \end{bmatrix}^T$

$\mathbf{y}^0$ is an (*m*,1) vector of approximate values of the functions, $\mathbf{y}^0 = \begin{bmatrix} y_1^0 & y_2^0 & \cdots & y_m^0 \end{bmatrix}^T$

$\Delta\mathbf{x}$ is an (*n*,1) vector of corrections to the approx. values, $\Delta\mathbf{x} = \begin{bmatrix} \Delta x_1 & \Delta x_2 & \cdots & \Delta x_n \end{bmatrix}^T$

J is the (m,n) the Jacobian matrix of partial derivatives

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} & \dfrac{\partial y_1}{\partial x_3} & \cdots & \dfrac{\partial y_1}{\partial x_n} \\ \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2} & \dfrac{\partial y_2}{\partial x_3} & \cdots & \dfrac{\partial y_2}{\partial x_n} \\ \vdots & & & & \vdots \\ \dfrac{\partial y_m}{\partial x_1} & \dfrac{\partial y_m}{\partial x_2} & \dfrac{\partial y_m}{\partial x_3} & \cdots & \dfrac{\partial y_m}{\partial x_n} \end{bmatrix}$$

7.3. The Derivation of some Common Surveying Observation Equations

Consider Figure 7.1. P_i is the instrument point and directions α_{ik} and distances s_{ik} have been observed to stations $P_1, P_2, P_3 \cdots P_k$. P_1 is the Reference Object (RO) and the direction $\alpha_{i1} = 0^\circ\,00'\,00''$. A bearing is assigned to the RO and bearings to all other stations may be obtained by adding the observed directions to the bearing of the RO.

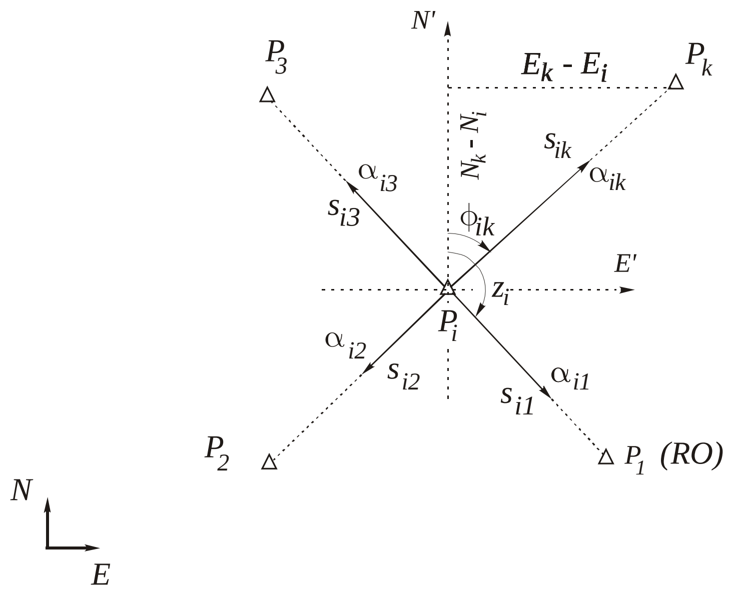


Figure 7.1 Observed directions α and distances s from P_i

α_{ik} observed direction P_i to P_k

ϕ_{ik} bearing P_i to P_k

s_{ik}	distance P_i to P_k
z_i	orientation constant for directions at P_i (bearing of the RO)
θ_{ik}	"observed" bearing $\theta_{ik} = \alpha_{ik} + z_i$
$E_i,\ N_i$	coordinates of P_i
$E_k,\ N_k$	coordinates of P_k

From Figure 7.1 the bearings ϕ_{ik} and distances s_{ik} are non-linear functions of the coordinates of points P_i and P_k

$$\phi_{ik} = \tan^{-1}\left(\frac{E_k - E_i}{N_k - N_i}\right) \tag{7.14}$$

$$s_{ik} = \sqrt{\left(E_k - E_i\right)^2 + \left(N_k - N_i\right)^2} \tag{7.15}$$

With $E = E^0 + \Delta E$ and $N = N^0 + \Delta N$ where $E^0,\ N^0$ are approximate values of the coordinates and $\Delta E,\ \Delta N$ are small corrections, linear approximations of ϕ_{ik} and s_{ik} can be written as

$$\phi_{ik} = \phi_{ik}^0 + \Delta E_k \frac{\partial \phi_{ik}}{\partial E_k} + \Delta N_k \frac{\partial \phi_{ik}}{\partial N_k} + \Delta E_i \frac{\partial \phi_{ik}}{\partial E_i} + \Delta N_i \frac{\partial \phi_{ik}}{\partial N_i} \tag{7.16}$$

$$s_{ik} = s_{ik}^0 + \Delta E_k \frac{\partial s_{ik}}{\partial E_k} + \Delta N_k \frac{\partial s_{ik}}{\partial N_k} + \Delta E_i \frac{\partial s_{ik}}{\partial E_i} + \Delta N_i \frac{\partial s_{ik}}{\partial N_i} \tag{7.17}$$

where ϕ_{ik}^0 and s_{ik}^0 are approximate bearings and distances respectively, obtained by substituting the approximate coordinates $E_k^0, N_k^0, E_i^0, N_i^0$ into equations (7.14) and (7.15).

The partial derivatives in equations (16) are evaluated in the following manner.

Using the relationships: $\dfrac{d}{dx}\tan^{-1} u = \dfrac{1}{1+u^2}\dfrac{du}{dx}$ and $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

The partial derivative $\dfrac{\partial \phi_{ik}}{\partial E_k}$

$$\frac{\partial \phi_{ik}}{\partial E_k} = \frac{1}{1+\left(\dfrac{E_k - E_i}{N_k - N_i}\right)^2} \frac{\partial}{\partial E_k}\left(\frac{E_k - E_i}{N_k - N_i}\right) = \frac{\left(N_k - N_i\right)^2}{\left(N_k - N_i\right)^2 + \left(E_k - E_i\right)^2} \frac{N_k - N_i}{\left(N_k - N_i\right)^2}$$

giving

$$\frac{\partial \phi_{ik}}{\partial E_k} = \frac{N_k - N_i}{\left(N_k - N_i\right)^2 + \left(E_k - E_i\right)^2} = \frac{N_k - N_i}{s_{ik}^2} = \frac{\cos\phi_{ik}}{s_{ik}} = b_{ik} \tag{7.18}$$

Similarly

$$\frac{\partial \phi_{ik}}{\partial N_k} = \frac{-\left(E_k - E_i\right)}{\left(N_k - N_i\right)^2 + \left(E_k - E_i\right)^2} = \frac{-\left(E_k - E_i\right)}{s_{ik}^2} = \frac{-\sin\phi_{ik}}{s_{ik}} = a_{ik} \tag{7.19}$$

$$\frac{\partial \phi_{ik}}{\partial E_i} = \frac{-\left(N_k - N_i\right)}{\left(N_k - N_i\right)^2 + \left(E_k - E_i\right)^2} = \frac{-\left(N_k - N_i\right)}{s_{ik}^2} = \frac{-\cos\phi_{ik}}{s_{ik}} = -b_{ik} \tag{7.20}$$

$$\frac{\partial \phi_{ik}}{\partial N_i} = \frac{\left(E_k - E_i\right)}{\left(N_k - N_i\right)^2 + \left(E_k - E_i\right)^2} = \frac{\left(E_k - E_i\right)}{s_{ik}^2} = \frac{\sin\phi_{ik}}{s_{ik}} = -a_{ik} \tag{7.21}$$

a_{ik} and b_{ik} are known as *direction coefficients.*

The partial derivatives of equation (7.17) are evaluated in the following manner

The partial derivative $\dfrac{\partial s_{ik}}{\partial E_k}$

$$\frac{\partial s_{ik}}{\partial E_k} = \frac{1}{2}\left[\left(E_k - E_i\right)^2 + \left(N_k - N_i\right)^2\right]^{-\frac{1}{2}} 2\left(E_k - E_i\right) = \frac{E_k - E_i}{s_{ik}} = \sin\phi_{ik} = d_{ik} \tag{7.22}$$

Similarly

$$\frac{\partial s_{ik}}{\partial N_k} = \frac{N_k - N_i}{s_{ik}} = \cos\phi_{ik} = c_{ik} \tag{7.23}$$

$$\frac{\partial s_{ik}}{\partial E_i} = \frac{-\left(E_k - E_i\right)}{s_{ik}} = -\sin\phi_{ik} = -d_{ik} \tag{7.24}$$

$$\frac{\partial s_{ik}}{\partial N_i} = \frac{-\left(N_k - N_i\right)}{s_{ik}} = -\cos\phi_{ik} = -c_{ik} \tag{7.25}$$

c_{ik} and d_{ik} are known as *distance coefficients.*

7.3.1. Observation equation for measured directions

An observation equation, relating observed directions to coordinates P_i and P_k can be written as

$$\alpha_{ik} + v_{ik} + z_i = \phi_{ik} = \tan^{-1}\left(\frac{E_k - E_i}{N_k - N_i}\right) \tag{7.26}$$

where v_{ik} are the residuals (small corrections) associated with observed directions. Using equation (7.16) together with the partial derivatives given in equations (7.18) to (7.21) gives a linear approximation of the observation equation for an observed direction

$$\alpha_{ik} + v_{ik} + z_i = a_{ik}\,\Delta N_k + b_{ik}\,\Delta E_k - a_{ik}\,\Delta N_i - b_{ik}\,\Delta E_i + \phi_{ik}^0 \tag{7.27}$$

where $a_{ik} = \dfrac{-\left(E_k - E_i\right)}{s_{ik}^2} = \dfrac{-\sin\phi_{ik}}{s_{ik}}$ and $b_{ik} = \dfrac{N_k - N_i}{s_{ik}^2} = \dfrac{\cos\phi_{ik}}{s_{ik}}$ are the <u>direction coefficients</u>

7.3.2. Observation equation for measured distances

An observation equation, relating observed distances to coordinates P_i and P_k can be written as

$$s_{ik} + v_{ik} = \sqrt{\left(E_k - E_i\right)^2 + \left(N_k - N_i\right)^2} \tag{7.28}$$

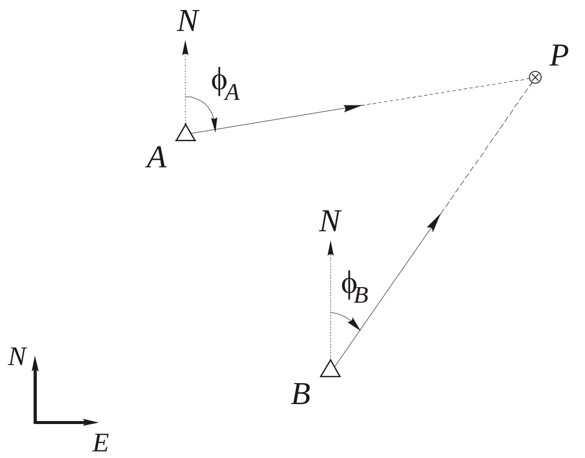
where v_{ik} are the residuals (small corrections) associated with observed distances. Using (7.17) together with the partial derivatives given in equations (7.22) to (7.25) gives a linear approximation of the observation equation for an observed distance

$$s_{ik} + v_{ik} = c_{ik}\,\Delta N_k + d_{ik}\,\Delta E_k - c_{ik}\,\Delta N_i - d_{ik}\,\Delta E_i + s_{ik}^0 \tag{7.29}$$

where $c_{ik} = \dfrac{N_k - N_i}{s_{ik}} = \cos\phi_{ik}$ and $d_{ik} = \dfrac{E_k - E_i}{s_{ik}} = \sin\phi_{ik}$ are the <u>distance coefficients</u>

7.4. An Example of Taylor's Theorem in Practice

Figure 7.2 shows a point *P*, whose coordinates are unknown, intersected by bearings from stations *A* and *B* whose coordinates are known.



Coordinates

A: 12273.910 E 29612.310 N *B:* 12875.270 E 28679.600 N

Bearings

$\phi_A = 81^\circ\, 01'\, 23''$ $\phi_B = 34^\circ\, 47'\, 52''$

Approximate coords *P:* 13677 E 29834 N

Figure 7.2 Bearing intersection

The information given above can be used to compute the coordinates of *P* by using an <u>iterative technique</u> employing linearized observation equations approximating the bearings ϕ_A and ϕ_B. These observation equations [see equation (7.27)] have been derived using Taylor's theorem.

In general, a bearing is a function of the coordinates of the ends of the line, i.e.,

$$\phi_{ik} = \tan^{-1}\left(\frac{E_k - E_i}{N_k - N_i}\right) = f\left(E_k, N_k, E_i, N_i\right) \tag{7.30}$$

where subscripts *i* and *k* represent instrument and target respectively. In this example (intersection) *A* and *B* are instrument points and are known and *P* is a target point and is unknown hence

$$\phi_{ik} = f\left(E_k, N_k\right)$$

is a non-linear function of the variables E_k and N_k only (the coordinates of *P*). Using equations (7.26) and (7.27) with modifications $\Delta E_i = \Delta N_i = 0$ since the coordinates of the instrument points are known gives

$$\phi_{ik} = a_{ik}\,\Delta N_k + b_{ik}\,\Delta E_k + \phi_{ik}^0 \tag{7.31}$$

$E_k = E_k^0 + \Delta E_k$, $N_k = N_k^0 + \Delta N_k$ and E_k^0, N_k^0, $\Delta E_k, \Delta N_k$ are approximate coordinates and small corrections respectively. $a_{ik} = \dfrac{-(E_k - E_i)}{s_{ik}^2} = \dfrac{-\sin\phi_{ik}}{s_{ik}}$ and $b_{ik} = \dfrac{N_k - N_i}{s_{ik}^2} = \dfrac{\cos\phi_{ik}}{s_{ik}}$ are <u>direction coefficients</u> and ϕ_{ik}^0 is an approximate bearing. Note that ϕ_{ik}^0 and the direction coefficients a_{ik} and b_{ik} are computed using the approximate coordinates of *P*.

Using equation (7.31), two equations for bearings ϕ_A and ϕ_B may be written as

$$\begin{aligned}\phi_A &= a_A \Delta N_P + b_A \Delta E_P + \phi_A^0 \\ \phi_B &= a_B \Delta N_P + b_B \Delta E_P + \phi_B^0\end{aligned}$$

These equations can be rearranged and expressed in matrix form as

$$\begin{bmatrix} a_A & b_A \\ a_B & b_B \end{bmatrix} \begin{bmatrix} \Delta N_P \\ \Delta E_P \end{bmatrix} = \begin{bmatrix} \phi_A - \phi_A^0 \\ \phi_B - \phi_B^0 \end{bmatrix}$$

or $$\mathbf{Cx} = \mathbf{u}$$

where **C** is a matrix of direction coefficients, **x** is the vector of corrections to the approximate coordinates of *P* and **u** is a vector of numeric terms (observed bearing – computed bearing).

The solution for the corrections in vector **x** is given by

$$\mathbf{x} = \mathbf{C}^{-1}\mathbf{u}$$

From the information given with Figure 7.2 the computed bearings $\left(\phi_{ik}^0\right)$ and distances $\left(s_{ik}^0\right)$ using the approximate coordinates of *P* are

AP:	81° 01' 17.1"	*BP:*	34° 46' 47.8"
	1420.5 m		*1405.5* m

and the numeric terms in vector **u** are

$$\begin{aligned}u_A &= \phi_A - \phi_A^0 \\ &= 81^\circ\,01'23'' - 81^\circ\,01'17.1'' \\ &= 5.9''\end{aligned}$$

$$\begin{aligned}u_B &= \phi_B - \phi_B^0 \\ &= 34^\circ\,47'52'' - 34^\circ\,46'47.8'' \\ &= 64.2''\end{aligned}$$

With the elements of **x** (the corrections to the approximate coordinates of *P*) in centimetres and the elements of **u** (the differences in observed and computed bearings) in seconds of arc

$$\begin{array}{ccc} \mathbf{Cx} & = & \mathbf{u} \\ \downarrow & & \searrow \\ \text{cm's} & & \text{seconds} \end{array}$$

the elements of the coefficient matrix **C** will be computed in sec/cm (seconds per centimetre) to maintain consistency of units so that

$$\begin{array}{ccccc} & \mathbf{x} = & \mathbf{C}^{-1} & \mathbf{u} & \\ \swarrow & & \downarrow & & \searrow \\ \text{cm} & & \text{cm}/\text{sec} & & \text{sec} \end{array}$$

Note that if the units (or dimensions) of the elements of **C** are sec/cm then the units of the elements of the inverse $\mathbf{C}^{-1}$ are cm/sec.

The elements of **C** are the direction coefficients and with distances s_{ik}^0 in centimetres

$$a_{ik} = \frac{-\sin\phi_{ik}^0}{s_{ik}^0} \times \rho'' \qquad \text{and} \qquad b_{ik} = \frac{\cos\phi_{ik}^0}{s_{ik}^0} \times \rho'' \qquad \text{where} \qquad \rho'' = \frac{180}{\pi} \times 3600$$

giving

$$a_A = \frac{-\sin\left(81^\circ\, 01'17.1''\right)}{(1420.5)(100)} \times \rho'' = -1.43426 \ \text{sec}/\text{cm}$$

$$b_A = \frac{\cos\left(81^\circ\, 01'17.1''\right)}{(1420.5)(100)} \times \rho'' = 0.22662 \ \text{sec}/\text{cm}$$

$$a_B = -0.83714 \ \text{sec}/\text{cm}$$

$$b_B = 1.20538 \ \text{sec}/\text{cm}$$

The matrix equation $\mathbf{Cx} = \mathbf{u}$ is

$$\begin{bmatrix} -1.43426 & 0.22662 \\ -0.83714 & 1.20538 \end{bmatrix} \begin{bmatrix} \Delta N_P \\ \Delta E_P \end{bmatrix} = \begin{bmatrix} 5.9 \\ 64.2 \end{bmatrix}$$

and the solution $\mathbf{x} = \mathbf{C}^{-1}\mathbf{u}$ is

$$\begin{bmatrix} \Delta N_P \\ \Delta E_P \end{bmatrix} = \begin{bmatrix} -0.78316 & 0.14724 \\ -0.54391 & 0.93187 \end{bmatrix} \begin{bmatrix} 5.9 \\ 64.2 \end{bmatrix} = \begin{bmatrix} 4.83 \\ 56.62 \end{bmatrix} \begin{array}{l} \text{cm} \\ \text{cm} \end{array}$$

giving the "adjusted" coordinates of P as

$$\begin{aligned} N_P &= N_P^0 + \Delta N_P = 29834.400 + 0.048 = 29834.048 \\ E_P &= E_P^0 + \Delta E_P = 13677.000 + 0.566 = 13677.566 \end{aligned}$$

These are the "new" approximate coordinates for P. A further iteration will show that the corrections to these values are less than 0.5 mm, hence the values above could be regarded as exact.