# FLUID THEORY RELATING TO WATER

## DEFINITION OF A FLUID

Fluids are substances which are capable of flowing and which under suitable temperature conditions, conform to the shape of containing vessels. When in equilibrium, fluids cannot sustain tangential or shearing forces. All fluids have some degree of compressibility and offer little resistance to change of form.

Fluids can be roughly divided into *liquids* and *gasses.* The main differences between them are:
(i) liquids are practically incompressible whereas gasses are compressible and
(ii) liquids occupy definite volumes and have free surfaces under the action of the Earth's gravity field whereas a given mass of gas expands until it occupies all portions of any containing vessel.

*Plastics* are different from liquids and gasses in that they possess a yield stress, which must be exceeded before flow commences. Substances such as tomato sauce, butter and many other organic substances are such fluids. Blood, which has a very low yield stress, is also regarded as a plastic.

## INTERNATIONAL SYSTEM OF UNITS (SI)

The *International System of Units* (SI) is a coherent set of units, one for each physical quantity. In this system the fundamental mechanical dimensions are *length* (the metre) *mass* (the kilogram) and *time* (the second). SI units have been used in Australia since 1970 (superseding the British Engineering system) and unless otherwise indicated, are used throughout these notes.

| *Physical quantity* | *Name* | *Symbol* |
|---|---|---|
| length | metre | m |
| mass | kilogram | kg |
| time interval | second | s |
| electric current | ampere | A |
| thermodynamic temperature | kelvin | K |
| amount of substance | mole | mol |
| luminous intensity | candela | cd |

SI derived units with special names (italics) and compound names relevant to hydraulics

| *Physical quantity* | *Name* | *Symbol* |
|---|---|---|
| force, weight[1] | *newton* | N (kg m/s$^2$) |
| pressure, stress | *pascal* | Pa (N/m$^2$) |
| energy, work, quantity of heat | *joule* | J (Nm) |
| area | square metre | m$^2$ |
| volume | cubic metre | m$^3$ |
| volumetric flow rate | cubic metre per second | m$^3$/s |
| velocity (linear) | metre per second | m/s |
| density (mass density) | kilogram per cubic metre | kg/m$^3$ |

Table 1

## BRITISH ENGINEERING SYSTEM OF UNITS

In this system of units (still in wide use around the world) the fundamental mechanical dimensions are *length* (the foot) *force* (the pound or pound weight) and *time* (the second). The unit of *mass* in this system is the *slug* and is defined in the following manner using Newton's second law:

[1] The *mass* of an object is the quantity of matter it contains; this is constant and expressed in units related to the kilogram. The *mass* of an object differs from its *weight*, which is the measure of the force of gravity acting on the object; this is measured in newtons.

$$\text{force (pounds)} = \text{mass (slugs)} \times \text{acceleration (ft/sec}^2\text{)}$$

Then

$$\text{weight (pounds)} = \text{mass (slugs)} \times g \text{ (ft/sec}^2\text{)}$$

or

$$\text{mass } M \text{ (slugs)} = \frac{\text{weight } W \text{ (pounds)}}{g \text{ (ft/sec}^2\text{)}}$$

By this equation, the units of mass (slugs) in this system are lb-sec$^2$/ft.

Some British Engineering system quantities are set out below

| *Physical quantity* | *Name* | *Symbol* |
|---|---|---|
| length | foot | ft |
| mass | slug | lb-sec$^2$/ft |
| time interval | second | s |
| temperature | fahrenheit | F |
| force, weight | pound (or pound force) | lb (or lbf) |
| pressure, stress | pound per square inch | psi |
| energy, work | foot-pound-force | ft-lbf |
| | | |
| area | square foot | ft$^2$ |
| volume | cubic foot | ft$^3$ |
| volumetric flow rate | cubic feet per second | ft$^3$/s |
| velocity (linear) | feet per second | ft/s |
| density (mass density) | slug per cubic foot | slug/ft$^3$ |

Table 2

**CONVERSION FACTORS and USEFUL RELATIONSHIPS**

The following relationships can be used to determine conversion factors between quantities in both systems

Conversion Factors (exact relationships, Mechtly,1973)

1 foot = 0.3048 metres

1 pound = 0.45359237 kilograms

Useful Relationships

*Force*

Using the conversion factors above and the internationally accepted values of the acceleration due to gravity (see National Institute of Standards and Technology Reference on Constants, Units and Uncertainty, http://physics.nist.gov/cuu/index.html)

$$g = 9.80665 \text{ m/s}^2 = 32.17405 \text{ ft/s}^2$$

the following relationships for force (weight), mass and density are

1 lb = 4.44822 N

1 slug = 14.59390 kg

$$1 \text{ slug/ft}^3 = 515.37884 \text{ kg/m}^3$$

*Pressure*

$$
\begin{aligned}
1 \text{ millibar (mb)} &= 100 \text{ pascal (Pa)} \\
1 \text{ mm of Hg } (0^\circ\text{C}) &= 133.3224 \text{ Pa} \\
1 \text{ in of Hg } (32^\circ\text{F}) &= 3386.389 \text{ Pa} \\
1 \text{ mb} &= 0.750062 \text{ mm of Hg} \\
&= 0.029530 \text{ in of Hg} \\
1 \text{ std. atmosphere} &= 1013.25 \text{ mb (exactly)} \\
&= 760.0003 \text{ mm of Hg} \\
&= 29.9213 \text{ in of Hg} \\
1 \text{ psi} &= 6894.7573 \text{ Pa} \\
&= 6.8948 \text{ KPa}
\end{aligned}
$$

*Temperature*

The relationships (exact) between degrees Fahrenheit (ºF) and degrees Celsius (ºC) are

$$
\begin{aligned}
{}^\circ\text{C} &= (5/9)\left({}^\circ\text{F} - 32\right) \\
{}^\circ\text{F} &= (9/5)\,{}^\circ\text{C} + 32
\end{aligned}
$$

## DIMENSIONAL ANALYSIS

Dimensional analysis is the mathematics of *dimensions* of quantities and is a useful tool for modern fluid mechanics. In any equation expressing a physical relationship between quantities, absolute numerical and dimensional equality must exist. In general all such physical relationships can be reduced to the fundamental quantities of force $F$, length $L$ and time $T$ (or mass $M$, length $L$ and time $T$). Dimensional analysis is useful in converting one system of units to another. The following table shows dimensions of various quantities

(a) in terms of force $F$, length $L$ and time $T$ and
(b) in terms of mass $M$, length $L$ and time $T$.

| *Quantity* | *Symbol* | (a) *F-L-T* | (b) *M-L-T* |
|---|---|---|---|
| Area $A$ in m$^2$ | $A$ | $L^2$ | $L^2$ |
| Volume $v$ in m$^3$ | $v$ | $L^3$ | $L^3$ |
| Velocity $V$ or $v$ | $V$ or $v$ | $L\,T^{-1}$ | $L\,T^{-1}$ |
| Acceleration $a$ or $g$ in m/s$^2$ | $a$ or $g$ | $L\,T^{-2}$ | $L\,T^{-2}$ |
| Force $F$ (mass $\times$ acceleration) in N | $F$ | $F$ | $M\,L\,T^{-2}$ |
| Mass $M$ in kg | $M$ | $F\,T^2 L^{-1}$ | $M$ |
| Density $\rho$ in kg/m$^3$ | $\rho$ | $F\,T^2 L^{-4}$ | $M\,L^{-3}$ |
| Pressure $p$ in N/m$^2$ or Pa | $p$ | $F L^{-2}$ | $M L^{-1}\,T^{-2}$ |
| Absolute viscosity $\mu$ in Pa s | $\mu$ | $F\,T\,L^{-2}$ | $M L^{-1}\,T^{-1}$ |
| Rate of discharge $Q$ in m$^3$/s | $Q$ | $L^3\,T^{-1}$ | $L^3\,T^{-1}$ |
| Shearing stress $\tau$ in N/m$^2$ or Pa | $\tau$ | $F L^{-2}$ | $M L^{-1}\,T^{-2}$ |
| Weight $W$ in N | $W$ | $F$ | $M\,L\,T^{-2}$ |

Table 3

**MASS DENSITY OF A SUBSTANCE** $\rho$ (rho)

The density of a substance is its *mass* per unit volume.

$$\rho = \frac{dm}{dv} \tag{1}$$

where $dm$ is an element of mass
$dv$ is an element of volume

The density of water is 1000 kg/m$^3$ at 4°C (Giles 1976) and may be regarded as constant for practical changes in pressure. In the British Engineering system, the relative density of water is taken as 1.94 slug/ft$^3$ at 40°F

When dealing with liquids the variable $\gamma$ (gamma) is used where

$$\gamma = \rho g \tag{2}$$

and $g$ is the gravitational acceleration; often taken as having an average value of 9.81 m/s$^2$.

> In the British Engineering system $\gamma$ is known as the *specific weight* although in the SI system, the use of the term specific is confined to descriptions of properties per unit mass. The term specific weight is still widely used in hydraulics although it is not defined in the SI system.

**RELATIVE DENSITY OF A BODY**

The relative density of a body is the dimensionless number denoting the ratio of the mass of the body to the mass of an equal volume of a substance taken as a standard. Relative densities of liquids are referred to water at 4°C as a standard.

$$\begin{aligned}\text{relative density of a substance} &= \frac{\text{mass of the substance}}{\text{mass of an equal volume of water}} \\ &= \frac{\text{density of substance}}{\text{density of water}}\end{aligned} \tag{3}$$

The relative density of water is 1.00 and of mercury is 13.57. Relative density is commonly known as *specific gravity* but for the reasons mentioned above the term specific is not used in the SI system.

**VISCOSITY OF A FLUID**

It is well known that fluids such as olive oil or sauce flow more slowly through a given tube than would water or petrol. This difference is ascribed to the presence of an internal fluid friction known as *viscosity;* olive oil being more viscous than water. Viscosity of a fluid is the amount of resistance to a shearing force and is due primarily to interaction between fluid molecules.

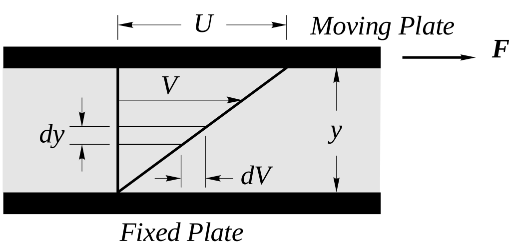



Figure 1

Referring to Figure 1, consider two large parallel plates at a small distance $y$ apart, the space between the plates filled with a fluid. The bottom plate is fixed and the upper plate is moving with a constant velocity $U$ due to a constant force $F$. Experiments have shown that whenever a fluid flows in a channel the layer of fluid at the channel wall actually adheres to it and is stationary, ie, there is no slip. So the fluid in contact with the upper plate will adhere to it and will move at a velocity $U$. The fluid in contact with the lower plate will have a velocity of zero. If distance $y$ and velocity $U$ are not to great, the velocity variation (or gradient) will be a straight line.

It is known (from experiments) that the force $F$ varies with the area of the plate, with velocity $U$, and inversely with distance $y$. Since, by similar triangles, $U/y = dV/dy$ then

$$F \propto \frac{AU}{y} \quad \text{or} \quad \frac{F}{A} \propto \frac{dV}{dy}$$

The ratio $F/A$ is known as the *shear stress* and is denoted by the symbol $\tau$ (tau). The units of shear stress ( $\tau$ ) are Newtons per square metre (N/m$^2$) or Pascals (Pa). If a constant of proportionality $\mu$ (mu) , called the *absolute viscosity* (or dynamic viscosity) is introduced into the equation above, then

$$\tau = \mu \frac{dV}{dy} \tag{4}$$

The units of absolute viscosity $(\mu)$ are Pascal-second (Pa s).

Equation (4) is *Newton's law of viscosity*, and fluids according with this equation are known as Newtonian fluids. In the second book of his famous *Principia* (*Philosophiae naturalis principia mathematica* – The Mathematical Principles of Natural Philosophy – published in 1687), Newton considered the circular motion of fluids as part of his studies of the planets and wrote

> *hupothesis*
>
> The resistance arising from the want of lubricity in the parts of a fluid, is, other things being equal, proportional to the velocity with which the parts of a fluid are separated from one another.

(Streeter 1971)

Viscosities of liquids decrease with temperature, but are not appreciably affected by pressure.

## PRESSURE $p$

A fluid under pressure exerts a force on any surface in contact with it. Fluid pressure is transmitted with equal intensity in all directions and acts normal to any plane. In the same horizontal plane, the pressure intensities in a liquid are equal.

Consider an element of area $\Delta A$ of a surface on which a force $\Delta F$ is exerted, the pressure $p$ is

$$p = \lim_{A \to 0} \frac{\Delta F}{\Delta A} = \frac{dF}{dA} \tag{5}$$

For conditions where the force $F$ is uniformly distributed over an area $A$ then

$$p = \frac{F}{A} \tag{6}$$

The units of pressure are newtons per square metre (N/m$^2$) or pascals[2] (Pa).

---

[2] A unit of pressure in the SI system named in honour of Blaise Pascal (1623–1662) a French mathematician, physicist, religious philosopher, and master of French prose. Pascal laid the foundation for the modern theory of probabilities, formulated what came to be known as Pascal's law of pressure, and propagated a religious doctrine that taught the experience of God through the heart rather than through reason. Pascal tested the theories of Galileo and Evangelista Torricelli (an Italian physicist who discovered the principle of the barometer). To do so, he reproduced and amplified experiments on atmospheric pressure by constructing mercury barometers and measuring air pressure, both in Paris and on the top of a mountain overlooking Clermont-Ferrand. These tests paved the way for further studies in hydrodynamics and hydrostatics. Pascal invented the syringe and created the hydraulic press, an instrument based upon the principle that became known as Pascal's Law: *pressure applied to a confined liquid is transmitted undiminished through the liquid in all directions regardless of the area to which the pressure is applied*. His publications on the problem of the vacuum (1647-48) added to his reputation (Encyclopaedia Britannica 1999).

**VARIATION OF PRESSURE WITH DEPTH**

Consider Figure 2 which shows a small (horizontal) element of fluid mass *dm* at rest, submerged within a fluid body.

The weight (force = mass $\times$ acceleration) of the element is $dW = g\,dm = g\rho\,dv$ where $dv = dA\,dh$ is the elemental volume. Hence

$$dW = \rho g\,dA\,dh \tag{7}$$

The forces acting on or by the fluid element must be in equilibrium, so

$$dF_1 + dW = dF_2 \tag{8}$$

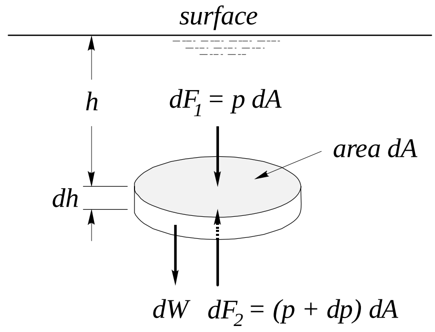



Figure 2

Substituting (7) into (8) and using (5) gives

$$p\,dA + \rho g\,dA\,dh = (p + dp)\,dA$$

Cancelling terms and rearranging gives

$$dp = \rho g\,dh \tag{9}$$

Assuming that $g$ and $\rho$ are constant, integrating (9) gives

$$\int_{p_1}^{p_2} dp = \rho g \int_{h_1}^{h_2} dh$$

Thus, the difference in pressure between any two points at different levels in a liquid is given by

$$p_2 - p_1 = \rho g\,(h_2 - h_1) \tag{10}$$

If point 1 is on the free surface of the liquid and $h$ is positive downward (10) becomes

$$p = \rho g h \tag{11}$$

where $p$ (in Pa) is known as gauge pressure. Gauge pressures are often expressed in *bar* where $1 \text{ bar} = 10^5$ Pa or *millibar* (mb) where $1 \text{ mb} = 100 \text{ Pa} = 1 \text{ hPa}$. In the British Engineering system, the units of pressure are pounds per square inch (psi).

**ABSOLUTE AND GAUGE PRESSURE**

If pressure is measured above absolute zero, it is known as *absolute* pressure. If pressure is measured either above or below or below atmospheric pressure as a base, it is called a *gauge* pressure. This is because practically all pressure gauges read zero when open to the atmosphere and measure only the difference between the pressure of the fluid to which they are connected and that of the surrounding air (Daugherty & Ingersoll 1954).

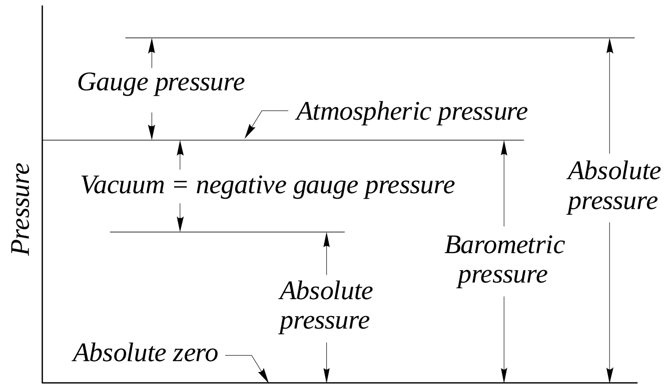



Figure 3

If the pressure is below that of the atmosphere, it is designated as a *vacuum* and its gauge value is negative. All values of absolute pressure for fluids are positive and atmospheric pressure is that measured by a *barometer* (barometric pressure). Atmospheric pressure varies with altitude, and for a fixed location, varies slightly from time to time (high- and low-pressure weather systems). For reference purposes, a *standard atmosphere* has a value of 1013.25 mb. Figure 3 shows the relationship between different types of pressure.

Example: At a depth of 60 metres below a free water surface, what will be the water pressure (gauge pressure)? (use density of water $\rho$ = 1000 kg/m$^3$ and acceleration due to gravity $g$ = 9.81 m/s$^2$)

$$\begin{aligned} p &= \rho g\, h \\ &= (1000)(9.81)(60) \\ &= 588,600 \text{ Pa} \\ &= 588.6 \text{ KPa} \end{aligned}$$

**PRESSURE HEAD** ***h***

Pressure head $h$ represents the height of a column of homogeneous fluid of density $\rho$ that will produce a given intensity of pressure. Then

$$h = \frac{p}{\rho g} \tag{12}$$

Example: What depth of oil (relative density 0.750) will produce a pressure of 2.75 bar? What depth of water? (use average value of 9810 N/m$^3$ for $\rho g$ of water)

Solution:

$$h_{oil} = \frac{p}{\rho_{oil}\, g} = \frac{2.75 \times 10^5}{0.750 \times 9810} = 37.4 \text{ m}$$

$$h_{water} = \frac{p}{\rho_{water}\, g} = \frac{2.75 \times 10^5}{9810} = 28 \text{ m}$$

## THE FLOW OF FLUIDS

Fluid flow is complex and not always subject to exact mathematical analysis. A fluid in flow experiences, in addition to gravity, pressure forces, viscous and turbulent shear resistances, boundary resistance, and forces due to surface tension and compressibility effects of the fluid. The presence of such a complex system of forces in real fluid flow makes the analysis very complicated. However, a simplifying approach to problems may be made by assuming the fluid to be *ideal* or perfect. Ideal fluids are non-viscous (frictionless) and incompressible. Water has a relatively low viscosity and is practically incompressible, and is found to behave like an ideal fluid. Therefore, simplifications can be made in the development of formulae and the application of those formulae to practical hydraulic problems. Some definitions and terminology are given below.

### Laminar and Turbulent Flow – Reynolds' Experiment

In 1883, the British engineer Osborne Reynolds (see citation below) demonstrated that there were two distinctly different types of fluid flow. He injected a fine, threadlike stream of coloured liquid at the entrance of a large glass tube through which water was flowing from a tank. A valve at the discharge end permitted him to vary the flow. When the velocity in the tube was small, the coloured liquid was visible as a straight line or *streamline* throughout the length of the glass tube, demonstrating that the particles of water moved in parallel straight lines. As the velocity of the water increased (by opening the valve) there was a point at which the flow changed. The line would first become wavy, and then at a short distance from the entrance would break into numerous vortices beyond which the colour would be uniformly diffused so that no streamlines could be distinguished. With closure of the valve, the process was reversed. The first type of flow (the coloured dye as a streamline) is known as *laminar* or *streamline* flow. These terms arise from (i) the fluid appears to move by the sliding of laminations of infinitesimal thickness relative to adjacent layers and (ii) individual particles of water appear to move in definite paths, which combined, form streamlines. The second type of flow (the coloured dye dispersed) is known as *turbulent* flow and is characterised by irregular velocities of individual particles.

Reynolds also found that the nature of flow depends on the ratio of the forces of *inertia* and fluid *friction* due to viscosity. For an incompressible fluid of density $\rho$ and viscosity $\mu$ flowing through a pipe of diameter *D* at an average velocity *V*, the ratio of these forces is a dimensionless number known as the Reynolds number $N_R$

*Reynolds number*
$$N_R = \frac{\rho D V}{\mu} \tag{13}$$

Reynolds established from the experiment that the transition from laminar to turbulent flow occurs when $N_R \approx 2000$ (Streeter 1971).

> **Reynolds**, Osborne (b. Aug. 23, 1842, Belfast, Ire.–d. Feb. 21, 1912, Watchet, Somerset, Eng.), British engineer, physicist, and educator best known for his work in hydraulics and hydrodynamics. **Reynolds** was born into a family of Anglican clerics. He gained early workshop experience by apprenticing with a mechanical engineer, and he graduated at Queens' College, Cambridge, in mathematics in 1867. In 1868, he became the first professor of engineering at Owens College, Manchester, a position he held until his retirement in 1905. He became a fellow of the Royal Society in 1877 and received a Royal Medal in 1888. Though his earliest professional research dealt with such properties as magnetism, electricity, and heavenly bodies, **Reynolds** soon began to concentrate on fluid mechanics. In this area he made a number of significant contributions. His studies of condensation and heat transfer between solids and fluids brought radical revision in boiler and condenser design, while his work on turbine pumps permitted their rapid development. He formulated the theory of lubrication (1886) and in 1889 developed the standard mathematical framework used in turbulence work. He also studied wave engineering and tidal motions in rivers and made pioneering contributions to the concept of group velocity. Among his other contributions were the explanation of the radiometer and an early absolute determination of the mechanical equivalent of heat. His paper on the law of resistance in parallel channels (1883) is a classic. The "Reynolds stress" in fluids with turbulent motion and the "Reynolds number" used for modelling in fluid flow experiments are named for him.
>
> 

**Steady and unsteady flow**

Flow parameters such as velocity $V$, pressure $p$ and density $\rho$ of a fluid flow are independent of time in a *steady* flow. For *unsteady* flow, $V$, $p$ and $\rho$, all or singly, vary with time. For example

$$\frac{dV}{dt} = 0 \quad \text{for steady flow}$$
$$\frac{dV}{dt} \neq 0 \quad \text{for unsteady flow}$$

In reality, these parameters are generally time dependent but often remain constant, on average, over a reasonable time period. In the solution of many practical hydraulic problems, fluid flow is assumed to be steady.

In steady flow the streamline has a fixed direction at every point; and is therefore fixed in space. A particle always moves tangentially to the streamline, hence in steady flow, the path of a particle is a streamline.

**Uniform and non-uniform flow**

Flow is uniform if its characteristics at any given instant remain the same at different points in the direction of flow; otherwise the flow is non-uniform. If $s$ is a distance measured along the flow, uniform and non-uniform flow is expressed mathematically as

$$\frac{dV}{ds} = 0 \quad \text{for uniform flow}$$
$$\frac{dV}{ds} \neq 0 \quad \text{for non-uniform flow}$$

The flow of water through a long uniform pipe at a constant rate is steady uniform flow. If the flow rate is changing then it is unsteady uniform flow.

The flow of water through a non-uniform pipe (eg, changing diameter) at a constant rate is steady non-uniform. If the flow rate varies, then the flow is unsteady non-uniform flow.

**Streamlines and Streamtubes**

Streamlines are imaginary curves drawn through a fluid to indicate the direction of motion of individual particles. A tangent to a streamline represents the instantaneous velocity of the fluid particles at that point. Streamrubes represent elementary portions of a flowing fluid bounded by a group of streamlines which confine the flow. If the cross-sectional area of the streamtube is sufficiently small, the velocity of the midpoint may be taken as the mean velocity for the section as a whole. The streamtube is used to derive the equation of continuity.

## EQUATION OF CONTINUITY

The equation of continuity results from the principle of conservation of mass. For steady flow, the mass of fluid passing all sections in a stream of fluid (a streamtube), per unit mass of time is the same.

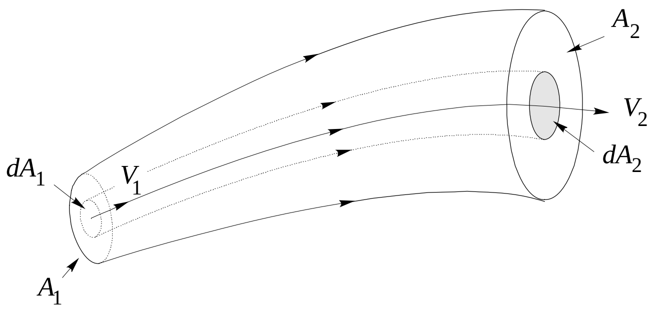



Figure 4

Considering Figure 4, which shows a streamtube, we may write

$$\frac{\text{mass entering stream tube}}{\text{second}} \text{ at section 1} = \frac{\text{mass leaving stream tube}}{\text{second}} \text{ at section 2}$$

Since density = mass / volume, volume = area $\times$ length and velocity = length / time then the equation of continuity can be written as

$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2 = \text{ constant} \tag{14}$$

For *incompressible* fluids and where the density is constant ( $\rho_1 = \rho_2$ ), then for all practical purposes the equation of continuity (14) can be expressed as

$$Q = A_1 V_1 = A_2 V_2 = \text{ constant} \tag{15}$$

where

- $Q$ is the discharge in cubic metres per second $\text{m}^3/\text{s}$
- $A$ is the cross-sectional area in $\text{m}^2$
- $V$ is the mean velocity of the section in m/s

## ENERGY EQUATION

The energy equation results from application of the principle of conservation of energy to fluid flow. In scientific usage, a body is said to possess energy if it is capable of doing work and the amount of work it can do is a measure of its energy. The energy possessed by a flowing liquid consists of internal energy and energies due to pressure, velocity and position. In the direction of flow, the energy principle is summarised in the following equation

$$\begin{bmatrix} \text{Energy at} \\ \text{Section 1} \end{bmatrix} + \begin{bmatrix} \text{Energy} \\ \text{Added} \end{bmatrix} - \begin{bmatrix} \text{Energy} \\ \text{Lost} \end{bmatrix} - \begin{bmatrix} \text{Energy} \\ \text{Exctacted} \end{bmatrix} = \begin{bmatrix} \text{Energy at} \\ \text{Section 2} \end{bmatrix}$$

*Energy Added* to fluid flow is generally from mechanical devices such as pumps, *Energy Lost* is usually due to friction forces and *Energy Extracted* from the flow is usually extracted by mechanical devices such as turbines.

For steady flow of incompressible fluids in which the change in internal energy is negligible, this equation simplifies to

*Bernoulli's theorem*

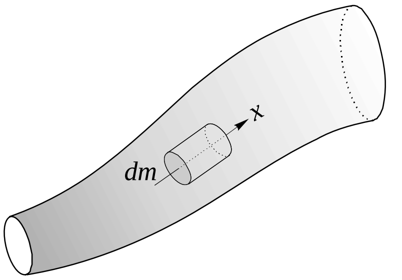
$$\left( \frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 \right) + H_A - H_L - H_E = \left( \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 \right) \tag{16}$$

In Bernoulli's theorem, all the terms are in units of length and are expressed as either "heads" or heights above a datum. The term $\dfrac{p}{\rho g}$ is the *pressure head* and the term $\dfrac{V^2}{2g}$ is the *velocity head*. $H_L$ is known as *lost head*.

A proof of Bernoulli's theorem is set out below, and follows the derivation given by Giles (1976).

## DERIVATION OF BERNOULLI'S THEOREM

Consider Figure 5(a) showing, as a free body, an elementary mass of fluid *dm*, contained within a cylinder of length *dl* and end-area *dA*. The fluid is flowing in the pipe at a steady rate and the direction of motion is the *x-axis*. Figure 5(b) shows a sectional view of the fluid element; the *x-axis* in the plane of the paper and inclined to the horizontal at an angle $\theta$ . The forces acting in the *x*-direction are due to (i) the pressure acting on the end areas, (ii) the component of the weight and (iii) the shearing forces $dF_S$ exerted by the adjacent fluid. Forces normal to the direction of motion have not been shown acting on the free body *dm.*



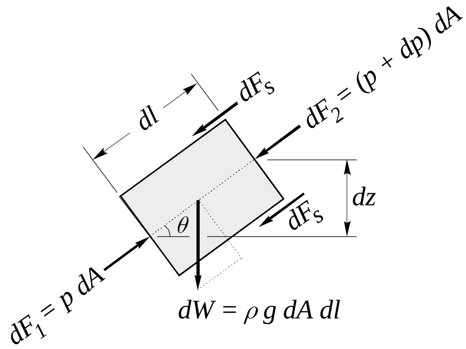




Figure 5(a) Figure 5(b)

Considering forces in the *x*-direction, Newton's second law $(\text{force = mass} \times \text{acceleration})$ can be written as $\sum F_x = dm\, a_x$, and with the acceleration in the *x*-direction $a_x = dV/dt$ we obtain

$$\left( dF_1 - dF_2 - dW \sin\theta - dF_S \right) = dm \frac{dV}{dt} \tag{17}$$

Remembering that pressure *p* is force divided by area, $dF_1$ and $dF_2$ equal $p\,dA$ and $(p + dp)\,dA$ respectively. The weight (force = mass $\times$ acceleration) of the element is $dW = g\,dm$ and $dm = \rho\, dv = \rho\, dA\, dl$, hence $dW = \rho g\, dA\, dl$. Making these substitutions, (17) becomes

$$\left( p\,dA - (p + dp)\,dA - \rho g dA dl \sin\theta - dF_S \right) = \rho dA dl \frac{dV}{dt} \tag{18}$$

Dividing (18) by $\rho g\, dA$, cancelling terms, then replacing $dl/dt$ with velocity *V* and $dl \sin\theta$ with *dz* gives

$$-\frac{dp}{\rho g} - dz - \frac{dF_S}{\rho g\, dA} = \frac{VdV}{g} \tag{19}$$

The term $\frac{dF_S}{\rho g\, dA}$ represents the resistance to flow in length *dl*. The shear stress $\tau = \text{force}/\text{area} = dF_S/dA$ and the area $(\text{perimeter} \times \text{length})$ is $dA = dP\, dl$, hence $dF_S = \tau dP\, dl$ and the term

$$\frac{dF_S}{\rho g\, dA} = \frac{\tau dP\, dl}{\rho g\, dA} = \frac{\tau dl}{\rho g\, R} \tag{20}$$

The term *R* in (20) is known as the *hydraulic radius* and is defined as the cross-sectional area divided by the wetted surface perimeter. In this case, $R = dA/dP$.

The sum of all the shearing forces is the measure of energy lost due to the flow, remembering that work is force by distance and a body is said to possess energy if it is capable of doing work. This energy loss is called *lost head* $h_L$. Shearing forces are often also called friction forces, and the lost head $h_L$ is also known as *friction head* $h_f$

$$\text{lost head } dh_L = \frac{\tau dl}{\rho g\, R} \tag{21}$$

The units of head loss (or friction loss) are metres if other quantities are in SI units. This can be verified by dimensional analysis

$$dh_L = \frac{\tau dl}{\rho g\, R} = \frac{\left( ML^{-1}T^{-2} \right)(L)}{\left( ML^{-3} \right)\left( LT^{-2} \right)(L)} = L = \text{metres}$$

Substituting (20) and (21) into (19) and re-arranging gives the equation of motion for steady flow of any fluid

$$\frac{dp}{\rho g} + \frac{VdV}{g} + dz + dh_L = 0 \tag{22}$$

This differential equation for steady flow is a fundamental fluid flow equation. When applied to an ideal fluid (lost head = 0) it is known as *Euler's equation.*

For *incompressible* fluids (and water is practically incompressible) the integration of (22) is as follows

$$\int_{p_1}^{p_2} \frac{dp}{\rho g} + \int_{V_1}^{V_2} \frac{VdV}{g} + \int_{z_1}^{z_2} dz + \int_1^2 dh_L = 0$$

The evaluation of the last integral yields the *total head lost* $H_L$. Integrating and substituting the limits gives

$$\left( \frac{p_2}{\rho g} - \frac{p_1}{\rho g} \right) + \left( \frac{V_2^2}{2g} - \frac{V_1^2}{2g} \right) + \left( z_2 - z_1 \right) - H_L = 0 \tag{23}$$

Re-arranging (23) gives the customary form of *Bernouilli's theorem* (when no energies are added or extracted)

$$\left( \frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 \right) - H_L = \left( \frac{p_1}{\rho g} + \frac{V_2^2}{2g} + z_2 \right) \tag{24}$$

Note that in equations (23) and (24) all terms are in units of length, a fact that can be verified by dimensional analysis, eg

$$\frac{p}{\rho g} + \frac{V^2}{2g} + z = \frac{ML^{-1}T^{-2}}{\left(ML^{-3}\right)\left(LT^{-2}\right)} + \frac{\left(LT^{-1}\right)^2}{LT^{-2}} + L = L + L + L$$

For an ideal fluid ( $H_L = 0$ ) and when no other energies are added or extracted Bernoulli's theorem can be expressed as

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{ constant} \tag{25}$$

In this form, Bernoulli's theorem can explain some seemingly paradoxical practical results of fluid flow (air and water). The following extract from Encyclopaedia Britannica (1994-1999) is illuminating.

> Bernoulli's theorem indicates that, if an ideal fluid is flowing along a pipe of varying cross section, then the pressure is relatively low at constrictions where the velocity is high and relatively high where the pipe opens out and the fluid stagnates. Many people find this situation paradoxical when they first encounter it. Surely, they say, a constriction should increase the local pressure rather than diminish it? The paradox evaporates as one learns to think of the pressure changes along the pipe as cause and the velocity changes as effect, instead of the other way around; it is only because the pressure falls at a constriction that the pressure gradient upstream of the constriction has the right sign to make the fluid accelerate. Paradoxical or not, predictions based on Bernoulli's theorem are well-verified by experiment. Try holding two sheets of paper so that they hang vertically two centimetres or so apart and blow downward so that there is a current of air between them. The sheets will be drawn together by the reduction in pressure associated with this current. Ships are drawn together for much the same reason if they are moving through the water in the same direction at the same speed with a small distance between them. In this case, the current results from the displacement of water by each ship's bow, which has to flow backward to fill the space created as the stern moves forward, and the current between the ships, to which they both contribute, is stronger than the current moving past their outer sides. As another simple experiment, listen to the hissing sound made by a tap that is almost, but not quite, turned off. What happens in this case is that the flow is so constricted and the velocity within the constriction so high that the pressure in the constriction is actually negative. Assisted by the dissolved gases that are normally present, the water cavitates as it passes through, and the noise that is heard is the sound of tiny bubbles collapsing as the water slows down and the pressure rises again on the other side.

## AN APPLICATION OF BERNOULLI'S THEOREM

Figure 6 shows a pipe with water flowing under pressure, at a mean velocity equal to *V*. A *piezometer* and a *Pitot tube* are attached to the pipe. The piezometer is a simple device for measuring fluid pressure. It is a thin tube inserted into the pipe at right angles and water will rise in the tube to a height equal to the *static* pressure head. The Pitot tube is a bent tube with one end pointed directly into the stream flow. The water rises in the tube until all its kinetic energy is converted into potential energy. At *A*, the fluid streamlines are parallel but divide as they approach the blunt end of the Pitot tube at *B*. At this point, there is complete stagnation (zero velocity), since the fluid at this point is not moving up, nor down, nor neither to the right or left. The water in the Pitot tube will rise to a height equal to the *stagnation* pressure head. The difference *h* between the static pressure head and stagnation pressure head is the *velocity head*.

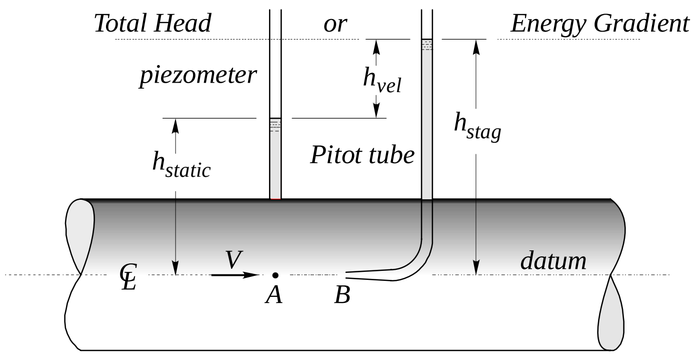



Figure 6

Applying Bernoulli's theorem from the pipe cross section through *A* in the undisturbed fluid flow to the section through *B*, considering the centre-line of the pipe as the datum yields

$$\left(\frac{p_A}{\rho g}+\frac{V^2}{2g}+0\right)-\begin{matrix}\text{no loss}\\\text{(assumed)}\end{matrix}=\left(\frac{p_B}{\rho g}+0+0\right)$$

The static pressure head is $h_{static}=\frac{p_A}{\rho g}$, the stagnation pressure head is $h_{stag}=\frac{p_B}{\rho g}$ and the difference between them is the velocity head $h_{vel}$, hence for an ideal "frictionless" fluid

$$h_{vel}=\frac{V^2}{2g} \tag{26}$$

Note, that in the case of water flowing in an open channel or stream, the static pressure head is zero (since the local pressure is zero gauge). The height to which water will rise in a Pitot tube, in this case, measures the velocity head. In stream gauging measurements using weirs, formulae are derived based on an assumption that the velocity of flow *v* through a thin strip is a function of the head *h* of water above the strip. This assumption can be deduced from a rearrangement of (26); $v=\sqrt{2gh}$

## APPLICATION OF THE BERNOULLI THEOREM IN HYDRAULIC DESIGN PROBLEMS

In hydraulic design problems, the Bernoulli theorem should be applied in the following manner

(1) Draw a sketch of the system, choosing and labelling all cross sections of the stream (or pipe) under consideration.
(2) Apply the Bernoulli equation in the direction of flow. That is, section numbers increase in the direction of flow. Select a datum line such that all quantities (heads and elevations) are positive.
(3) Evaluate the energy upstream at Section 1 expressing pressure head and velocity head in metres of fluid. If required, pressures should be expressed in gauge units. All velocities can be assumed to be average or mean velocities of flow.

(4) Add, in metres of the fluid, any energy contributed by mechanical devices such as pumps.
(5) Subtract, in metres of the fluid, any energy lost during flow (friction losses).
(6) Subtract, in metres of the fluid, any energy extracted by mechanical devices, such as turbines.
(7) Equate this summation of energy to the sum of the pressure head, velocity head and elevation head in Section 2.
(8) If the two velocity heads are unknown, they can be related by the equation of continuity.

## ENERGY GRADIENT

The energy gradient (or energy line or energy grade line) is a graphical representation of the energy at each section. With respect to a datum, the total energy (in metres of fluid) can be plotted at each section and the line obtained (passing through these points) is a valuable tool in many flow problems. The energy gradient will slope downwards in the direction of flow, unless energy is added by mechanical means, or unless no losses are assumed. In the case of no losses assumed, the energy gradient will be a horizontal line.

## HYDRAULIC GRADIENT

The hydraulic gradient (or hydraulic grade line) lies vertically below the energy gradient by an amount equal to the velocity head at each section. The two lines (energy gradient and hydraulic gradient) are parallel for all sections of equal cross sectional area. The vertical distance between the centre of the stream and the hydraulic gradient is the pressure head at the particular section.

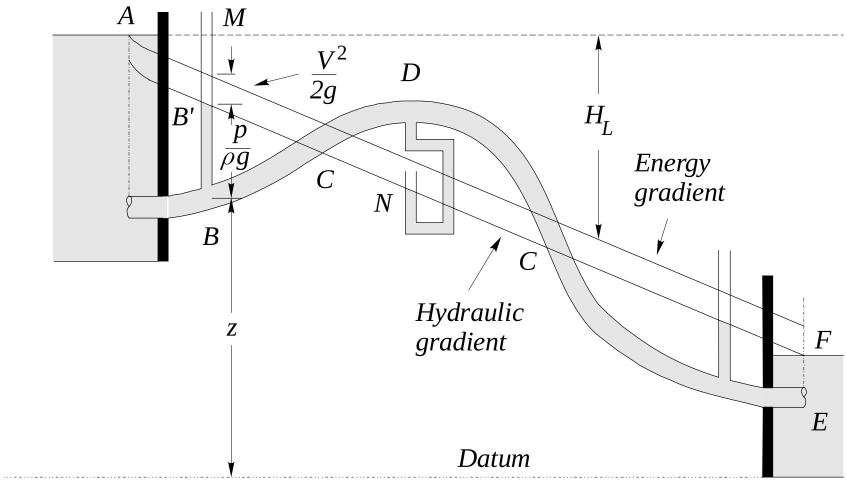



Figure 7

Consider Figure 7 which shows two bodies of fluid connected by a pipe of constant cross sectional area. If a piezometer tube were erected at *B* the fluid would rise in it to *BB′* equal to the pressure head $p/\rho g$ existing at that section. If the end of the pipe at *E* were closed so that no flow would occur, the height of the column would be *BM*. The drop from *M* to *B′* when flow occurs, is due to two factors,

(i) a portion of the pressure head has been converted into velocity head $V^2/2g$ which the fluid has at *B*, and
(ii) there has been a loss of head $H_L$ due to fluid friction between *A* and *B*.

If a series of piezometers were erected along the pipe, the fluid would rise in them to various levels. The line drawn through the summits of these imaginary series of fluid columns is the *hydraulic gradient.* The hydraulic gradient represents the pressure along the pipe, as at any point the vertical distance from the pipe centre line to the hydraulic gradient is the pressure head at that point, assuming the profile has been drawn to scale.

At *C*, this distance is zero, indicating that the pressure is atmospheric at both locations. At *D*, the pipe is above the hydraulic gradient, indicating that the pressure head is $-DN$, or a vacuum of *DN*.

If the profile of a pipeline is drawn to scale, the hydraulic gradient enables the pressure head to be determined at any section by measurement on the diagram. In addition, the hydraulic gradient shows the variation of pressure in the entire length of the pipe. The hydraulic gradient is a straight line only if the pipe is straight and of constant cross sectional area. If pipe sizes change, then there will be abrupt jumps in the hydraulic gradient.

At *A*, there is slight curvature in the hydraulic gradient and energy gradient due to streamlines converging as they enter the pipe. For long pipelines, the hydraulic gradient can be drawn from water surface to water surface without any appreciable loss in accuracy.

## FRICTION LOSSES IN PIPES

Bernoulli's theorem (the energy equation) is used in the solution of practical pipe flow problems in various branches of engineering. Flow of real fluids is more complex than flow of ideal fluids; shear forces between fluid particles and the boundary walls and between fluids themselves, result from the fluid's viscosity. Equations that might account for the flow (Euler's differential equation) have no general solution and recourse must be made to experimentation and semi-empirical methods to solve flow problems in pipes.

An important problem in determining the flow of fluids in pipes (using Bernoulli's theorem) is determining head loss $H_L$ due to the friction of fluids flowing in pipes. A formula for determining head loss is the *Darcy-Weisbach Formula*

$$\begin{aligned} H_L &= \text{friction factor } f \times \frac{\text{pipe length } L}{\text{pipe diameter } D} \times \text{velocity head } \frac{V^2}{2g} \\ &= f\left(\frac{L}{D}\right)\left(\frac{V^2}{2g}\right) \end{aligned} \tag{27}$$

where

- $f$ is a pipe friction factor
- $L$ is the length of the pipe in metres
- $D$ is the diameter of the pipe
- $V$ is the mean velocity of the flow
- $g$ is the acceleration due to gravity

The friction factor *f* in the Darcy-Weisbach formula can be evaluated using a formula developed by Colebrook (1939) known as the *Colebrook-White equation* which is regarded as reliable for all pipes (rough or smooth) (Giles 1976, Featherstone & Nalluri 1988)

*Colebrook-White equation*

$$\frac{1}{\sqrt{f}} = -2\log_{10}\left[\frac{\varepsilon}{3.7D} + \frac{2.51}{N_R\sqrt{f}}\right] \tag{28}$$

where

- $\varepsilon/D$ is the relative roughness of the pipe
- $N_R$ is the Reynolds number

In the Colebrook-White equation, $\varepsilon/D$ is the ratio of the height of protuberances $\varepsilon$ of small particles attached to the inside surface of the pipe to its diameter *D*. Studies of flow in pipes, whether *laminar* or *turbulent* (see Reynolds' experiment), have shown that there is a very thin layer of fluid attached to the sides of the pipe known as the *boundary layer* of thickness $\delta$ (generally $\delta < 0.02$ mm). In the early part of the 20th century, the German engineers Nikuradse and Prandtl (see citation below) experimented with pipes of varying diameter by coating them with sand grains of uniform diameter $\varepsilon$ and then subjecting them to flows of varying velocity. Their experiments showed that if the coating thickness $\varepsilon$ was greater than the thickness of the boundary layer $\delta$, the surface was regarded as hydraulically *rough*. And if the coating thickness $\varepsilon$, was less than the thickness of the boundary layer $\delta$, the surface was regarded as hydraulically *smooth*.

Their equations for determining the friction factor *f*, improved and modified by others are quoted in the literature as the Kármán-Prandtl equations (Giles, 1976, Featherstone & Nalluri, 1988)

*smooth* pipe surface $(\delta > 1.7\varepsilon)$
$$\frac{1}{\sqrt{f}} = 2\,\log_{10}\left(N_R\sqrt{f}\right) - 0.8 \tag{29}$$

*rough* pipe surface $(\delta < 0.08\varepsilon)$
$$\frac{1}{\sqrt{f}} = 2\,\log_{10}\left(\frac{D}{\varepsilon}\right) + 1.14 \tag{30}$$

The Colebrook-White equation (28) – the addition of equations (29) and (30) – has been found to accord with observations of flow in commercial pipes over a wide range of laminar and turbulent flows and is regarded as sufficient for determining the friction factor *f* in pipes (Featherstone & Nalluri 1988).

The derivation of the Darcy-Weisbach formula is an interesting study in the usefulness of *dimensional analysis* and is given below.

> **Prandtl**, Ludwig (b. Feb. 4, 1875, Freising, Ger.—d. Aug. 15, 1953, Göttingen), German physicist who is considered to be the father of aerodynamics.
> In 1901 **Prandtl** became professor of mechanics at the University of Hannover, where he continued his earlier efforts to provide a sound theoretical basis for fluid mechanics. He served as professor of applied mechanics at the University of Göttingen from 1904 to 1953 and there established a school of aerodynamics and hydrodynamics that achieved world renown. In 1925 he became director of the Kaiser Wilhelm (later the Max Planck) Institute for Fluid Mechanics. His discovery (1904) of the boundary layer, which adjoins the surface of a body moving in air or water, led to an understanding of skin friction drag and of the way in which streamlining reduces the drag of airplane wings and other moving bodies. His work on wing theory, which followed similar work by a British physicist, Frederick W. Lanchester, but was carried out independently, elucidated the process of airflow over airplane wings of finite span. That body of work is known as the Lanchester-**Prandtl** wing theory. **Prandtl** made decisive advances in boundary-layer and wing theories, and his work became the fundamental material of aerodynamics. He was an early pioneer in streamlining dirigibles, and his advocacy of monoplanes greatly advanced heavier-than-air aviation. He contributed the **Prandtl**-Glaubert rule for subsonic airflow to describe the compressibility effects of air at high speeds. In addition to his important advances in the theories of supersonic flow and turbulence, he made notable innovations in the design of wind tunnels and other aerodynamic equipment. He also devised a soap-film analogy for analysing the torsion forces of structures with non-circular cross sections.
>
> 

## DERIVATION OF THE DARCY-WEISBACH FORMULA

In the development of Bernoulli's theorem, the lost head for the elemental distance *dl* was shown as equation (5) $dh_L = \tau \dfrac{dl}{\rho g R}$ and for the total length of pipe *L* the lost head $H_L$ can be expressed

$$H_L = \tau \frac{L}{\rho g\, R} \tag{31}$$

where

- $\tau$ is shear stress
- $L$ is the length of the pipe in metres
- $R$ is the hydraulic radius in metres
- $\rho$ is the density in kg/m$^3$
- $g$ is the acceleration due to gravity

The shear stress $\tau$ is known to be a function of flow velocity *V*, pipe diameter *D*, density $\rho$, viscosity $\mu$ and a coefficient *K*, the relative roughness of the pipe. $K = \varepsilon/D$ where $\varepsilon$ is the size of the protuberances on the inside of the pipe. Thus, we may write

$$\tau = f\left(V, D, \rho, \mu, K\right) \tag{32}$$

The exact form of (32) is unknown but can be expressed in a general form

$$\tau = C V^a D^b \rho^c \mu^d K^e \tag{33}$$

where *C* is a dimensionless coefficient and the unknown powers of the variables are expressed as the indices *a*, *b*, *c*, *d* and *e*. Expressing (33) in terms of the dimensions of each of the variables gives (ignoring *C* since it is dimensionless)

$$\begin{aligned} ML^{-1}T^{-2} &= \left(LT^{-1}\right)^a \left(L\right)^b \left(ML^{-3}\right)^c \left(ML^{-1}T^{-1}\right)^d \left(LL^{-1}\right)^e \\ &= L^a T^{-a} L^b M^c L^{-3c} M^d L^{-d} T^{-d} \end{aligned}$$

Equating the indices of the dimensions *M*, *L* and *T* gives the following three equations

$$\begin{aligned} 1 &= c + d \\ -1 &= a + b - 3c - d \\ -2 &= -a - d \end{aligned}$$

Solving in terms of *d*, which will allow *V*, *D*, $\rho$ and $\mu$ to be combined, gives

$$\begin{aligned} a &= 2 - d \\ c &= 1 - d \\ b &= -d \end{aligned}$$

Substituting these values into (33) and simplifying gives

$$\begin{aligned} \tau &= C V^{2-d} D^{-d} \rho^{1-d} \mu^d K^e \\ &= C\left(\frac{\rho D V}{\mu}\right)^{-d} \rho K^e V^2 \end{aligned}$$

The term $\dfrac{\rho D V}{\mu}$ is the Reynolds number $N_R$. Combining the Reynolds number, the coefficient *C* and the relative roughness of the pipe $K = \varepsilon/D$ together in a coefficient $C_f$ gives

$$\tau = C_f \frac{\rho V^2}{2} \tag{34}$$

Substituting (34) into (31) gives

$$H_L = C_f \frac{LV^2}{2gR} \tag{35}$$

where $C_f = 2CK^e N_R^{-d}$ .

The hydraulic radius *R* is the cross-sectional area divided by the wetted surface perimeter

*Hydraulic Radius*

$$R = \frac{\text{cross-sectional area}}{\text{wetted surface perimeter}} = \frac{A}{P} \tag{36}$$

For a circular pipe flowing full, the cross-sectional area $A = \left(\pi D^2\right)/4$, the wetted surface perimeter is the pipe circumference $P = \pi D$ and the hydraulic radius is $R = D/4$. Substituting this expression into (35) gives

$$\begin{aligned} H_L &= C_f \frac{L}{R}\frac{V^2}{2g} \\ &= C_f \frac{4L}{D}\frac{V^2}{2g} \end{aligned}$$

Again, gathering terms into another coefficient – the friction factor *f* – gives the *Darcy-Weisbach* formula

$$H_L = f\left(\frac{L}{D}\right)\left(\frac{V^2}{2g}\right) \tag{37}$$

The friction factor *f* is a function of the Reynolds Number $N_R$, the relative roughness of the pipe $K = \varepsilon/D$ and the original coefficient *C* in (33)

$$f = 4C_f = 8C\,K^e N_R^{-d} \tag{38}$$

The friction factor *f* is given by the empirically derived *Colebrook-White* equation (28).

**MINOR LOSSES**

In addition to head loss $H_L$ due to fluid friction, other head losses occur at entrances and exits of pipes, bends in pipes, changes in pipe size, at valves and other pipe fittings. These minor losses (or local losses), due to changes in either direction or magnitude (or both) of the flow velocity *V* can be expressed in the general form

$$\text{minor head loss} = K_e \frac{V_2^2}{2g}$$

where $K_e$ is the *kinetic energy correction factor*, $V_2$ is the velocity in the downstream section of diameter $D_2$; $D_1$ is the diameter upstream. Featherstone & Nalluri (1988) have the following table of representative values of $K_e$ for changes in pipe diameters

| $\frac{D_2}{D_1}$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| $K_e$ | 0.5 | 0.45 | 0.38 | 0.28 | 0.14 | 0 |

Table 4

Note that the value $K_e = 0.5$ relates to the abrupt entry from a reservoir into a circular pipe. Minor losses of this type are called entry losses.

Other values for minor losses cited in Featherstone & Nalluri (1988) are

Head loss at abrupt enlargement $= \dfrac{V_2^2}{2g}\left(\dfrac{A_2}{A_1} - 1\right)^2$

Head loss 90º elbow $= 1.0\dfrac{V^2}{2g}$

Head loss at 90º smooth bend $= \dfrac{V^2}{2g}$

Discharge (exit) loss $= \dfrac{V^2}{2g}$

Head loss at a valve $= K_V \dfrac{V^2}{2g}$

where $K_V$ depends upon the type of valve and percentage of closure

## SOLUTION OF PIPE-FLOW PROBLEMS

In the solution of pipe flow problems using Bernoulli's theorem, three basic cases arise

| *Case* | *Given* | *To find* |
|---|---|---|
| I | $Q$ or $V$, $L$, $D$, $\mu$, $\varepsilon$ | $H_L$ |
| II | $H_L$, $L$, $D$, $\mu$, $\varepsilon$ | $Q$ or $V$ |
| III | $H_L$, $Q$ or $V$, $L$, $\mu$, $\varepsilon$ | $D$ |

Examples 1, 2 and 3 below describe the solutions to these three basic cases. In these cases the following equations, in various combinations, are used, numbered as in the text

*Reynolds Number* $$N_R = \frac{\rho D V}{\mu} \tag{13}$$

*Continuity* equation $$Q = A_1 V_1 = A_2 V_2 = \text{ constant} \tag{15}$$

*Darcy-Weisbach* equation $$H_L = f\left(\frac{L}{D}\right)\left(\frac{V^2}{2g}\right) \qquad \text{(27) and (37)}$$

*Colebrook-White* equation $$\frac{1}{\sqrt{f}} = -2\log_{10}\left[\frac{\varepsilon}{3.7D} + \frac{2.51}{N_R\sqrt{f}}\right] \tag{28}$$

To solve problems of type I, the Reynolds Number, Darcy-Weisbach and Colebrook-White equations can be combined to give an explicit equation for the flow velocity $V$

$$V = -2\sqrt{2g\,D\frac{H_L}{L}}\,\log_{10}\left[\frac{\varepsilon}{3.7D} + \frac{2.51\mu}{\rho D\sqrt{2g\,D\dfrac{H_L}{L}}}\right] \tag{39}$$

The following table for the size of protuberances $\varepsilon$ (in mm's) for pipes is useful. The values in the table have been taken from Diagram A1 in the Appendix of Giles et al (1994) where they were originally given in feet.

| **Kind of Pipe or Lining (new)** | **Values of $\varepsilon$ in mm** | |
|---|---|---|
| | **Range** | **Design Value** |
| Brass | 0.0015 | 0.0015 |
| Copper | 0.0015 | 0.0015 |
| Concrete | 0.30 – 3.0 | 1.2 |
| Cast Iron – uncoated | 0.12 – 0.61 | 0.250 |
| – asphalt dipped | 0.06 – 0.18 | 0.125 |
| – cement lined | 0.0024 | 0.0024 |
| – bituminous lined | 0.0024 | 0.0024 |
| – centrifugally spun | 0.0030 | 0.0030 |
| Galvanised Iron | 0.06 – 0.245 | 0.1520 |
| Wrought Iron | 0.03 – 0.09 | 0.06 |
| Commercial & Welded Steel | 0.03 – 0.09 | 0.06 |
| Riveted Steel | 0.9 –9.0 | 1.80 |

Table 5
(Taken from Diagram A1, Giles et al (1994), original values in feet)

**Example 1** (**Case I:** given $Q$ or $V$, $L$, $D$, $\mu$, $\varepsilon$ to find $H_L$)

Determine the lost head $H_L$ in 300 metres of new uncoated 300 mm inside diameter cast iron pipe when water at 15ºC flows at 1.5 m/s.

Use: water at 15ºC has density $\rho = 1000 \text{ kg/m}^3$ and viscosity $\mu = 1.130 \times 10^{-3}$ Pa s
$\varepsilon = 0.250$ mm (Table 5)
$g = 9.81 \text{ m/s}^2$

**Solution**

1. Compute the Reynolds Number from (13)

$$N_R = \frac{\rho DV}{\mu} = \frac{(1000)(0.3)(1.5)}{1.130 \times 10^{-3}} = 398,231$$

2. Compute the friction factor $f$ from the Colebrook-White equation (28)

$$\frac{1}{\sqrt{f}} = -2 \log_{10}\left[\frac{\varepsilon}{3.7D} + \frac{2.51}{N_R\sqrt{f}}\right]$$

Note that $f$ must be computed by iteration since it appears on the Left-Hand Side (LHS) and Right-Hand Side (RHS) of (28). A table of iterations is shown below

| Iteration $n$ | $f_n$ | RHS | $f_{n+1}$ |
|---|---|---|---|
| 1 | 0.1 | 7.221113 | 0.019177 |
| 2 | 0.019177 | 7.134898 | 0.019644 |
| 3 | 0.019644 | 7.136643 | 0.019634 |
| 4 | 0.019634 | 7.136608 | 0.019634 |

Adopt $f$ = 0.019634

3. Compute the lost head $H_L$ from the Darcy-Weisbach equation (27)

$$H_L = f\left(\frac{L}{D}\right)\left(\frac{V^2}{2g}\right) = 0.019634\left(\frac{300}{0.3}\right)\left(\frac{1.5^2}{2\times 9.81}\right) = 2.25 \text{ m}$$

**Example 2** (**Case II:** given $H_L$, $L$, $D$, $\mu$, $\varepsilon$ to find $V$)

Water at 15ºC flows through a 300 mm (inside diameter) steel pipe with a head loss $H_L$ of 6 metres in a pipe of length 300 metres. Calculate the flow.

Use: water at 15ºC has density $\rho = 1000 \text{ kg/m}^3$ and viscosity $\mu = 1.130\times 10^{-3}$ Pa s
$\varepsilon = 0.06$ mm (Table 5)
$g = 9.81 \text{ m/s}^2$

**Solutions**

**Method 1** (iterative)

1. Assume a friction factor $f$
2. Calculate $V$ from the Darcy-Weisbach equation (27)

$$H_L = f\left(\frac{L}{D}\right)\left(\frac{V^2}{2g}\right)$$

giving

$$V = \sqrt{2g\left(\frac{H_L}{f}\right)\left(\frac{D}{L}\right)}$$

3. Compute the Reynolds Number from (13)

$$N_R = \frac{\rho D V}{\mu}$$

4. Compute a "new" friction factor by iteration using the Colebrook-White equation (28)

$$\frac{1}{\sqrt{f}} = -2\log_{10}\left[\frac{\varepsilon}{3.7D} + \frac{2.51}{N_R\sqrt{f}}\right]$$

5. Repeat steps 2 to 4 until there is no change in $V$.

**Method 2** (direct)

Use equation (39), which is a combination of the Reynolds number, Darcy-Weisbach and Colebrook-White equations giving an explicit equation in $V$

$$V = -2\sqrt{2gD\frac{H_L}{L}}\log_{10}\left[\frac{\varepsilon}{3.7D} + \frac{2.51\mu}{\rho D\sqrt{2gD\frac{H_L}{L}}}\right] = 2.81 \text{ m/s}$$

**Example 3** (**Case III:** given $H_L$, $Q$ or $V$, $L$, $\mu$, $\varepsilon$ to find $D$ )

What size of new uncoated cast iron pipe, 2500 metres long will deliver 1 m$^3$/s of water at 15ºC with a drop in the hydraulic grade line of 65 metres?

Use: water at 15ºC has density $\rho = 1000\ \text{kg/m}^3$ and viscosity $\mu = 1.130 \times 10^{-3}$ Pa s
$\varepsilon$ = 0.250 mm (Table 5)
$g = 9.81\ \text{m/s}^2$

**Solution**

In this case, with $D$ unknown, there are three unknowns in the Darcy-Weisbach equation ($f$, $V$ and $D$), three unknowns in the Reynolds number ( $N_R$, $V$ and $D$ ) and the relative roughness of the pipe $\varepsilon/D$ is also unknown. Using the Continuity equation (15) to eliminate the velocity in the Darcy-Weisbach equation (27) and in the Reynolds number will simplify the solution.

Since $Q = A_1V_1 = A_2V_2 =$ constant (Continuity equation) then $Q = AV$ and area $A = \dfrac{\pi D^2}{4}$ then rearranging gives $V = \dfrac{Q}{A} = \dfrac{4Q}{\pi D^2}$ and $V^2 = \dfrac{16Q^2}{\pi^2 D^4}$. The expressions for $V$ and $V^2$ can be substituted into the Darcy-Weisbach and Reynolds number equations to give

$$H_L = f\,\frac{8LQ^2}{\pi^2 g\, D^5} \tag{40}$$

and

$$N_R = \frac{4\rho Q}{\pi\,\mu D} \tag{41}$$

The solution, using equations (40) and (41) and the Colebrook-White equation, is iterative

Note: Normally, only one or two iterations are required since standard pipe sizes are usually selected. The next larger pipe-size diameter from that given by the computation is taken.

**Iteration A**

A1. Assume a friction factor $f$ say

$$f = 0.02$$

A2. Calculate $D$ from a rearrangement of (40) using the assumed friction factor

$$D = \sqrt[5]{f\,\frac{8LQ}{\pi^2 g\, H_L}} = \sqrt[5]{0.02\frac{8(2500)(1)}{\pi^2(9.81)(65)}} = 0.576283 \text{ m} \tag{42}$$

A3. Compute the Reynolds Number from (41)

$$N_R = \frac{4\rho Q}{\pi\,\mu D} = \frac{4(1000)(1)}{\pi\left(1.130\times10^{-3}\right)(0.576283)} = 1{,}955{,}222.10594$$

A4. Compute the friction factor $f$ from the Colebrook-White equation (28)

$$\frac{1}{\sqrt{f}} = -2\,\log_{10}\left[\frac{\varepsilon}{3.7D} + \frac{2.51}{N_R\sqrt{f}}\right]$$

Note that $f$ must be computed by iteration since it appears on the Left-Hand Side (LHS) and Right-Hand Side (RHS) of (28). A table of iterations is shown below

| Iteration $n$ | $f_n$ | RHS | $f_{n+1}$ |
|---|---|---|---|
| 1 | 0.02 | 7.797024 | 0.016449 |
| 2 | 0.016449 | 7.790639 | 0.016476 |
| 3 | 0.016476 | 7.790695 | 0.016476 |

Adopt $f$ = 0.016476

Repeat the above steps until the value of the friction factor does not change. When this occurs, all equations are satisfied.

**Iteration B**

B1. Adopt the "new" friction factor $f$ from Iteration A

$$f = 0.016476$$

B2. Calculate $D$ from (42) using the new friction factor

$$D = \sqrt[5]{f \frac{8LQ}{\pi^2 g\, H_L}} = 0.554369 \text{ m}$$

B3. Compute the Reynolds Number from (41)

$$N_R = \frac{4\rho Q}{\pi\, \mu D} = 2{,}032{,}509.48955$$

B4. Compute the friction factor $f$ by iteration using the Colebrook-White equation (28).

| Iteration $n$ | $f_n$ | RHS | $f_{n+1}$ |
|---|---|---|---|
| 1 | 0.016476 | 7.762130 | 0.016597 |
| 2 | 0.016597 | 7.762363 | 0.016596 |

Adopt $f$ = 0.016596

**Iteration C**

C1. Adopt the "new" friction factor $f$ from Iteration B

$$f = 0.016596$$

C2. Calculate $D$ from (42) using the new friction factor

$$D = \sqrt[5]{f \frac{8LQ}{\pi^2 g\, H_L}} = 0.555178 \text{ m}$$

C3. Compute the Reynolds Number from (41)

$$N_R = \frac{4\rho Q}{\pi\, \mu D} = 2{,}029{,}549.61912$$

C4. Compute the friction factor $f$ by iteration using the Colebrook-White equation (28).

| Iteration $n$ | $f_n$ | RHS | $f_{n+1}$ |
|---|---|---|---|
| 1 | 0.016596 | 7.763442 | 0.016592 |
| 2 | 0.016592 | 7.763443 | 0.016592 |

Adopt $f$ = 0.016592

The friction factor has changed by 0.003524 between Iterations A and B, and only 0.000004 between Iterations B and C. It may be assumed that further iterations would produce no change. Thus we may adopt the computed pipe diameter from the last iteration $D$ = 0.555 m.

In practice, a pipe diameter computed in this manner will generally not accord with "standard" pipe sizes. In such cases, the nearest larger diameter is chosen. This will usually deliver a greater quantity of water, which could be regulated by the introduction of a valve to give the desired discharge $Q$.

**Example 4**

A uniform pipeline, 5000 m long, 200 mm in diameter and roughness size 0.03 mm, conveys water at 15ºC between two reservoirs $A$ and $B$. The difference in water level between the reservoirs is maintained at 50 m. In addition to the entry loss of $0.5\dfrac{V^2}{2g}$ a valve on the pipeline produces a head loss of $10\dfrac{V^2}{2g}$.

Determine the steady discharge $Q$ between the reservoirs using the Colebrook-White equation.

Use: water at 15ºC has density $\rho = 1000\ \text{kg/m}^3$ and viscosity $\mu = 1.130 \times 10^{-3}$ Pa s
$g = 9.81\ \text{m/s}^2$

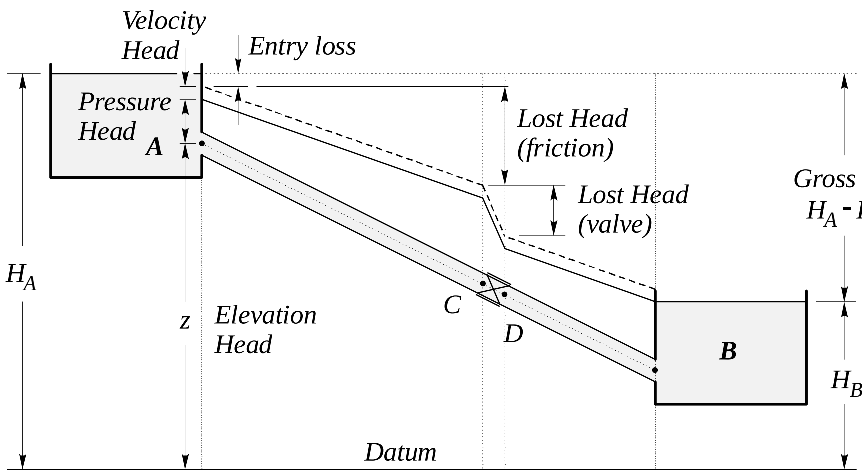



Figure 8

**Solution (Outline)**

The reservoir surface at $A$ is the hydraulic grade line and is also the energy grade line. At the square-edged entrance to the pipe the energy gradient drops by $0.5\dfrac{V^2}{2g}$ because of the entrance loss $(K_e = 0.5)$ and the hydraulic gradient drops $1.5\dfrac{V^2}{2g}$. This can be seen by applying Bernoulli's theorem (16)

$$\left(\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1\right) + H_{ADDED} - H_{LOST} - H_{EXTRACTED} = \left(\frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2\right) \tag{16}$$

between the reservoir surface and a point just downstream of the pipe entrance at $A$.

$$0 + 0 + H_A - 0.5\frac{V_A^2}{2g} = \frac{p_A}{\rho g} + \frac{V_A^2}{2g} + z_A \tag{i}$$

and
$$\frac{p_A}{\rho g} + z_A = H_A - 1.5\frac{V_A^2}{2g}$$

Repeating the procedure between *A* and *C*, a point just upstream of the valve, then between *C* and *D*, a point just downstream of the valve, then between *D* and *B*, a point just upstream of the exit, and finally between *B* and the reservoir surface gives

$$\frac{p_A}{\rho g} + \frac{V_A^2}{2g} + z_A - H_{L1} = \frac{p_C}{\rho g} + \frac{V_C^2}{2g} + z_C \qquad \text{(ii)}$$

$$\frac{p_C}{\rho g} + \frac{V_C^2}{2g} + z_C - 10\frac{V_C^2}{2g} = \frac{p_D}{\rho g} + \frac{V_D^2}{2g} + z_D \qquad \text{(iii)}$$

$$\frac{p_D}{\rho g} + \frac{V_D^2}{2g} + z_D - H_{L2} = \frac{p_B}{\rho g} + \frac{V_B^2}{2g} + z_B \qquad \text{(iv)}$$

$$\frac{p_B}{\rho g} + \frac{V_B^2}{2g} + z_B - \begin{matrix}\text{no exit loss}\\ \text{(assumed)}\end{matrix} = 0 + 0 + H_B \qquad \text{(v)}$$

$H_{L1}$ and $H_{L2}$ are head losses due to fluid friction in the length of pipe $L_1$ between *A* and the valve *C* and the length of pipe $L_2$ between the valve *D* and *B*. The total length of pipe is $L = L_1 + L_2$. Adding equations (i), (ii), (iii), (iv) and (v) gives

$$H_A - H_B = \text{Gross Head} = 0.5\frac{V_A^2}{2g} + 10\frac{V_B^2}{2g} + H_{L1} + H_{L2}$$

From the Continuity equation $Q = A_1V_1 = A_2V_2 = \cdots A_nV_n =$ constant and since the pipe diameter (and hence cross sectional area) is constant, we may say that $V_A = V_B = V$, and from the Darcy-Weisbach equation

$$\text{Gross Head} = 10.5\frac{V^2}{2g} + f\frac{L_1}{D}\frac{V^2}{2g} + f\frac{L_2}{D}\frac{V^2}{2g} = 10.5\frac{V^2}{2g} + f\frac{L}{D}\frac{V^2}{2g} \qquad \text{(vi)}$$

In equation (vi), Gross Head, *L*, *D* and *g* are known; the pipe friction factor *f* and the velocity *V* are unknown. The solution of (vi) requires iteration. Two "methods" are available

**Method 1** (iterative)

1. Assume a friction factor *f*
2. Calculate *V* from equation (vi)
3. Calculate the Reynolds number from (13)
4. Calculate "new" friction factor *f* by iteration from the Colebrook-White equation (28)
5. Repeat steps 2, 3 and 4 until the friction factor *f* remains unchanged

**Method 2** (iterative)

In most practical problems, the lost head due to friction $H_L$ will be very much larger than minor losses (entry loss, valve loss, etc). If the minor losses are ignored, we may assume that the Gross Head is the lost head due to friction and we have (see Example 2 above) the case where $H_L$, $L$, $D$, $\mu$, $\varepsilon$ are given and *V* can be determined directly. This value will be a good starting approximation of *V* in an iterative process.

1. Calculate *V* from equation (39), with an approximation of the lost head due to friction $H_L$
2. Calculate the minor losses in (vi) and then a "new" lost head
4. Repeat steps 1 and 2 until the velocity *V* remains unchanged

This method is simpler and converges more rapidly than Method 1.

**Solution** (numerical, using Method 2)

**Iteration A**

A1. Assume the lost head due to friction $H_L$ is equal to the Gross head and use equation (39), which is a combination of the Reynolds number, Darcy-Weisbach and Colebrook-White equations to solve for $V$

$$V = -2\sqrt{2g\,D\frac{H_L}{L}}\,\log_{10}\left[\frac{\varepsilon}{3.7D} + \frac{2.51\mu}{\rho D\sqrt{2g\,D\frac{H_L}{L}}}\right] = 1.565026\ \text{m/s}$$

A2. Calculate minor losses in (vi)

$$H_{MINOR} = 10.5\frac{V^2}{2g} = 1.310791$$

Calculate lost head due to friction

$$H_L = \text{Gross Head} - H_{MINOR} = 48.689209\ \text{m}$$

**Iteration B**

B1. Use the "new" lost head due to friction $H_L$ in equation (39) to compute a new $V$

$$V = 1.542932\ \text{m/s}$$

B2. Calculate minor losses in (vi)

$$H_{MINOR} = 10.5\frac{V^2}{2g} = 1.274043\ \text{m}$$

Calculate lost head due to friction

$$H_L = \text{Gross Head} - H_{MINOR} = 48.725957\ \text{m}$$

**Iteration C**

C1. Use the "new" lost head due to friction $H_L$ in equation (39) to compute a new $V$

$$V = 1.543556\ \text{m/s}$$

C2. Calculate minor losses in (vi)

$$H_{MINOR} = 10.5\frac{V^2}{2g} = 1.275072\ \text{m}$$

Calculate lost head due to friction

$$H_L = \text{Gross Head} - H_{MINOR} = 48.724928\ \text{m}$$

Since the change in $V$ between Iterations B and C has only been 0.000624 m/s we may assume the following values as correct

$$V = 1.544\ \text{m/s}$$
$$H_{MINOR} = 1.28\ \text{m}$$
$$H_L = 48.72\ \text{m}$$

The discharge $Q = AV = \dfrac{\pi D^2 V}{4} = 0.04851\ \text{m}^3/\text{s}$ (48.51 litres/second)

**PIPES IN SERIES**

Where a pipe is made up of sections of different diameters, the Continuity equation and Energy equations (Bernoulli's theorem) establish the following two simple relationships which must be satisfied:

*Discharge* $Q = Q_1 = Q_2 = \cdots = Q_n$

*Head Loss due to friction* $H_L = H_{L1} + H_{L2} + \cdots + H_{Ln}$

**Example 5**

Reservoir *A* delivers water to reservoir *B* through two uniform pipelines *AJ*:*JB* of diameters 300 mm and 200 mm respectively. Length of pipe *AJ* is 3000 m and pipe *JB* is 4000 m; the effective roughness size of both pipes is $\varepsilon = 0.015$ mm and the gross head *H* is 25 m.

Assuming an entry loss of $0.5\dfrac{V^2}{2g}$ and a head loss due to the contraction of the pipe sizes of $K_e\dfrac{V^2}{2g}$ (use $K_e = 0.28$ from Table 4) determine the discharge *Q* at *B*.

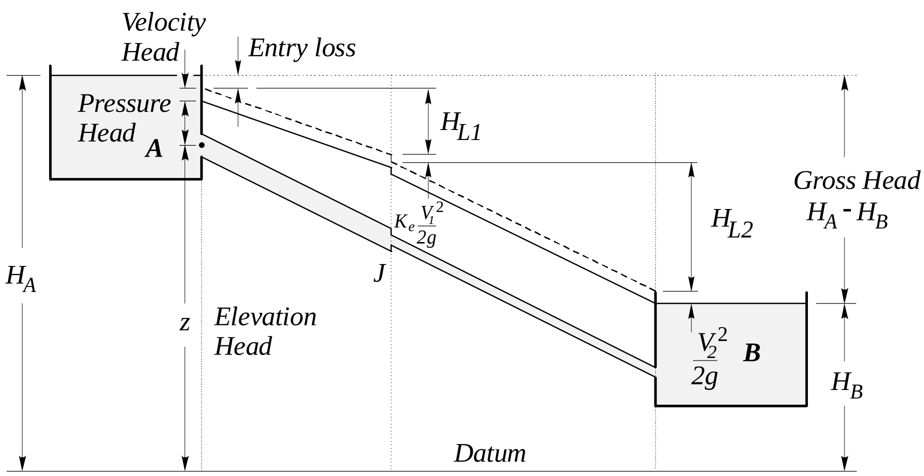



Figure 9

**Solution**

Applying Bernoulli's theorem between *A* and *B* gives

$$\text{Gross Head} = H = 0.5\frac{V_1^2}{2g} + f_1\frac{L_1}{D_1}\frac{V_1^2}{2g} + 0.28\frac{V_1^2}{2g} + f_2\frac{L_2}{D_2}\frac{V_2^2}{2g} + \frac{V_2^2}{2g} \qquad \text{(i)}$$

Since the friction factors, $f_1$ and $f_2$ are initially unknown the simplest method of solution is to input a series of trial values of the discharge $Q_1$. If $Q_1$ is assumed, the velocity $V_1$ is given by

$$V_1 = \frac{Q_1}{A_1} \qquad \text{(ii)}$$

From the continuity equation $Q_1 = Q_2 = Q$ and since $Q = AV$ the velocity $V_2$ is linked to $V_1$ by

$$V_2 = \frac{A_1}{A_2}V_1 = \left(\frac{D_1}{D_2}\right)^2 V_1 \qquad \text{(iii)}$$

The Reynolds numbers $N_R$ can be computed for the pipe velocities $V_1$ and $V_2$ from (13) and (41)

$$N_R = \frac{\rho DV}{\mu} = \frac{4\rho Q}{\pi\,\mu D} \tag{iv}$$

The friction factors $f$ can be computed by iteration from the Colebrook-White equation (28)

$$\frac{1}{\sqrt{f}} = -2\,\log_{10}\left[\frac{\varepsilon}{3.7D} + \frac{2.51}{N_R\sqrt{f}}\right] \tag{v}$$

The minor losses $H_m$ and head losses $H_{L1}$ and $H_{L2}$ can be computed for the trial value of $Q_1$.

Table 6 is part of an Excel spreadsheet, and shows trial values of $Q_1$ (discharge in litres/second) for 20 l/s to 50 l/s in intervals of 5 l/s. These values are in the left-hand column. Friction factors have been computed by iteration from the Colebrook-White equation and head losses computed. The total head loss $H_L$ is shown in the right-hand column.

Figure 10 is an Excel plot of Discharge v Total Head Loss, ie the left-hand column of Table 6 versus the right-hand column. A "trendline" has been fitted to the seven data points (an Excel graph option). Using this equation, discharge $Q$ = 36.79 l/s for a head loss of 25 m is obtained. This is the last value in Table 6.

| pipe flows | | pipe velocities | | pipe friction factors | | Head Losses | | | Total |
|---|---|---|---|---|---|---|---|---|---|
| $Q_1$ (l/s) | $Q_2$ (l/s) | $V_1$ (m/s) | $V_2$ (m/s) | $f_1$ | $f_2$ | $H_m$ | $H_{L1}$ | $H_{L2}$ | $H_L$ |
| 20 | 20 | 0.2829 | 0.6366 | 0.019338 | 0.017974 | 0.024 | 0.789 | 7.426 | 8.239 |
| 25 | 25 | 0.3537 | 0.7958 | 0.018485 | 0.017249 | 0.037 | 1.179 | 11.135 | 12.350 |
| 30 | 30 | 0.4244 | 0.9549 | 0.017835 | 0.016700 | 0.054 | 1.637 | 15.524 | 17.215 |
| 35 | 35 | 0.4951 | 1.1141 | 0.017316 | 0.016265 | 0.073 | 2.164 | 20.579 | 22.816 |
| 40 | 40 | 0.5659 | 1.2732 | 0.016888 | 0.015908 | 0.095 | 2.756 | 26.289 | 29.140 |
| 45 | 45 | 0.6366 | 1.4324 | 0.016527 | 0.015609 | 0.121 | 3.414 | 32.646 | 36.181 |
| 50 | 50 | 0.7074 | 1.5915 | 0.016216 | 0.015352 | 0.149 | 4.135 | 39.640 | 43.925 |
| | | | | | | | | | |
| 36.79 | 36.79 | 0.5205 | 1.1711 | 0.017154 | 0.016129 | 0.081 | 2.368 | 22.547 | 24.997 |

Table 6

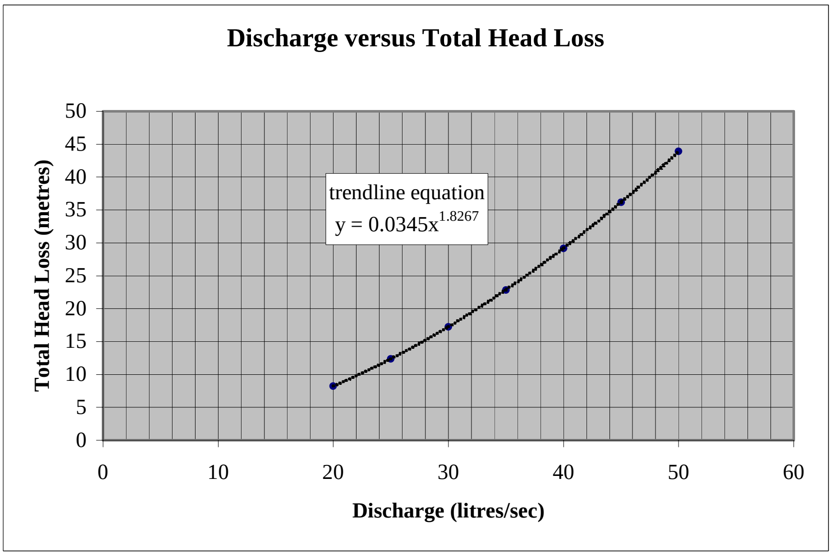



Figure 10

## PIPES IN PARALLEL

For flow through two or more parallel pipes of equal or unequal diameter, again the Continuity equation and Energy equations (Bernoulli's theorem) establish the following two simple relationships which must be satisfied:

*Discharge* $Q = Q_1 = Q_2 = \cdots = Q_n$

*Head Loss due to friction* $H_L = H_{L1} + H_{L2} + \cdots + H_{Ln}$

Solutions of flow problems for pipes in parallel, are iterative in nature. Examples are not given in these notes, but several texts mentioned in the references (eg, Featherstone & Nalluri, 1988) have worked examples demonstrating the various problems and techniques of solution.

## PIPE NETWORK ANALYSIS

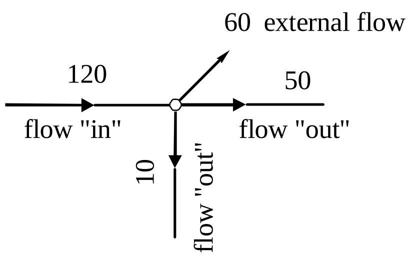
Pipe network analysis provides the basis for the design of new systems and extension of existing systems. Design criteria are that specified minimum flow rates must be attained at the outflow points of the network. The flows and pressure distributions across a network are affected by

- (a) the arrangement and sizes of the pipes and
- (b) the distribution of the outflows.

Since a change in diameter in one pipe length will affect the flow and pressure distribution everywhere, <u>pipe network analysis is not an explicit process</u>. Optimal design methods, which incorporate hydraulic analysis of the system in which pipe diameters are systematically altered, require computer software to properly analyse pipe networks.

Pipe network analysis involves the determination of pipe flow rates and pressure heads, which satisfy the *Continuity* and *Energy Conservation* conditions.

(1) *Continuity Conservation:* The algebraic sum of the flow rates (say litres/second) in the pipes meeting at a junction (or node), together with any external flows, is zero




(2) *Energy Conservation:* The algebraic sum of the head losses $H_L$ in the pipes around any closed loop formed by pipes is zero.

When the equation relating energy losses to pipe flow rates are introduced into the Continuity and Energy Conservation conditions (1) or (2), systems of non-linear equations are produced. There are <u>no direct solutions</u> of such sets of equations and iterative techniques are employed.

The earliest systematic method of pipe network analysis, due to Professor Hardy Cross (see citation below) and known by his name (the *Hardy Cross Method*), is applicable to systems in which the pipes form closed loops. Assumed pipe flow rates, complying with the continuity requirement are successively adjusted, loop by loop, until in every loop, the Energy Conservation condition is satisfied within some small, specified tolerance. The Hardy Cross Method is also known as the *Head Balance Method* or the *Loop Method.*

> **Cross**, Hardy (b. Feb. 10, 1885, Nansemond County, Va., U.S.--d. Feb. 11, 1959, Virginia Beach, Va.), U.S. professor of civil and structural engineering whose outstanding contribution was a method of calculating tendencies to produce motion (moments) in the members of a continuous framework, such as the skeleton of a building. **Cross** was appointed professor of structural engineering at the University of Illinois, Urbana, in 1930; seven years later he became full professor at Yale, retiring in 1951. Among other honours, he received the Institution of Structural Engineers' (British) gold medal. By the use of **Cross's** technique, known as the moment distribution method, or simply the **Hardy Cross** method, calculation can be carried to any required degree of accuracy by successive approximations, thus avoiding the immense labour of solving simultaneous equations that contain as many variables as there are rigid joints in a frame. He also successfully applied his mathematical methods to the solution of pipe network problems that arise in municipal water supply design; these methods have been extended to other pipe systems, such as gas pipelines.
>
> 

In the Hardy Cross Method, outflows from the system are generally assumed to occur at nodes (or junctions); this assumption results in uniform flow in the pipelines which simplifies the analysis.

REFERENCES

Colebrook, C.F., 1939. 'Turbulent flow in pipes with particular reference to the transition between smooth and rough pipe laws', *Journal of the Institute of Civil Engineers, London,* Vol. 11, pp. 133-156, Feb. 1939.

Daugherty, R.L. and A.C. Ingersoll, 1954. *Fluid Mechanics – With Engineering Applications,* McGraw-Hill Book Co., New York.

Featherstone, R.E. and C. Nalluri, 1988. *Civil Engineering Hydraulics*, BSP Professional Books, Oxford.

Giles, Ranald V., Jack B. Evett and Cheng Lui, 1994. *Theory and Problems of Fluid Mechanics and Hydraulics,* 3rd edn, Schaum's Outline Series, McGraw-Hill, New York

Giles, Ranald V., 1976. *Theory and Problems of Fluid Mechanics and Hydraulics,* 2nd edn (SI edition), Schaum's Outline Series, McGraw-Hill, New York

Mechtly, E.A., 1964. *The International System of Units: Physical Constants and Conversion Factors,* NASA SP7012. Second Revision, National Aeronautics and Space Administration, Washington, DC.

Streeter, Victor L., 1971. *Fluid Mechanics,* 5th edn, McGraw-Hill, London.