GEODESICS ON AN ELLIPSOID - BESSEL'S METHOD

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ABSTRACT

These notes provide a detailed derivation of the equations for computing the direct and inverse problems on the ellipsoid. These equations could be called *Bessel's method* and have a history dating back to F. W. Bessel's original paper on the topic titled: 'On the computation of geographical longitude and latitude from geodetic measurements', published in *Astronomische Nachrichten* (Astronomical Notes), Band 4 (Volume 4), Number 86, Speiten 241-254 (Columns 241-254), Altona 1826. The equations developed here are of a slightly different form than those presented by Bessel, but they lead directly to equations presented by Rainsford (1955) and Vincenty (1975) and the method of development closely follows that shown in *Geometric Geodesy* (Rapp, 1981). An understanding of the methods introduced in the following pages, in particular the evaluation of elliptic integrals by series expansion, will give the student an insight into other geodetic calculations.

INTRODUCTION

The direct and inverse problems on the ellipsoid are fundamental geodetic operations and can be likened to the equivalent operations of plane surveying; radiations (computing coordinates of points given bearings and distances radiating from a point of known coordinates) and joins (computing bearings and distances between points having known coordinates). In plane surveying, the coordinates are 2-Dimensional (2D) rectangular coordinates, usually designated East and North and the reference surface is a plane, either a local horizontal plane or a map projection plane.

In geodesy, the reference surface is an ellipsoid, the coordinates are latitudes and longitudes, directions are known as azimuths and distances are geodesic arc lengths.

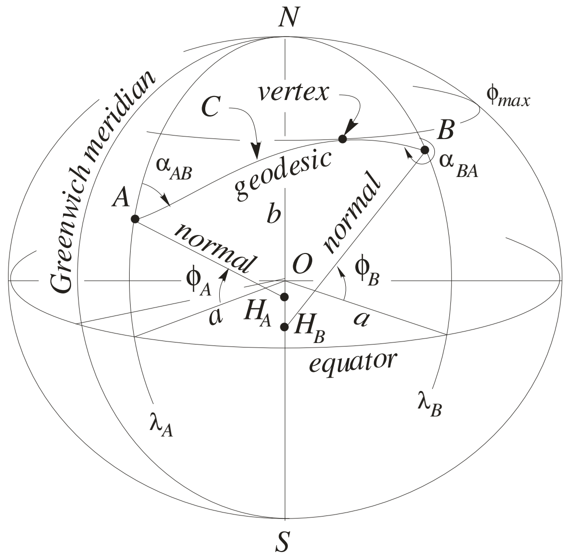


Fig. 1: Geodesic curve on an ellipsoid

The geodesic is a unique curve on the surface of an ellipsoid defining the shortest distance between two points. A geodesic will cut meridians of an ellipsoid at angles α, known as azimuths and measured clockwise from north 0^{o} to 360°. Figure 1 shows a geodesic curve C between two points A (ϕ_A, λ_A) and B (ϕ_B, λ_B) on an ellipsoid. ϕ, λ are latitude and longitude respectively and an ellipsoid is taken to mean a surface of revolution created by rotating an ellipse about its minor axis, *NS*. The geodesic curve C of length s from A to B has a forward azimuth α_{AB} measured at A and a reverse azimuth α_{BA} measured at B.

The direct problem on an ellipsoid is: given latitude and longitude of A and azimuth α_{AB} and geodesic distance s, compute the latitude and longitude of B and the reverse azimuth α_{BA}.

The inverse problem is: given the latitudes and longitudes of A and B, compute the forward and reverse azimuths α_{AB}, α_{BA} and the geodesic distance s.

Formula for computing geodesic distances and longitude differences between points connected by geodesic curves are derived from solutions of elliptic integrals and in Bessel's method, these elliptic integrals are solutions of equations connecting differential elements on the ellipsoid with corresponding elements on an auxiliary sphere. These integrals do not have direct solutions but instead are solved by expanding them into trigonometric series and integrating term-by-term. Hence the equations developed here are series-type formula truncated at a certain number of terms that give millimetre precision for any length of line not exceeding 180^{o} in longitude difference.

These formulae were first developed by Bessel (1826) who gave examples of their use using 10-place logarithms. A similar development is given in *Handbuch der Vermessungskunde* (Handbook of Geodesy) by Jordan/Eggert/Kneissl, 1958.

The British geodesist Hume Rainsford (1955) presented equations and computational methods for the direct and inverse problems that were applicable to machine computation of the mid 20th century. His formulae and iterative method for the inverse case were similar to Bessel's, although his equations contained different ellipsoid constants and geodesic curve parameters, but his equations for the direct case, different from Bessel's, were based on a direct technique given by G.T. McCaw (1932-33) which avoided iteration. For many years Rainsford's (and McCaw's) equations were the standard method of solving the direct and inverse problems on the ellipsoid when millimetre precision was required, even though they involved iteration and lengthy long-hand machine computation. In 1975, Thaddeus (Tom) Vincenty (1975-76), then working for the Geodetic Survey Squadron of the US Air Force, presented a set of compact nested equations that could be conveniently programmed on the then new electronic computers. His method and equations were based on Rainsford's inverse method combined with techniques developed by Professor Richard H. Rapp of the Ohio State University. Vincenty's equations for the direct and inverse problems on the ellipsoid have become a standard method of solution.

Vincenty's method (following on from Rainsford and Bessel) is not the only method of solving the direct and inverse problems on the ellipsoid. There are other techniques; some involving elegant solutions to integrals using recurrence relationships, e.g., Pittman (1986) and others using numerical integration techniques, e.g., Kivioja (1971) and Jank & Kivioja (1980).

In this paper, we present a development following Rapp (1981) and based on Bessel's method which yields Rainsford's equations for the inverse problem. We then show how Vincenty's equations are obtained and how they are used in practice. In addition, certain ellipsoid relationships are given, the mathematical definition of a geodesic is discussed and the characteristic equation of a geodesic derived. The characteristic equation of a geodesic is fundamental to all solutions of the direct and inverse problems on the ellipsoid.

SOME ELLIPSOID RELATIONSHIPS

The size and shape of an ellipsoid is defined by one of three pairs of parameters: (i) a, b where a and b are the semi-major and semi-minor axes lengths of an ellipsoid respectively, or (ii) a, f where f is the flattening of an ellipsoid, or (iii) a, e^2 where e^2 is the square of the first eccentricity of an ellipsoid. The ellipsoid parameters a, b, f, e^2 are related by the following equations

$$f = \frac{a-b}{a} = 1 - \frac{b}{a} \tag{1}$$

$$b = a(1-f) \tag{2}$$

$$e^2 = \frac{a^2 - b^2}{a^2} = 1 - \frac{b^2}{a^2} = f(2-f) \tag{3}$$

$$1 - e^2 = \frac{b^2}{a^2} = 1 - f(2-f) = (1-f)^2 \tag{4}$$

The second eccentricity e' of an ellipsoid is also of use and

$$e'^2 = \frac{a^2 - b^2}{b^2} = \frac{a^2}{b^2} - 1 = \frac{e^2}{1-e^2} = \frac{f(2-f)}{(1-f)^2} \tag{5}$$

$$e^2 = \frac{e'^2}{1+e'^2} \tag{6}$$

In Figure 1 the normals to the surface at A and B intersect the rotational axis of the ellipsoid (NS line) at H_A and H_B making angles ϕ_A, ϕ_B with the equatorial plane of the ellipsoid. These are the latitudes of A and B respectively. The longitudes λ_A, λ_B are the angles between the Greenwich meridian plane (a reference plane) and the meridian planes $ONAH_A$ and $ONBH_B$ containing the normals through A and B. ϕ and λ are curvilinear coordinates and meridians of longitude (curves of constant λ) and parallels of latitude (curves of constant ϕ) are parametric curves on the ellipsoidal surface.

For a general point P on the surface of the ellipsoid (see Fig. 2), planes containing the normal to the ellipsoid intersect the surface creating elliptical sections known as normal sections. Amongst the infinite number of possible normal sections at a point, each having a certain radius of curvature, two are of interest: (i) the meridian section, containing the axis of revolution of the ellipsoid and having the least radius of curvature, denoted by ρ, and (ii) the prime vertical section, perpendicular to the meridian plane and having the greatest radius of curvature, denoted by ν.

$$\rho = \frac{a\left(1-e^2\right)}{\left(1-e^2\sin^2\phi\right)^{\frac{3}{2}}} = \frac{a\left(1-e^2\right)}{W^3} \tag{7}$$

$$\nu = \frac{a}{\left(1-e^2\sin^2\phi\right)^{\frac{1}{2}}} = \frac{a}{W} \tag{8}$$

$$W^2 = 1 - e^2\sin^2\phi \tag{9}$$

The centres of the radii of curvature of the prime vertical sections at A and B are at H_A and H_B, where H_A and H_B are the intersections of the normals at A and B and the rotational axis, and $\nu_A = PH_A$, $\nu_B = PH_B$. The centres of the radii of curvature of the meridian sections at A and B lie on the normals between P and H_A and P and H_B.

Alternative equations for the radii of curvature ρ and ν are given by

$$\rho = \frac{a^2}{b\left(1+e'^2\cos^2\phi\right)^{\frac{3}{2}}} = \frac{c}{V^3} \tag{10}$$

$$\nu = \frac{a^2}{b\left(1+e'^2\cos^2\phi\right)^{\frac{1}{2}}} = \frac{c}{V} \tag{11}$$

$$c = \frac{a^2}{b} = \frac{a}{1-f} \tag{12}$$

$$V^2 = 1 + e'^2\cos^2\phi \tag{13}$$

and c is the polar radius of curvature of the ellipsoid.

The latitude functions W and V are related as follows

$$W^2 = \frac{V^2}{1+e'^2} \quad \text{and} \quad W = \frac{V}{\left(1+e'^2\right)^{\frac{1}{2}}} = \frac{b}{a}V \tag{14}$$

Points on the ellipsoidal surface have curvilinear coordinates ϕ, λ and Cartesian coordinates x,y,z where the x-z plane is the Greenwich meridian plane, the x-y plane is the equatorial plane and the y-z plane is a meridian plane 90° east of the Greenwich meridian plane. Cartesian and curvilinear coordinates are related by

$$\begin{aligned} x &= \nu\cos\phi\cos\lambda \\ y &= \nu\cos\phi\cos\lambda \\ z &= \nu\left(1-e^2\right)\sin\phi \end{aligned} \tag{15}$$

Note that $\nu\left(1-e^2\right)$ is the distance along the normal from a point on the surface to the point where the normal cuts the equatorial plane.

THE DIFFERENTIAL RECTANGLE ON THE ELLIPSOID

The derivation of equations relating to the geodesic requires an understanding of the connection between differentially small quantities on the surface of the ellipsoid. These relationships can be derived from the differential rectangle, with diagonal PQ in Figure 2 which shows P and Q on an ellipsoid, having semi-major axis a, flattening f, separated by differential changes in latitude $d\phi$ and longitude $d\lambda$. P and Q are connected by a curve of length ds making an angle α (the azimuth) with the meridian through P. The meridians λ and $\lambda + d\lambda$, and the parallels ϕ and $\phi + d\phi$ form a differential rectangle on the surface of the ellipsoid. The differential distances dp along the parallel ϕ and dm along the meridian λ are

$$dp = w\,d\lambda = \nu \cos\phi\, d\lambda \tag{16}$$

$$dm = \rho\, d\phi \tag{17}$$

where ρ and ν are radii of curvature in the meridian and prime vertical planes respectively and $w = \nu\cos\phi$ is the perpendicular distance from the rotational axis.

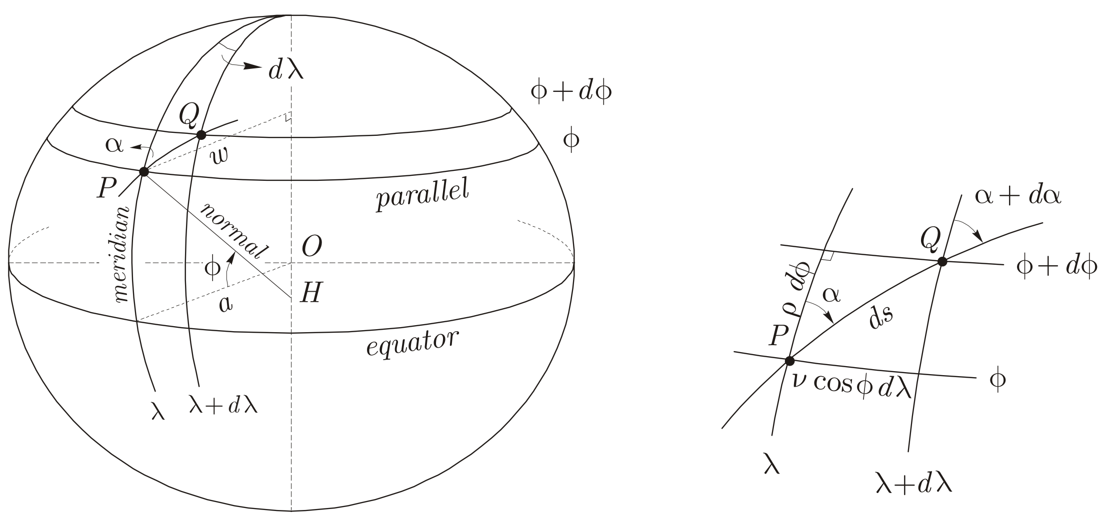


Figure 2: Differential rectangle on the ellipsoid

The differential distance ds is given by

$$ds = \sqrt{dp^2 + dm^2} = \sqrt{\left(\nu\cos\phi\, d\lambda\right)^2 + \left(\rho\, d\phi\right)^2} \tag{18}$$

and so

$$\frac{ds}{d\phi} = \sqrt{\nu^2 \cos^2\phi \left(\frac{d\lambda}{d\phi}\right)^2 + \rho^2} \qquad \text{or} \qquad \frac{ds}{d\lambda} = \sqrt{\nu^2 \cos^2\phi + \rho^2 \left(\frac{d\phi}{d\lambda}\right)^2}$$

while

$$\sin\alpha = \nu\cos\phi \frac{d\lambda}{ds} \qquad \text{and} \qquad \cos\alpha = \rho\frac{d\phi}{ds} \tag{19}$$

MATHEMATICAL DEFINITION OF A GEODESIC

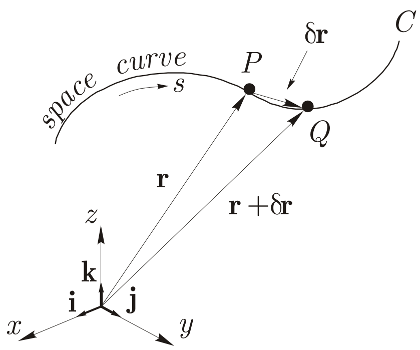


Figure 3: Space curve C

A geodesic can be defined mathematically by considering concepts associated with space curves and surfaces. A space curve may be defined as the locus of the terminal points P of a position vector $\mathbf{r}(t)$ defined by a single scalar parameter t,

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k} \tag{20}$$

$\mathbf{i}, \mathbf{j}, \mathbf{k}$ are fixed unit Cartesian vectors in the directions of the x,y,z coordinate axes. As the parameter t varies the terminal point P of the vector sweeps out the space curve C.

Let s be the arc-length of C measured from some convenient point on C, so that $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$ or $s = \int \sqrt{\frac{d\mathbf{r}}{dt} \bullet \frac{d\mathbf{r}}{dt}}\, dt$. Hence s is a function of t and x,y,z are functions of s. Let Q, a small distance δs along the curve from P, have a position vector $\mathbf{r} + \delta\mathbf{r}$. Then $\delta\mathbf{r} = \overrightarrow{PQ}$ and $|\delta\mathbf{r}| \simeq |\delta s|$. Both when δs is positive or negative $\frac{\delta\mathbf{r}}{\delta s}$ approximates to a unit vector in the direction of s increasing and $\frac{d\mathbf{r}}{ds}$ is a tangent vector of unit length denoted by $\hat{\mathbf{t}}$; hence

$$\hat{\mathbf{t}} = \frac{d\mathbf{r}}{ds} = \frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} + \frac{dz}{ds}\mathbf{k} \tag{21}$$

Since $\hat{\mathbf{t}}$ is a unit vector then $\hat{\mathbf{t}} \bullet \hat{\mathbf{t}} = 1$ and differentiating with respect to s leads to $\hat{\mathbf{t}} \bullet \frac{d\hat{\mathbf{t}}}{ds} = 0$ from which we deduce that $\frac{d\hat{\mathbf{t}}}{ds}$ is orthogonal to $\hat{\mathbf{t}}$ and write

$$\frac{d\hat{\mathbf{t}}}{ds} = \kappa\hat{\mathbf{n}}\,, \qquad \kappa > 0 \tag{22}$$

$\dfrac{d\hat{\mathbf{t}}}{ds}$ is called the curvature vector $\mathbf{k}$, $\hat{\mathbf{n}}$ is a unit vector called the principal normal vector, κ the curvature and $\dfrac{1}{\kappa} = \rho$ is the radius of curvature. The circle through P, tangent to $\hat{\mathbf{t}}$ with this radius ρ is called the osculating circle. Also $\hat{\mathbf{n}} \bullet \dfrac{d\hat{\mathbf{t}}}{ds} = \kappa$; i.e., $\hat{\mathbf{n}}$ is the unit vector in the direction of $\mathbf{k}$. Let $\hat{\mathbf{b}}$ be a third unit vector defined by the vector cross product

$$\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}} \tag{23}$$

thus $\hat{\mathbf{t}}, \hat{\mathbf{b}}$ and $\hat{\mathbf{n}}$ form a right-handed triad. Differentiating equation (23) with respect to s gives

$$\frac{d\hat{\mathbf{b}}}{ds} = \frac{d}{ds}\left(\hat{\mathbf{t}} \times \hat{\mathbf{n}}\right) = \frac{d\hat{\mathbf{t}}}{ds} \times \hat{\mathbf{n}} + \hat{\mathbf{t}} \times \frac{d\hat{\mathbf{n}}}{ds} = \kappa \hat{\mathbf{n}} \times \hat{\mathbf{n}} + \hat{\mathbf{t}} \times \frac{d\hat{\mathbf{n}}}{ds} = \hat{\mathbf{t}} \times \frac{d\hat{\mathbf{n}}}{ds}$$

then

$$\hat{\mathbf{t}} \bullet \frac{d\hat{\mathbf{b}}}{ds} = \hat{\mathbf{t}} \bullet \left(\hat{\mathbf{t}} \times \frac{d\hat{\mathbf{n}}}{ds}\right) = \frac{d\hat{\mathbf{n}}}{ds} \bullet \left(\hat{\mathbf{t}} \times \hat{\mathbf{t}}\right) = 0$$

so that $\dfrac{d\hat{\mathbf{b}}}{ds}$ is orthogonal to $\hat{\mathbf{t}}$. But from $\hat{\mathbf{b}} \bullet \hat{\mathbf{b}} = 1$ it follows that $\hat{\mathbf{b}} \bullet \dfrac{d\hat{\mathbf{b}}}{ds} = 0$ so that $\dfrac{d\hat{\mathbf{b}}}{ds}$ is orthogonal to $\hat{\mathbf{b}}$ and so is in the plane containing $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$. Since $\dfrac{d\hat{\mathbf{b}}}{ds}$ is in the plane of $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$ and is orthogonal to $\hat{\mathbf{t}}$, it must be parallel to $\hat{\mathbf{n}}$. The direction of $\dfrac{d\hat{\mathbf{b}}}{ds}$ is opposite $\hat{\mathbf{n}}$ as it must be to ensure the cross product $\dfrac{d\hat{\mathbf{b}}}{ds} \times \hat{\mathbf{t}}$ is in the direction of $\hat{\mathbf{b}}$. Hence

$$\frac{d\hat{\mathbf{b}}}{ds} = -\tau \hat{\mathbf{n}}, \qquad \tau > 0 \tag{24}$$

We call $\hat{\mathbf{b}}$ the unit binormal vector, τ the torsion, and $\dfrac{1}{\tau}$ the radius of torsion. $\hat{\mathbf{t}}$, $\hat{\mathbf{n}}$ and $\hat{\mathbf{b}}$ form a right-handed set of orthogonal unit vectors along a space curve.

The plane containing $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$ is the osculating plane, the plane containing $\hat{\mathbf{n}}$ and $\hat{\mathbf{b}}$ is the normal plane and the plane containing $\hat{\mathbf{t}}$ and $\hat{\mathbf{b}}$ is the rectifying plane. Figure 4 shows these orthogonal unit vectors for a space curve.

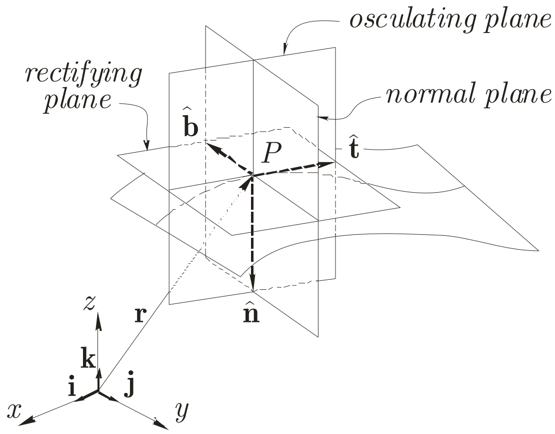


Figure 4: The tangent $\hat{\mathbf{t}}$, principal normal $\hat{\mathbf{n}}$ and binormal $\hat{\mathbf{b}}$ to a space curve

Also $\hat{\mathbf{n}} = \hat{\mathbf{b}} \times \hat{\mathbf{t}}$ and the derivative with respect to s is

$$\frac{d\hat{\mathbf{n}}}{ds} = \frac{d}{ds}\left(\hat{\mathbf{b}} \times \hat{\mathbf{t}}\right) = \frac{d\hat{\mathbf{b}}}{ds} \times \hat{\mathbf{t}} + \hat{\mathbf{b}} \times \frac{d\hat{\mathbf{t}}}{ds} = -\tau\hat{\mathbf{n}} \times \hat{\mathbf{t}} + \hat{\mathbf{b}} \times \kappa\hat{\mathbf{n}} = \tau\hat{\mathbf{b}} - \kappa\hat{\mathbf{t}} \tag{25}$$

Equations (22), (24) and (25) are known as the Frenet-Serret formulae.

$$\begin{aligned} \frac{d\hat{\mathbf{t}}}{ds} &= \kappa\hat{\mathbf{n}} \\ \frac{d\hat{\mathbf{b}}}{ds} &= -\tau\hat{\mathbf{n}} \\ \frac{d\hat{\mathbf{n}}}{ds} &= \tau\hat{\mathbf{b}} - \kappa\hat{\mathbf{t}} \end{aligned} \tag{26}$$

These formulae, derived independently by the French mathematicians Jean-Frédéric Frenet (1816–1900) and Joseph Alfred Serret (1819–1885) describe the dynamics of a point moving along a continuous and differentiable curve in three-dimensional space. Frenet derived these formulae in his doctoral thesis at the University of Toulouse; the latter part of which was published as 'Sur quelques propriétés des courbes à double courbure', (Some properties of curves with double curvature) in the *Journal de mathématiques pures et appliqués* (Journal of pure and applied mathematics), Vol. 17, pp.437-447, 1852. Frenet also explained their use in a paper titled 'Théorèmes sur les courbes gauches' (Theorems on awkward curves) published in 1853. Serret presented an independent derivation of the same formulae in 'Sur quelques formules relatives à la théorie des courbes à double courbure' (Some formulas relating to the theory of curves with double curvature) published in the *J. de Math.* Vol. 16, pp.241-254, 1851 (DSB 1971).

A geodesic may be defined in the following manner:

A curve drawn on a surface so that its osculating plane at any point contains the normal to the surface at the point is a **geodesic**. It follows that the principal normal at any point on the curve is the normal to the surface and the geodesic is the shortest distance between two points on a surface.

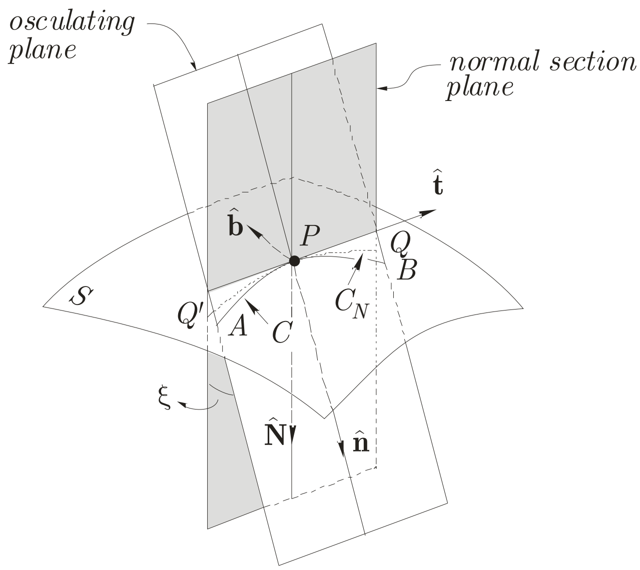


Figure 5: The osculating plane of a geodesic

To understand that the geodesic is the shortest path on a surface requires the use of *Meusnier's theorem,* a fundamental theorem on the nature of surfaces. Jean-Baptiste-Marie-Charles Meusnier de la Place (1754 - 1793) was a French mathematician who, in a paper titled *Mémoire sur la corbure des surfaces* (Memoir on the curvature of surfaces), read at the Paris Academy of Sciences in 1776 and published in 1785, derived his theorem on the curvature, at a point of a surface, of plane sections with a common tangent (DSB 1971). His theorem can be stated as:

> Between the radius ρ of the osculating circle of a plane slice C and the radius ρ_N of the osculating circle of a normal slice C_N, where both slices have the same tangent at P, there exists the relation
>
> $$\rho = \rho_N \cos \xi$$
>
> where ξ is the angle between the unit principal normals $\hat{\mathbf{n}}$ and $\hat{\mathbf{N}}$ to curves C and C_N at *P.*

In Figure 5, an infinitesimal arc PQ of a geodesic coincides with the section of the surface S by a plane containing $\hat{\mathbf{t}}$ and $\hat{\mathbf{N}}$ where $\hat{\mathbf{N}}$ is a unit vector normal to the surface at P.

This plane is a normal section plane through P and by Meusnier's theorem, the geodesic arc PQ is the arc of least curvature through P and Q; or the shortest distance on the surface between two adjacent points P and Q is along the geodesic through the points. In Figure 5, curve C (the arc APB) will have a smaller radius of curvature at P than curve C_N the normal section arc $Q'PQ$.

THE CHARACTERISTIC EQUATION OF A GEODESIC USING DIRECTION COSINES

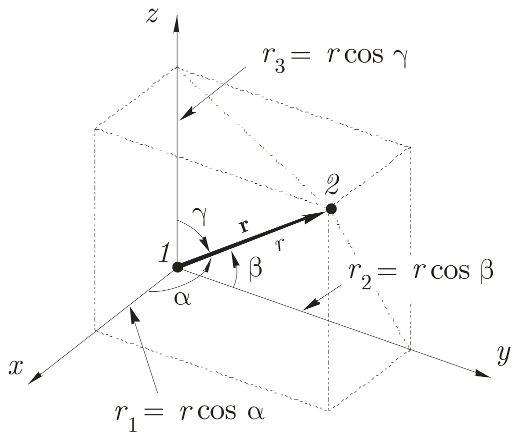


Figure 6: Direction cosines

The characteristic equation of a geodesic can be derived from relationships between the direction cosines of the principal normal to a curve and the normal to the surface. In Figure 6, $\mathbf{r} = r_1\,\mathbf{i} + r_2\,\mathbf{j} + r_3\,\mathbf{k}$ is a vector between two points in space having a magnitude $r = \sqrt{r_1^2 + r_2^2 + r_3^2}$. $\hat{\mathbf{r}} = \dfrac{\mathbf{r}}{r} = \dfrac{r_1}{r}\mathbf{i} + \dfrac{r_2}{r}\mathbf{j} + \dfrac{r_3}{r}\mathbf{k}$ is a unit vector and the scalar components $\dfrac{r_1}{r} = \cos\alpha$, $\dfrac{r_2}{r} = \cos\beta$ and $\dfrac{r_3}{r} = \cos\gamma$. $l = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$ are known as direction cosines and the unit vector can be expressed as $\hat{\mathbf{r}} = l\,\mathbf{i} + m\,\mathbf{j} + n\,\mathbf{k}$.

From equations (20) and (22) we may write the unit principal normal vector $\hat{\mathbf{n}}$ of a curve C as

$$\hat{\mathbf{n}} = \frac{1}{\kappa}\frac{d^2\mathbf{r}}{ds^2} = \frac{x''}{\kappa}\mathbf{i} + \frac{y''}{\kappa}\mathbf{j} + \frac{z''}{\kappa}\mathbf{k} = \rho\,x''\,\mathbf{i} + \rho\,y''\,\mathbf{j} + \rho\,z''\,\mathbf{k} \tag{27}$$

where $\rho = \dfrac{1}{\kappa}$. $x' = \dfrac{dx}{ds}$ and $x'' = \dfrac{d^2x}{ds^2}$ are first and second derivatives with respect to arc length respectively and similarly for y', z', y'', z''.

The unit normal $\hat{\mathbf{N}}$ to the ellipsoid surface is $\hat{\mathbf{N}} = \dfrac{N_1}{\nu}\mathbf{i} + \dfrac{N_2}{\nu}\mathbf{j} + \dfrac{N_3}{\nu}\mathbf{k}$ where N_1, N_2, N_3 are the Cartesian components of the normal vector $\overrightarrow{PH}$ and ν is the magnitude. $\dfrac{N_1}{\nu} = \cos\alpha$, $\dfrac{N_2}{\nu} = \cos\beta$ and $\dfrac{N_3}{\nu} = \cos\gamma$ are the direction cosines *l*, *m* and *n*. Note that the direction of the unit normal to the ellipsoid is towards the centre of curvature of normal sections passing through *P*.

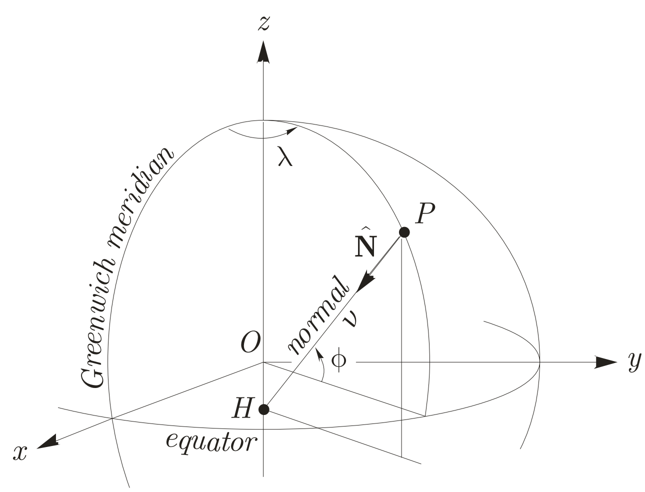


Figure 7: The unit normal $\hat{\mathbf{N}}$ to the ellipsoid

The unit normal $\hat{\mathbf{N}}$ to the ellipsoid surface is given by

$$\hat{\mathbf{N}} = \left(\frac{-x}{\nu}\right)\mathbf{i} + \left(\frac{-y}{\nu}\right)\mathbf{j} + \left(\frac{-\nu\sin\phi}{\nu}\right)\mathbf{k} \tag{28}$$

To ensure that the curve C is a geodesic, i.e., the unit principal normal $\hat{\mathbf{n}}$ to the curve must be coincident with the unit normal $\hat{\mathbf{N}}$ to the surface, the coefficients in equations (27) and (28) must be equal, thus

$$\frac{-x}{\nu} = \rho\, x''; \quad \frac{-y}{\nu} = \rho\, y''; \quad \frac{-\nu\sin\phi}{\nu} = \rho\, z''$$

This leads to

$$\frac{\rho\, x''}{x/\nu} = \frac{\rho\, y''}{y/\nu} = \frac{\rho\, z''}{\nu\sin\phi/\nu} \tag{29}$$

From the first two equations of (29) we have $\rho\, x''\,\dfrac{\nu}{x} = \rho\, y''\,\dfrac{\nu}{y}$ giving the second-order differential equation (provided $\rho\nu \neq 0$)

$$xy'' - yx'' = 0$$

which can be written as $\dfrac{d}{ds}\left(xy' - yx'\right) = 0$ and so a first integral is

$$xy' - yx' = C \tag{30}$$

where C is an arbitrary constant. Now, from equations (15), x and y are functions of ϕ and λ, and the chain rule gives

$$\begin{aligned} x' &= \frac{\partial x}{\partial \phi}\frac{d\phi}{ds} + \frac{\partial x}{\partial \lambda}\frac{d\lambda}{ds} \\ y' &= \frac{\partial y}{\partial \phi}\frac{d\phi}{ds} + \frac{\partial y}{\partial \lambda}\frac{d\lambda}{ds} \end{aligned} \tag{31}$$

Differentiating the first two equations of (15) with respect to ϕ, bearing in mind that ν is a function of ϕ gives

$$\begin{aligned} \frac{\partial x}{\partial \phi} &= -\nu\sin\phi\cos\lambda + \cos\phi\cos\lambda\frac{d\nu}{d\phi} \\ &= -\nu\sin\phi\cos\lambda + \cos\phi\cos\lambda\frac{ae^2\sin\phi\cos\phi}{\left(1 - e^2\sin^2\phi\right)^{\frac{3}{2}}} \end{aligned}$$

Using equation (8) and simplifying yields

$$\frac{\partial x}{\partial \phi} = -\rho\sin\phi\cos\lambda$$

Similarly

$$\frac{\partial y}{\partial \phi} = -\nu\sin\phi\sin\lambda + \cos\phi\sin\lambda\frac{d\nu}{d\phi} = -\rho\sin\phi\sin\lambda$$

Placing these results, together with the derivatives $\dfrac{\partial x}{\partial \lambda}$ and $\dfrac{\partial y}{\partial \lambda}$ into equations (31) gives

$$\begin{aligned} x' &= -\rho\sin\phi\cos\lambda\frac{d\phi}{ds} - \nu\cos\phi\sin\lambda\frac{d\lambda}{ds} \\ y' &= -\rho\sin\phi\sin\lambda\frac{d\phi}{ds} + \nu\cos\phi\cos\lambda\frac{d\lambda}{ds} \end{aligned}$$

These values of x' and y' together with x and y from equations (15) substituted into equation (30) gives

$$\nu^2\cos^2\phi\,\frac{d\lambda}{ds} = C \tag{32}$$

which can be re-arranged to give an expression for the differential distance ds

$$ds = \frac{\nu^2 \cos^2 \phi}{C} d\lambda$$

ds is also given by equation (18) and equating the two and simplifying gives the differential equation of the geodesic (Thomas 1952)

$$C^2 \rho^2 d\phi^2 + \nu^2 \cos^2 \phi \left(C^2 - \nu^2 \cos^2 \phi \right) d\lambda^2 = 0 \tag{33}$$

From equation (19), $\sin \alpha = \nu \cos \phi \dfrac{d\lambda}{ds}$ and substituting into equation (32) gives the characteristic equation of the geodesic on the ellipsoid

$$\boxed{\nu \cos \phi \sin \alpha = C} \tag{34}$$

Equation (34) is also known as *Clairaut's equation* in honour of the French mathematical physicist Alexis-Claude Clairaut (1713-1765). In a paper in 1733 titled *Détermination géométrique de la perpendiculaire à la méridienne, tracée par M. Cassini, avec plusieurs methods d'en tirer la grandeur et la figure de la terre* (Geometric determination of the perpendicular to the meridian, traced by Mr. Cassini, … on the figure of the Earth.) Clairaut made an elegant study of the geodesics of quadrics of rotation. It included the property already pointed out by Johann Bernoulli: the osculating plane of the geodesic is normal to the surface (DSB 1971).

The characteristic equation of a geodesic shows that the geodesic on the ellipsoid has the intrinsic property that at any point, the product of the radius w of the parallel of latitude and the sine of the azimuth of the geodesic at that point is a constant. This means that as $w = \nu \cos \phi$ decreases in higher latitudes, in both the northern and southern hemispheres, $\sin \alpha$ increases until it reaches a maximum or minimum of ± 1, noting that the azimuth of a geodesic at a point will vary between 0° and 180° if the point is moving along a geodesic in an easterly direction or between 180° and 360° if the point is moving along a geodesic in a westerly direction. At the point when $\sin \alpha = \pm 1$, which is known as the *vertex*, w is a minimum and the latitude ϕ will be a maximum value ϕ_0, known as the geodetic latitude of the vertex. Thus the geodesic oscillates over the surface of the ellipsoid between two parallels of latitude having a maximum in the northern and southern hemispheres and crossing the equator at nodes; but as we will demonstrate later, due to the eccentricity of the ellipsoid the geodesic will not repeat after a complete cycle.

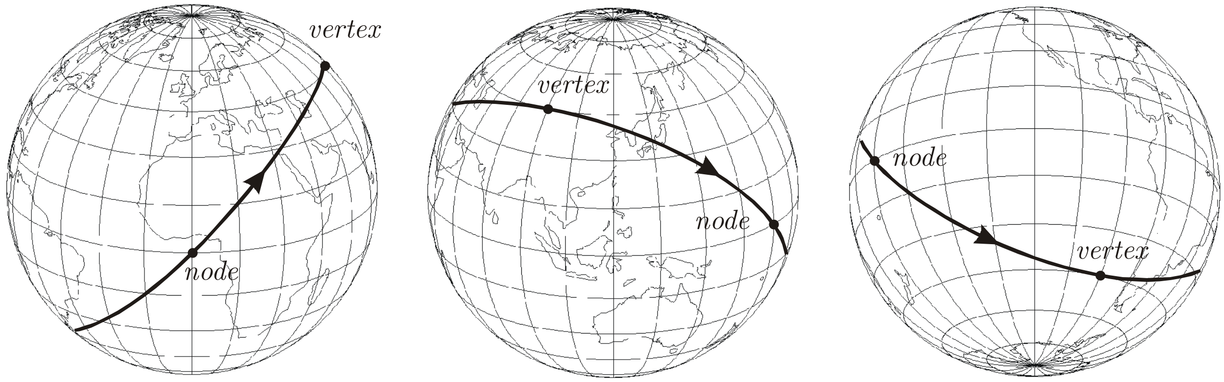


Figure 8a Figure 8b Figure 8c

Figure 8: A single cycle of a geodesic on the Earth

Figures 8a, 8b and 8c show a single cycle of a geodesic on the Earth. This particular geodesic reaches maximum latitudes of approximately $\pm 45^{\circ}$ and has an azimuth of approximately 45° as it crosses the equator at longitude 0°.

Figure 9 shows a schematic representation of the oscillation of a geodesic on an ellipsoid. P is a point on a geodesic that crosses the equator at A, heading in a north-easterly direction reaching a maximum northerly latitude $\phi_{\max}$ at the vertex P_0 (north), then descends in a south-easterly direction crossing the equator at B, reaching a maximum southerly latitude $\phi_{\min}$ at P_0 (south), then ascends in a north-easterly direction crossing the equator again at A'. This is one complete cycle of the geodesic, but $\lambda_{A'}$ does not equal λ_A due to the eccentricity of the ellipsoid, hence we say that the geodesic curve does not repeat after a complete cycle.

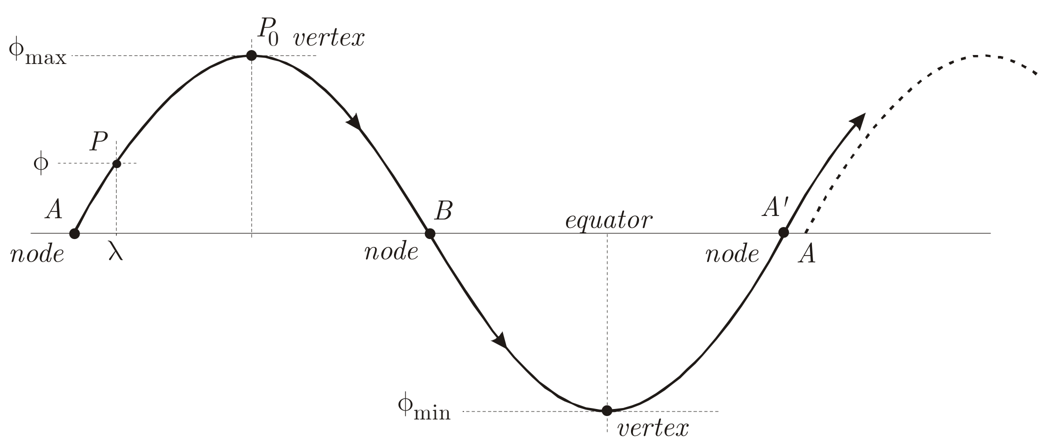


Figure 9: Schematic representation of the oscillation of a geodesic on an ellipsoid

RELATIONSHIPS BETWEEN PARAMETRIC LATITUDE ψ AND GEODETIC LATITUDE ϕ

The development of formulae is simplified if parametric latitude ψ is used rather than geodetic latitude ϕ. The connection between the two latitudes can be obtained from the following relationships.

Figure 10 shows a portion of a meridian NPE of an ellipsoid having semi-major axis $OE = a$ and semi-minor axis $ON = b$. P is a point on the ellipsoid and P' is a point on an auxiliary circle centred on O of radius a. P and P' have the same perpendicular distance w from the axis of revolution ON. The normal to the ellipsoid at P cuts the major axis at an angle ϕ (the geodetic latitude) and intersects the rotational axis at H. The distance $PH = \nu$. The angle $P'OE = \psi$ is the parametric latitude

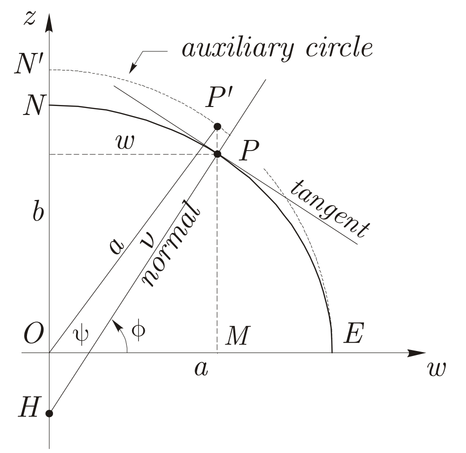


Figure 10: Meridian section of ellipsoid

The Cartesian equation of the ellipse and the auxiliary circle of Figure 10 are $\dfrac{w^2}{a^2} + \dfrac{z^2}{b^2} = 1$ and $w^2 + z^2 = a^2$ respectively. Now, since the w-coordinate of P and P' are the same then $a^2 - \dfrac{a^2}{b^2} z_P^2 = w_P^2 = w_{P'}^2 = a^2 - z_{P'}^2$ which leads to $z_P = \dfrac{b}{a} z_{P'}$. Using this relationship

$$\begin{aligned} w &= OM = a\cos\psi \\ z &= MP = b\sin\psi \end{aligned} \tag{35}$$

Note that writing equations (35) as $\dfrac{w}{a} = \cos\psi$ and $\dfrac{z}{b} = \sin\psi$ then squaring and adding gives $\dfrac{w^2}{a^2} + \dfrac{z^2}{b^2} = \cos^2\psi + \sin^2\psi = 1$ which is the Cartesian equation of an ellipse.

From Figure 10

$$w = \nu\cos\phi = a\cos\psi \tag{36}$$

and from the third of equations (15) $z = \nu\left(1 - e^2\right)\sin\phi$, hence using equations (35) we may write

$$\begin{aligned} w &= a\cos\psi = \nu\cos\phi \\ z &= b\sin\psi = \nu\left(1-e^2\right)\sin\phi \end{aligned} \tag{37}$$

from which the following ratios are obtained

$$\frac{z}{w} = \frac{b}{a}\tan\psi = \left(1-e^2\right)\tan\phi$$

Since $e^2 = \dfrac{a^2-b^2}{a^2} = 1-\dfrac{b^2}{a^2}$ then $1-e^2 = \dfrac{b^2}{a^2}$ and we may define parametric latitude ψ by

$$\tan\psi = \frac{b}{a}\tan\phi = \left(1-e^2\right)^{\frac{1}{2}}\tan\phi = (1-f)\tan\phi \tag{38}$$

Alternatively, using equations (36) and (8) we may define the parametric latitude ψ by

$$\cos\psi = \frac{\cos\phi}{\left(1-e^2\sin^2\phi\right)^{\frac{1}{2}}} \tag{39}$$

or equivalently by

$$\sin\phi = \frac{\sin\psi}{\left(1-e^2\cos^2\psi\right)^{\frac{1}{2}}} \tag{40}$$

These three relationships are useful in the derivation of formulae for geodesic distance and longitude difference that follow.

THE LATITUDES ϕ_0 AND ψ_0 OF THE GEODESIC VERTEX

Now Clairaut's equation (34) is $\nu\cos\phi\sin\alpha = \text{constant} = C$, where $\nu = \dfrac{a}{\left(1-e^2\sin^2\phi\right)^{\frac{1}{2}}}$.

The term $\nu\cos\phi$ will be a minimum (and the latitude ϕ will be a maximum in the northern and southern hemispheres) when $|\sin\alpha|$ is a maximum of 1, and this occurs when $\alpha = 90^\circ$ or 270°. This point is known as the geodesic vertex.

Let $\nu_0\cos\phi_0$ be this smallest value, then

$$\nu_0\cos\phi_0 = C = \nu\cos\phi\sin\alpha \tag{41}$$

ϕ_0 is called the maximum geodetic latitude and the value of ψ corresponding to ϕ_0 is called the maximum parametric latitude and is denoted by ψ_0. Using this correspondence and equations (36) and (41) gives

$$a\cos\psi_0 = \nu\cos\phi\sin\alpha = a\cos\psi\sin\alpha \tag{42}$$

From this we may define the parametric latitude of the vertex ψ_0 as

$$\boxed{\cos\psi_0 = \cos\psi\sin\alpha \qquad (43)}$$

and the azimuth α of the geodesic as

$$\cos\alpha = \frac{\sqrt{\cos^2\psi - \cos^2\psi_0}}{\cos\psi} \qquad (44)$$

From equation (43) we see that if the azimuth α of a geodesic is known at a point P having parametric latitude ψ, the parametric latitude ψ_0 of the vertex P_0 can be computed. Conversely, given ψ and ψ_0 of points P and P_0 the azimuth of the geodesic between them may be computed from equation (44).

THE ELLIPSOID, THE AUXILIARY SPHERE AND THE DIFFERENTIAL EQUATIONS

The derivation of Bessel's formulae (or Rainsford's and Vincenty's equations) begins by developing relationships between the ellipsoid and a sphere. The sphere is an auxiliary surface and not an approximation of the ellipsoid; its radius therefore is immaterial and can be taken to be 1 (unit radius).

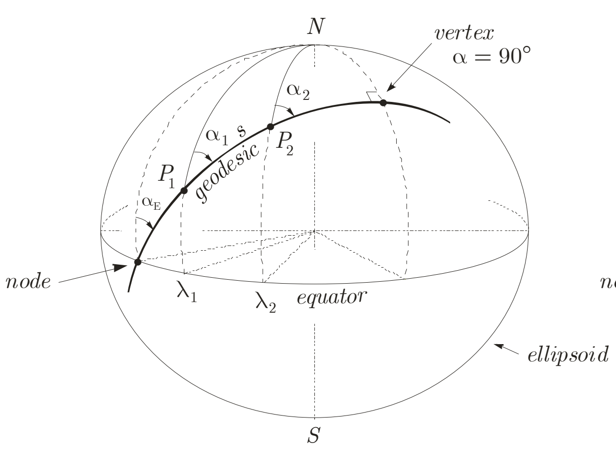


Figure 11a: The geodesic passing through P_1 and P_2 on the ellipsoid.

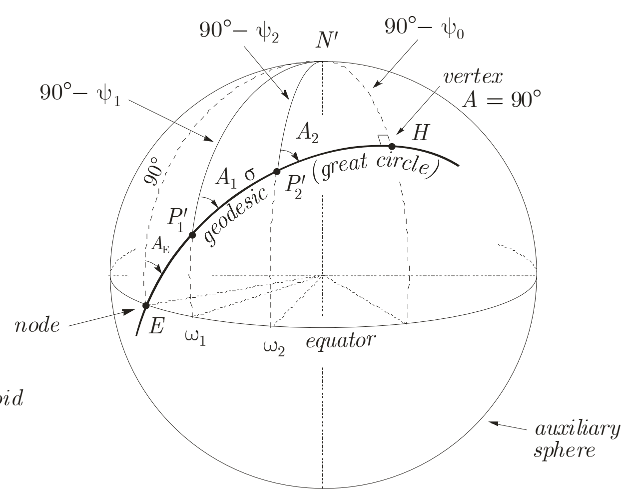


Figure 11b: The great circle passing through P_1' and P_2' on the auxiliary sphere.

Figure 11a shows a geodesic passing through P_1 and P_2 on an ellipsoid. The geodesic has azimuths α_E where it crosses the equator (a node), α_1 and α_2 at P_1 and P_2 respectively and reaches a maximum latitude at the vertex where its azimuth is $\alpha = 90^\circ$. The length of the geodesic between P_1 and P_2 is s and the longitudes of P_1 and P_2 are λ_1 and λ_2. Using equation (43) we may write

$$\cos\psi_1 \sin\alpha_1 = \cos\psi_2 \sin\alpha_2 = \cos\psi_0 \tag{45}$$

Figure 11b shows P_1' and P_2' on an auxiliary sphere (of unit radius) where latitudes on this sphere are defined to be equal to parametric latitudes on the ellipsoid. The geodesic, a great circle on a sphere, passing through P_1' and P_2' has azimuths A_E at the equator E, A_1 and A_2 at P_1' and P_2' respectively and $A = 90^\circ$ at the vertex H. The length of the great circle between P_1' and P_2' is σ and the longitudes of P_1' and P_2' are ω_1 and ω_2. Again, using equation (43), which holds for all geodesics (or great circles on auxiliary spheres) we may write

$$\cos\psi_1 \sin A_1 = \cos\psi_2 \sin A_2 = \cos\psi_0 \tag{46}$$

Now, since parametric latitudes are defined to be equal on the auxiliary sphere and the ellipsoid, equations (45) and (46) show that on these two surfaces $A = \alpha$, i.e., <u>azimuths of great circles on the auxiliary sphere are equal to azimuths of geodesics on the ellipsoid</u>.

Now, consider the differential rectangle on the ellipsoid and sphere shown in Figures 12a and 12b below

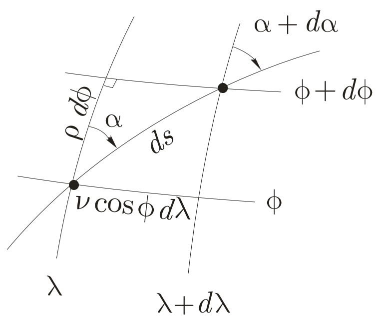


Figure 12a: Differential rectangle on ellipsoid

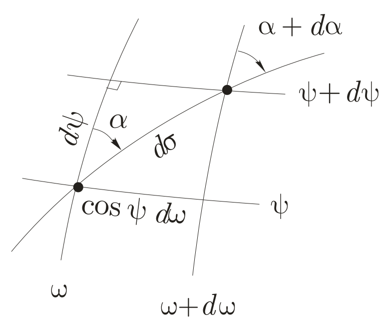


Figure 12b: Differential rectangle on sphere

We have for the ellipsoid [see Figure 2 and equations (19)]

$$\begin{aligned} ds\cos\alpha &= \rho\, d\phi \\ ds\sin\alpha &= \nu\cos\phi\; d\lambda \end{aligned} \tag{47}$$

and for the sphere

$$\begin{aligned} d\sigma\cos\alpha &= d\psi \\ d\sigma\sin\alpha &= \cos\psi\ d\omega \end{aligned} \tag{48}$$

Dividing equations (47) by equations (48) gives

$$\frac{ds\cos\alpha}{d\sigma\cos\alpha} = \frac{\rho\,d\phi}{d\psi}; \qquad \frac{ds\sin\alpha}{d\sigma\sin\alpha} = \frac{\nu\cos\phi\ d\lambda}{\cos\psi\ d\omega}$$

and noting from equation (36) that $\nu\cos\phi = a\cos\psi$, then cancelling terms gives

$$\frac{ds}{d\sigma} = \rho\frac{d\phi}{d\psi} = a\frac{d\lambda}{d\omega} \tag{49}$$

We may write these equations as two separate relationships

$$\frac{ds}{d\sigma} = \rho\frac{d\phi}{d\psi} \tag{50}$$

$$\frac{d\lambda}{d\omega} = \frac{1}{a}\frac{ds}{d\sigma} \tag{51}$$

and if we can obtain an expression for $\dfrac{d\phi}{d\psi}$ then we may develop two relatively simple differential equations; one involving distance $\dfrac{ds}{d\sigma}$ (s ellipsoid and σ sphere) and the other involving longitude $\dfrac{d\lambda}{d\omega}$ (λ ellipsoid and ω sphere). Integration yields equations that will enable us to compute geodesic lengths s on the ellipsoid given great circle distances σ on an auxiliary sphere, and equations to compute longitude differences $\Delta\lambda$ on the ellipsoid given longitude differences $\Delta\omega$ on the auxiliary sphere.

An expression for $\dfrac{d\phi}{d\psi}$ can be determined as follows.

From equation (38) we have

$$\tan\psi = \left(1-e^2\right)^{\frac{1}{2}}\tan\phi$$

and differentiating with respect to ψ gives

$$\frac{d}{d\psi}(\tan\psi) = \frac{d}{d\phi}\left\{\left(1-e^2\right)^{\frac{1}{2}}\tan\phi\right\}\frac{d\phi}{d\psi}$$

and

$$\sec^2\psi = \left(1-e^2\right)^{\frac{1}{2}}\sec^2\phi\frac{d\phi}{d\psi}$$

giving $$\frac{d\phi}{d\psi} = \frac{1}{\left(1-e^2\right)^{\frac{1}{2}}} \frac{\cos^2\phi}{\cos^2\psi} \tag{52}$$

Substituting equation (52) into equation (50) gives

$$\frac{ds}{d\sigma} = \frac{\rho}{\left(1-e^2\right)^{\frac{1}{2}}} \frac{\cos^2\phi}{\cos^2\psi} \tag{53}$$

and substituting equation (53) into equation (51) gives

$$\frac{d\lambda}{d\omega} = \frac{\rho}{a\left(1-e^2\right)^{\frac{1}{2}}} \frac{\cos^2\phi}{\cos^2\psi} \tag{54}$$

Now from equation (36) we may write

$$\frac{\cos\phi}{\cos\psi} = \frac{a}{\nu} \quad \text{and} \quad \frac{\cos^2\phi}{\cos^2\psi} = \frac{a^2}{\nu^2}$$

and using the relationships given in equations (4), (10), (11) and (12) we may write

$$\frac{\cos^2\phi}{\cos^2\psi} = \frac{a^2}{\nu^2} = \frac{b^2V^2}{a^2}; \quad \frac{\rho}{\left(1-e^2\right)^{\frac{1}{2}}} = \frac{c}{V^3}\frac{a}{b} = \frac{a^3}{b^2V^3}; \quad \frac{\rho}{a\left(1-e^2\right)^{\frac{1}{2}}} = \frac{a^2}{b^2V^3} \tag{55}$$

Substituting these results into equations (53) and (54) gives

$$\frac{ds}{d\sigma} = \frac{a}{V} \tag{56}$$

and $$\frac{d\lambda}{d\omega} = \frac{1}{V} \tag{57}$$

Now from equation (13) we may write $V^2 = 1 + e'^2\cos^2\phi$ and also from equation (55) we may write $\cos^2\phi = \frac{b^2V^2}{a^2}\cos^2\psi$. Using these gives

$$V^2 = 1 + e'^2\frac{b^2V^2}{a^2}\cos^2\psi$$

Now using equations (4) and (5) gives

$$\begin{aligned} V^2 &= 1 + \frac{e^2}{1-e^2}\left(1-e^2\right)V^2\cos^2\psi \\ &= 1 + e^2V^2\cos^2\psi \end{aligned}$$

and $V^2\left(1-e^2\cos^2\psi\right) = 1$ from which we obtain

$$V = \frac{1}{\left(1-e^2\cos^2\psi\right)^{\frac{1}{2}}} \tag{58}$$

Substituting equation (58) into equations (56) and (57) gives

$$\frac{ds}{d\sigma} = a\left(1 - e^2 \cos^2 \psi\right)^{\frac{1}{2}} \tag{59}$$

and

$$\frac{d\lambda}{d\omega} = \left(1 - e^2 \cos^2 \psi\right)^{\frac{1}{2}} \tag{60}$$

Equations (59) and (60) are the two differential equations from which we obtain distance s and longitude difference $\omega - \lambda$.

FORMULA FOR COMPUTATION OF GEODESIC DISTANCE *s*

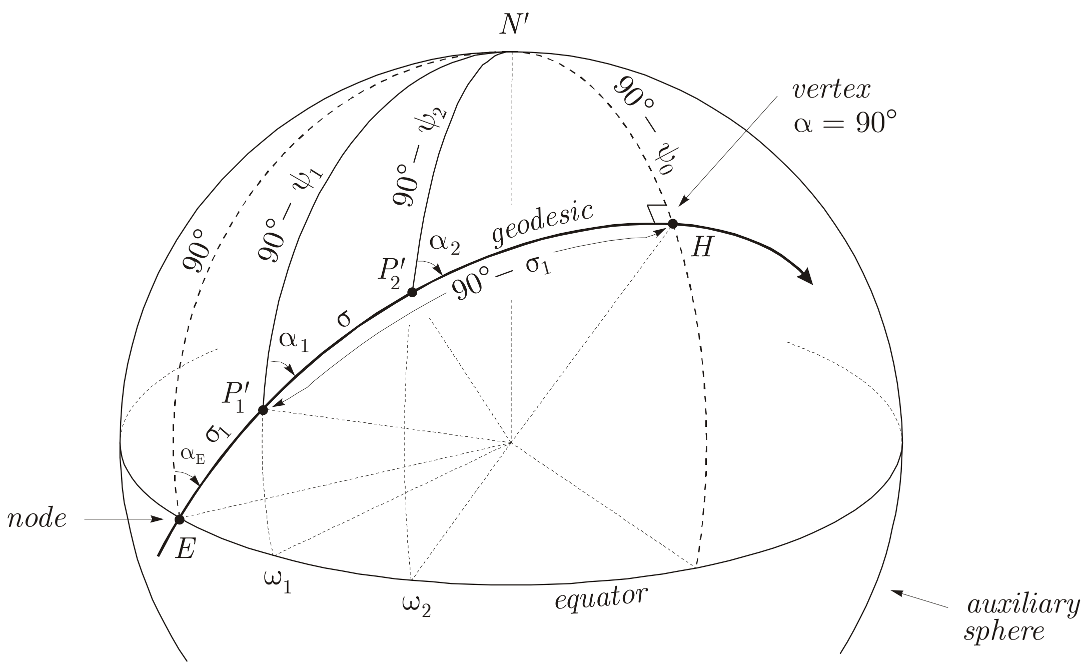
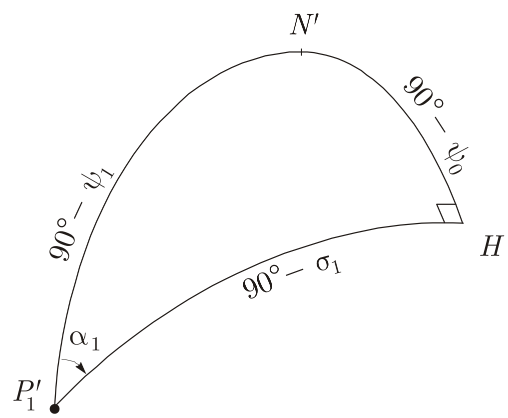


Figure 13: Geodesic on auxiliary sphere

Figure 13 shows P_1' and P_2' on an auxiliary sphere (of unit radius) where latitudes on this sphere are defined to be equal to parametric latitudes on the ellipsoid. The geodesic, a great circle on a sphere, passing through P_1' and P_2' has azimuths α_E at the equator E, α_1 at P_1', α_2 at P_2' and $\alpha = 90^\circ$ at the vertex *H*.

Note here that we have shown previously that for our auxiliary sphere, the azimuth of a great circle on the sphere is equal to the azimuth of the geodesic on the ellipsoid. The length of the great circle arc between P_1' and P_2' is σ and the longitudes of P_1' and P_2' are ω_1 and ω_2. Also note that σ_1 and σ_2 are angular distances along the great circle from the node E to P_1' and E to P_2' respectively and the angular distance from E to the vertex H is 90°. ψ_1, ψ_2 and ψ_0 are the parametric latitudes of P_1, P_2 and the vertex respectively, and they are also the latitudes of P_1', P_2' and the vertex H on the auxiliary sphere.

From the spherical triangle $P_1'N'H$ with the right-angle at H, using the sine rule (for spherical trigonometry)



$$\frac{\sin\alpha_1}{\sin(90^\circ-\psi_0)}=\frac{\sin(90^\circ)}{\sin(90^\circ-\psi_1)}$$

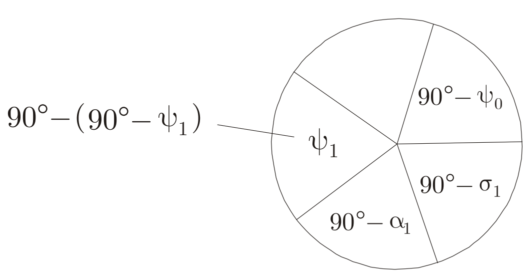
or
$$\frac{\sin\alpha_1}{\cos\psi_0}=\frac{1}{\cos\psi_1}$$

so
$$\sin\alpha_1\cos\psi_1=\cos\psi_0 \qquad (61)$$

Note that equation (61) can also be obtained from equation (43) and at the equator where $\psi = 90^\circ$ and $\cos\psi = 1$ we have

$$\boxed{\sin\alpha_E=\cos\psi_0} \qquad (62)$$

Using Napier's Rules for circular parts in the right-angled spherical triangle $P_1'N'H$

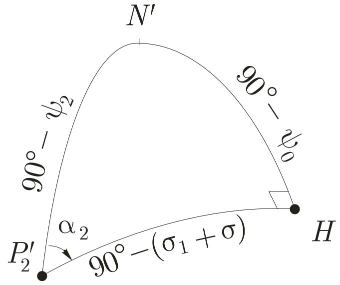


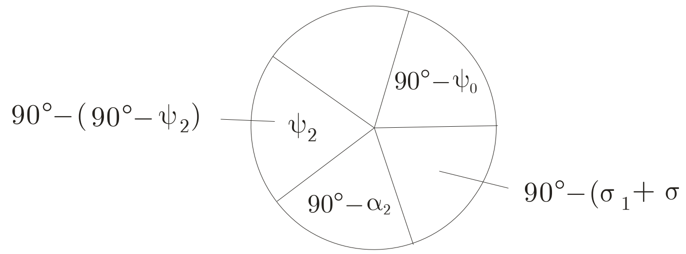
$$\sin(\text{mid-part}) = \text{product of } \tan(\text{adjacent-parts})$$
$$\sin(90^\circ-\alpha_1)=\tan\psi_1\tan(90^\circ-\sigma_1)$$
$$\cos\alpha_1=\tan\psi_1\cot\sigma_1$$
$$=\frac{\tan\psi_1}{\tan\sigma_1}$$

and

$$\boxed{\tan\sigma_1=\frac{\tan\psi_1}{\cos\alpha_1}} \qquad (63)$$

Using Napier's Rules for circular parts in the right-angled spherical triangle $P_2'N'H$





$$\sin(\text{mid-part}) = \text{product of } \cos(\text{opposite-parts})$$

$$\sin\psi_2 = \cos\big(90^\circ - (\sigma_1 + \sigma)\big)\cos(90^\circ - \psi_0)$$

$$\sin\psi_2 = \sin(\sigma_1 + \sigma)\sin\psi_0 \tag{64}$$

Note: The subscript 2 can be dropped and we can just refer to a general point P' and the distance from P_1' to P' is σ, hence

$$\boxed{\sin\psi = \sin(\sigma_1 + \sigma)\sin\psi_0} \tag{65}$$

Referring to equations (59) and (60), we need to develop an expression for $\cos^2\psi$. This can be achieved in the following manner.

Squaring both sides of equation (65) and using the trigonometric identity $\sin^2\psi + \cos^2\psi = 1$ we have

$$\sin^2\psi = 1 - \cos^2\psi = \sin^2(\sigma_1 + \sigma)\sin^2\psi_0$$

so that

$$\cos^2\psi = 1 - \sin^2(\sigma_1 + \sigma)\sin^2\psi_0 \tag{66}$$

Let

$$\boxed{x = \sigma_1 + \sigma} \tag{67}$$

and equation (66) becomes

$$\cos^2\psi = 1 - \sin^2 x \sin^2\psi_0 \tag{68}$$

We may now write equation (59) with $dx = d\sigma$ since σ_1 is constant, as

$$\begin{aligned} ds &= a\left(1 - e^2\cos^2\psi\right)^{\frac{1}{2}} d\sigma \\ &= a\left(1 - e^2\left[1 - \sin^2 x\sin^2\psi_0\right]\right)^{\frac{1}{2}} dx \\ &= a\left(1 - e^2 + e^2\sin^2 x\sin^2\psi_0\right)^{\frac{1}{2}} dx \end{aligned}$$

Now using equations (4), (5) and (6)

$$\begin{aligned} ds &= a\left(\frac{1}{1+e'^2}+\frac{e'^2}{1+e'^2}\sin^2 x\sin^2\psi_0\right)^{\frac{1}{2}}dx \\ &= \frac{a}{\left(1+e'^2\right)^{\frac{1}{2}}}\left(1+e'^2\sin^2 x\sin^2\psi_0\right)^{\frac{1}{2}}dx \\ &= b\left(1+e'^2\sin^2 x\sin^2\psi_0\right)^{\frac{1}{2}}dx \end{aligned}$$

Now, since e'^2 is a constant for the ellipsoid and ψ_0 is a constant for a particular geodesic we may write

$$\boxed{u^2 = e'^2\sin^2\psi_0 = e'^2\cos^2\alpha_E} \tag{69}$$

where α_E is the azimuth of the geodesic at the node or equator crossing, and

$$ds = b\left(1+u^2\sin^2 x\right)^{\frac{1}{2}}dx \tag{70}$$

The length of the geodesic arc s between P_1 and P_2 is found by integration as

$$s = b\int_{x=\sigma_1}^{x=\sigma_1+\sigma}\left(1+u^2\sin^2 x\right)^{\frac{1}{2}}dx \tag{71}$$

where the integration terminals are $x=\sigma_1$ and $x=\sigma_1+\sigma$ remembering that at P_1', $\sigma=0$ and $x=\sigma_1$, and at P_2', $x=\sigma_1+\sigma$.

Equation (71) is an elliptic integral and does not have a simple closed-form solution. However, the integrand $\left(1+u^2\sin^2 x\right)^{\frac{1}{2}}$ can be expanded in a series and then evaluated by term-by-term integration.

The integrand in equation (71) can be expanded by use of the binomial series

$$(1+x)^\beta = \sum_{n=0}^{\infty} B_n^\beta x^n \tag{72}$$

An infinite series where n is a positive integer, β is any real number and the binomial coefficients B_n^β are given by

$$B_n^\beta = \frac{\beta(\beta-1)(\beta-2)(\beta-3)\cdots(\beta-n+1)}{n!} \tag{73}$$

The binomial series (72) is convergent when $-1<x<1$. In equation (73) $n!$ denotes n-factorial and $n! = n(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1$. zero-factorial is defined as $0!=1$ and the binomial coefficient $B_0^\beta = 1$.

In the case where β is a positive integer, say k, the binomial series (72) can be expressed as the <u>finite</u> sum

$$(1+x)^k = \sum_{n=0}^{k} B_n^k \, x^n \tag{74}$$

where the binomial coefficients B_n^k in series (74) are given by

$$B_n^k = \frac{k!}{n!(k-n)!} \tag{75}$$

The binomial coefficients $B_n^{\frac{1}{2}}$ for the series (72) are given by equation (73) with the following results for $n = 0, 1, 2$ and 3

$n = 0 \qquad B_0^{\frac{1}{2}} = 1$

$n = 1 \qquad B_1^{\frac{1}{2}} = \dfrac{1}{2}$

$n = 2 \qquad B_2^{\frac{1}{2}} = \dfrac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!} = -\dfrac{1}{8}$

$n = 3 \qquad B_3^{\frac{1}{2}} = \dfrac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!} = \dfrac{1}{16}$

Inspecting the results above, we can see that the binomial coefficients $B_n^{\frac{1}{2}}$ form a sequence

$$1, \ \frac{1}{2}, \ -\frac{1\cdot 1}{2\cdot 4}, \ \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}, \ -\frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}, \ \frac{1\cdot 1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 10}, \ -\frac{1\cdot 1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12}, \cdots$$

Using these results

$$\begin{aligned}\left(1+u^2\sin^2 x\right)^{\frac{1}{2}} = 1 &+ \frac{1}{2}u^2\sin^2 x - \frac{1\cdot 1}{2\cdot 4}u^4\sin^4 x + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}u^6\sin^6 x \\ &- \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}u^8\sin^8 x + \frac{1\cdot 1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 10}u^{10}\sin^{10} x + \cdots\end{aligned} \tag{76}$$

To simplify this expression, and make the eventual integration easier, the powers of $\sin x$ can be expressed in terms of multiple angles using the standard form

$$\begin{aligned}\sin^{2n} x = \frac{1}{2^{2n}}\binom{2n}{n} + \frac{(-1)^n}{2^{2n-1}}\Bigg\{&\cos 2nx - \binom{2n}{1}\cos(2n-2)x + \binom{2n}{2}\cos(2n-4)x \\ &- \binom{2n}{3}\cos(2n-6)x + \cdots (-1)^n\binom{2n}{n-1}\cos 2x\Bigg\}\end{aligned} \tag{77}$$

Using equation (77) and the binomial coefficients $B_n^{2n} = \dbinom{2n}{n}$ computed using equation (75) gives

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$$

$$\sin^4 x = \frac{3}{8} + \frac{1}{8}\cos 4x - \frac{1}{2}\cos 2x$$

$$\sin^6 x = \frac{5}{16} - \frac{1}{32}\cos 6x + \frac{3}{16}\cos 4x - \frac{15}{32}\cos 2x$$

$$\sin^8 x = \frac{35}{128} + \frac{1}{128}\cos 8x - \frac{1}{16}\cos 6x + \frac{7}{32}\cos 4x - \frac{7}{16}\cos 2x$$

$$\sin^{10} x = \frac{63}{256} - \frac{1}{512}\cos 10x + \frac{5}{256}\cos 8x - \frac{45}{512}\cos 6x + \frac{15}{64}\cos 4x - \frac{105}{256}\cos 2x \tag{78}$$

Substituting equations (78) into equation (76) and arranging according to $\cos 2x$, $\cos 4x$, etc, we obtain (Rapp 1981, p. 7-8)

$$\left(1 + u^2 \sin^2 x\right)^{\frac{1}{2}} = A + B\cos 2x + C\cos 4x + D\cos 6x + E\cos 8x + F\cos 10x + \cdots \tag{79}$$

where the coefficients A, B, C, etc., are

$$\begin{array}{llllllll}
A = & 1 + \frac{1}{4}u^2 & -\frac{3}{64}u^4 & +\frac{5}{256}u^6 & -\frac{175}{16384}u^8 & +\frac{441}{65536}u^{10} & -\cdots \\
B = & -\frac{1}{4}u^2 & +\frac{1}{16}u^4 & -\frac{15}{512}u^6 & +\frac{35}{2048}u^8 & -\frac{735}{65536}u^{10} & +\cdots \\
C = & & -\frac{1}{64}u^4 & +\frac{3}{256}u^6 & -\frac{35}{4096}u^8 & +\frac{105}{16384}u^{10} & -\cdots \\
D = & & & -\frac{1}{512}u^6 & +\frac{5}{2048}u^8 & -\frac{315}{131072}u^{10} & +\cdots \\
E = & & & & -\frac{5}{16384}u^8 & +\frac{35}{65536}u^{10} & -\cdots \\
F = & & & & & -\frac{7}{131072}u^{10} & +\cdots
\end{array} \tag{80}$$

Substituting equation (79) into equation (71) gives

$$s = b\int_{\sigma_1}^{\sigma_1+\sigma} \left\{A + B\cos 2x + C\cos 4x + D\cos 6x + E\cos 8x + F\cos 10x + \cdots\right\} dx \tag{81}$$

or

$$\begin{aligned}
\frac{s}{b} &= A\int_{\sigma_1}^{\sigma_1+\sigma} dx + B\int_{\sigma_1}^{\sigma_1+\sigma} \cos 2x\, dx + C\int_{\sigma_1}^{\sigma_1+\sigma} \cos 4x\, dx + D\int_{\sigma_1}^{\sigma_1+\sigma} \cos 6x\, dx \\
&\quad + E\int_{\sigma_1}^{\sigma_1+\sigma} \cos 8x\, dx + F\int_{\sigma_1}^{\sigma_1+\sigma} \cos 10x\, dx \cdots
\end{aligned} \tag{82}$$

The evaluation of the integral

$$\int_{\sigma_1}^{\sigma_1+\sigma} \cos nx\, dx = \frac{1}{n}\left[\sin nx\right]_{\sigma_1}^{\sigma_1+\sigma} = \frac{1}{n}\left\{\sin n\left(\sigma_1+\sigma\right) - \sin n\sigma_1\right\} \tag{83}$$

combined with the trigonometric identity

$$\sin nX - \sin nY = 2\cos\left[\frac{n}{2}(X+Y)\right]\sin\left[\frac{n}{2}(X-Y)\right]$$

where $X = \sigma_1 + \sigma$ and $Y = \sigma_1$ so that $X + Y = 2\sigma_1 + \sigma$ and $X - Y = \sigma$ gives

$$\int_{\sigma_1}^{\sigma_1+\sigma} \cos nx\, dx = \frac{2}{n}\cos n\sigma_m \sin\frac{n}{2}\sigma \tag{84}$$

Noting that

$$\sin n\left(\sigma_1+\sigma\right) - \sin n\sigma_1 = 2\cos\frac{n}{2}\left(2\sigma_1+\sigma\right)\sin\frac{n}{2}\sigma$$

and with $\sigma = \sigma_2 - \sigma_1$, then $2\sigma_1 + \sigma = 2\sigma_1 + \left(\sigma_2 - \sigma_1\right) = \sigma_1 + \sigma_2$

and putting
$$\sigma_m = \frac{\sigma_1+\sigma_2}{2} \tag{85}$$

then

$$\boxed{2\sigma_m = 2\sigma_1 + \sigma} \tag{86}$$

and

$$\sin n\left(\sigma_1+\sigma\right) - \sin n\sigma_1 = 2\cos n\sigma_m \sin\frac{n}{2}\sigma \tag{87}$$

Using this result, equation (82) becomes

$$\begin{aligned}\frac{s}{b} &= A\sigma + B\left(\cos 2\sigma_m \sin\sigma\right) + C\left(\tfrac{1}{2}\cos 4\sigma_m \sin 2\sigma\right) + D\left(\tfrac{1}{3}\cos 6\sigma_m \sin 3\sigma\right)\\ &\quad + E\left(\tfrac{1}{4}\cos 8\sigma_m \sin 4\sigma\right) + F\left(\tfrac{1}{5}\cos 10\sigma_m \sin 5\sigma\right) + \cdots\end{aligned}$$

or re-arranged as (Rapp 1981, equation 39, p. 9)

$$\begin{aligned}s &= b\left\{A\sigma + B\cos 2\sigma_m \sin\sigma + \frac{C}{2}\cos 4\sigma_m \sin 2\sigma + \frac{D}{3}\cos 6\sigma_m \sin 3\sigma\right.\\ &\qquad \left. + \frac{E}{4}\cos 8\sigma_m \sin 4\sigma + \frac{F}{5}\cos 10\sigma_m \sin 5\sigma + \cdots\right\}\end{aligned} \tag{88}$$

Equation (88) may be modified by adopting another set of constants; defined as

$$B_0 = A; \quad B_2 = B; \quad B_4 = \frac{C}{2}; \quad B_6 = \frac{D}{3}; \quad B_8 = \frac{E}{4}; \quad B_{10} = \frac{F}{5} \tag{89}$$

to give

$$
\begin{aligned}
s &= b\{B_0\sigma + B_2\cos 2\sigma_m \sin\sigma + B_4\cos 4\sigma_m \sin 2\sigma + B_6\cos 6\sigma_m \sin 3\sigma \\
&\quad + B_8\cos 8\sigma_m \sin 4\sigma + B_{10}\cos 10\sigma_m \sin 5\sigma + \cdots \\
&\quad + B_{2n}\cos 2n\sigma_m \sin n\sigma + \cdots\}
\end{aligned} \tag{90}
$$

where the coefficients $B_0, B_2, B_4, \ldots$ are

$$
\begin{aligned}
B_0 &= 1 + \frac{1}{4}u^2 - \frac{3}{64}u^4 + \frac{5}{256}u^6 - \frac{175}{16384}u^8 + \frac{441}{65536}u^{10} - \cdots \\
B_2 &= -\frac{1}{4}u^2 + \frac{1}{16}u^4 - \frac{15}{512}u^6 + \frac{35}{2048}u^8 - \frac{735}{65536}u^{10} + \cdots \\
B_4 &= -\frac{1}{128}u^4 + \frac{3}{512}u^6 - \frac{35}{8192}u^8 + \frac{105}{32768}u^{10} - \cdots \\
B_6 &= -\frac{1}{1536}u^6 + \frac{5}{6144}u^8 - \frac{105}{131072}u^{10} + \cdots \\
B_8 &= -\frac{5}{65536}u^8 + \frac{35}{262144}u^{10} - \cdots \\
B_{10} &= -\frac{7}{655360}u^{10} + \cdots
\end{aligned}
$$

Since each of these convergent series is alternating, an upper bound of the error committed in truncating the series is the first term omitted – keeping terms up to u^8 only commits an error of order u^{10} – and equation (90) can be approximated by

$$
\begin{aligned}
s &= b\{B_0\sigma + B_2\cos 2\sigma_m \sin\sigma + B_4\cos 4\sigma_m \sin 2\sigma + B_6\cos 6\sigma_m \sin 3\sigma \\
&\quad + B_8\cos 8\sigma_m \sin 4\sigma\}
\end{aligned} \tag{91}
$$

where

$$
\begin{aligned}
B_0 &= 1 + \frac{1}{4}u^2 - \frac{3}{64}u^4 + \frac{5}{256}u^6 - \frac{175}{16384}u^8 \\
B_2 &= -\frac{1}{4}u^2 + \frac{1}{16}u^4 - \frac{15}{512}u^6 + \frac{35}{2048}u^8 \\
B_4 &= -\frac{1}{128}u^4 + \frac{3}{512}u^6 - \frac{35}{8192}u^8 \\
B_6 &= -\frac{1}{1536}u^6 + \frac{5}{6144}u^8 \\
B_8 &= -\frac{5}{65536}u^8
\end{aligned} \tag{92}
$$

The approximation (91) and the coefficients given by equations (92) are the same as Rainsford (1955, equations 18 and 19, p.15) and also Rapp (1981, equations 40 and 41, p. 9).

Equation (91) can be used in two ways which will be discussed in detail later. Briefly, however, the first way is in the direct problem – where s, u^2 and σ_1 are known – to solve iteratively for σ (and hence σ_m from $2\sigma_m = 2\sigma_1 + \sigma$; and $x = \sigma_1 + \sigma$) by using Newton-Raphson iteration for the real roots of the equation $f(\sigma) = 0$ given in the form of an iterative equation

$$\sigma_{(n+1)} = \sigma_{(n)} - \frac{f\left(\sigma_{(n)}\right)}{f'\left(\sigma_{(n)}\right)} \tag{93}$$

where n denotes the n^{th} iteration and $f(\sigma)$ can be obtained from equation (91) as

$$\begin{aligned} f(\sigma) = B_0\sigma &+ B_2 \cos 2\sigma_m \sin\sigma + B_4 \cos 4\sigma_m \sin 2\sigma + B_6 \cos 6\sigma_m \sin 3\sigma \\ &+ B_8 \cos 8\sigma_m \sin 4\sigma - \frac{s}{b} \end{aligned} \tag{94}$$

and the derivative $f'(\sigma) = \dfrac{d}{d\sigma}\{f(\sigma)\}$ is given by

$$f'\left(\sigma\right) = \left(1 + u^2 \sin^2 x\right)^{\frac{1}{2}} \tag{95}$$

[Note here that $f\left(\sigma\right)$ is the result of integrating the function $\left(1 + u^2 \sin^2 x\right)^{\frac{1}{2}}$ with respect to *dx;* so then the derivative $f'\left(\sigma\right)$ must be the original function.]

An initial value, $\sigma_{(1)}$ (σ for $n = 1$) can be computed from $\sigma_{(1)} = \dfrac{s}{B_0 b}$ and the functions $f\left(\sigma_{(1)}\right)$ and $f'\left(\sigma_{(1)}\right)$ evaluated from equations (94) and (95) using $\sigma_{(1)}$. $\sigma_{(2)}$ (σ for $n = 2$) can now be computed from equation (93) and this process repeated to obtain values $\sigma_{(3)}, \sigma_{(4)}, \ldots$. This iterative process can be concluded when the difference between $\sigma_{(n+1)}$ and $\sigma_{(n)}$ reaches an acceptably small value.

The second application of equation (91) is in the inverse problem where s is computed once σ has been determined by spherical trigonometry.

FORMULA FOR COMPUTATION OF LONGITUDE DIFFERENCE BETWEEN TWO POINTS ON A GEODESIC

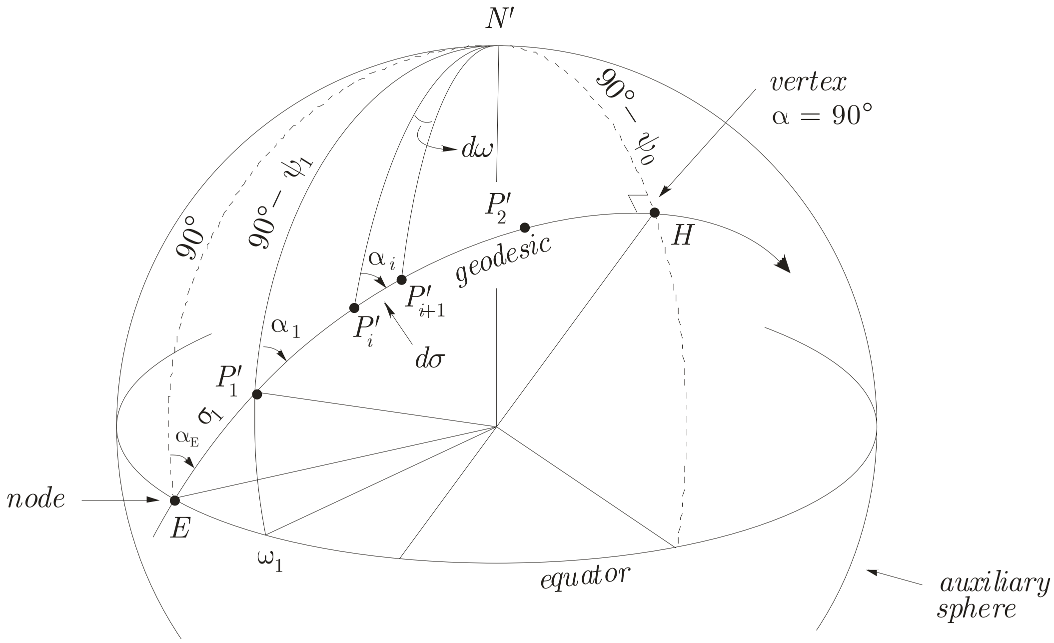


Figure 14: Geodesic on auxiliary sphere

Figure 14 shows P_1' and P_2' on an auxiliary sphere (of unit radius) where latitudes on this sphere are defined to be equal to parametric latitudes on the ellipsoid. P_i' and P_{i+1}' are arbitrary points on the geodesic (a great circle) between P_1' and P_2' separated by the angular distance $d\sigma$.

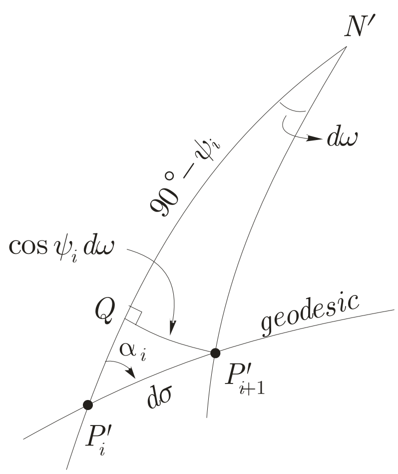


Figure 15

Figure 15 shows the differential spherical triangle $P_i' N' P_{i+1}'$ broken into two right-angled spherical triangles $P_i' Q P_{i+1}'$ and $Q N' P_{i+1}'$. The great circle arc $Q P_{i+1}'$ is defined as $\cos\psi_1\, d\omega$, which is the differential arc length of the parallel of parametric latitude ψ_1. Approximating the spherical triangle $P_i' Q P_{i+1}'$ with a plane right-angled triangle gives $\cos\psi_i\, d\omega = d\sigma \sin\alpha_i$ and

$$d\omega = \frac{\sin\alpha_i}{\cos\psi_i}\, d\sigma \tag{96}$$

From equation (43)

$$\sin\alpha_i = \frac{\cos\psi_0}{\cos\psi_i} \tag{97}$$

and substituting equation (97) into (96) gives the relationship (dropping the subscript i)

$$d\omega = \frac{\cos\psi_0}{\cos^2\psi}\, d\sigma \tag{98}$$

Substituting equation (98) into equation (60) and re-arranging gives

$$d\lambda = \cos\psi_0 \frac{\left(1 - e^2\cos^2\psi\right)^{\frac{1}{2}}}{\cos^2\psi}\, d\sigma \tag{99}$$

Subtracting equation (98) from equation (99) gives an expression for the difference between differentials of two measures of longitude; $d\omega$ on the auxiliary sphere and $d\lambda$ on the ellipsoid

$$d\lambda - d\omega = \cos\psi_0 \left[\frac{\left(1 - e^2\cos^2\psi\right)^{\frac{1}{2}}}{\cos^2\psi} - \frac{1}{\cos^2\psi}\right] d\sigma \tag{100}$$

Equation (100) can be simplified by expanding $\left(1 - e^2\cos^2\psi\right)^{\frac{1}{2}}$ using the binomial series (72)

$$\left(1 - e^2\cos^2\psi\right)^{\frac{1}{2}} = \sum_{n=0}^{\infty} B_n^{\frac{1}{2}} \left(-e^2\cos^2\psi\right)^n$$

and from the previous development, the binomial coefficients $B_n^{\frac{1}{2}}$ form a sequence

$$1,\ \frac{1}{2},\ -\frac{1\cdot 1}{2\cdot 4},\ \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6},\ -\frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8},\ \frac{1\cdot 1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 10},\ -\frac{1\cdot 1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12},\ldots$$

Using these results

$$\left(1-e^2\cos^2\psi\right)^{\frac{1}{2}} = 1-\frac{1}{2}e^2\cos^2\psi-\frac{1\cdot 1}{2\cdot 4}e^4\cos^4\psi-\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}e^6\cos^6\psi$$
$$-\frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}e^8\cos^8\psi-\frac{1\cdot 1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 10}e^{10}\cos^{10}\psi+\cdots \tag{101}$$

so that

$$\frac{\left(1-e^2\cos^2\psi\right)^{\frac{1}{2}}}{\cos^2\psi} = \frac{1}{\cos^2\psi}-\frac{1}{2}e^2-\frac{1}{8}e^4\cos^2\psi-\frac{1}{16}e^6\cos^4\psi$$
$$-\frac{5}{128}e^8\cos^6\psi-\frac{7}{256}e^{10}\cos^8\psi+\cdots \tag{102}$$

Now, subtracting $\dfrac{1}{\cos^2\psi}$ from both sides of equation (102) gives a new equation whose left-hand-side is the term inside the brackets [ ] in equation (100), and using this result we may write equation (100) as

$$d\lambda-d\omega=\cos\psi_0\left\{-\frac{1}{2}e^2-\frac{1}{8}e^4\cos^2\psi-\frac{1}{16}e^6\cos^4\psi\right.$$
$$\left.-\frac{5}{128}e^8\cos^6\psi-\frac{7}{256}e^{10}\cos^8\psi+\cdots\right\}d\sigma \tag{103}$$

which can be re-arranged as

$$d\omega-d\lambda-=\frac{e^2}{2}\cos\psi_0\left\{1+\frac{1}{4}e^2\cos^2\psi+\frac{1}{8}e^4\cos^4\psi\right.$$
$$\left.+\frac{5}{64}e^6\cos^6\psi+\frac{7}{128}e^8\cos^8\psi+\cdots\right\}d\sigma \tag{104}$$

From equations (65) and (67) we have $\sin\psi=\sin\left(\sigma_1+\sigma\right)\sin\psi_0$ and $x=\sigma_1+\sigma$ respectively, which gives $\sin\psi=\sin x\sin\psi_0$ and $\sin^2\psi=\sin^2 x\sin^2\psi_0=1-\cos^2\psi$. This result can be re-arranged as

$$\cos^2\psi=1-\sin^2\psi_0\sin^2 x$$

Now $\cos^4\psi=\left(1-\sin^2\psi_0\sin^2 x\right)^2$, $\cos^6\psi=\left(1-\sin^2\psi_0\sin^2 x\right)^3$, $\cos^8\psi=\left(1-\sin^2\psi_0\sin^2 x\right)^4$, etc., and using the binomial series (74) we may write

$$\cos^4\psi=1-2\sin^2\psi_0\sin^2 x+\sin^4\psi_0\sin^4 x$$

$$\cos^6\psi=1-3\sin^2\psi_0\sin^2 x+3\sin^4\psi_0\sin^4 x-\sin^6\psi_0\sin^6 x$$

$$\cos^8\psi=1-4\sin^2\psi_0\sin^2 x+6\sin^4\psi_0\sin^4 x-4\sin^6\psi_0\sin^6 x+\sin^8\psi_0\sin^8 x$$

Substituting these relationships into equation (104) and noting that $dx = d\sigma$ gives

$$\begin{aligned}
d\omega - d\lambda - = {} & \frac{e^2}{2}\cos\psi_0 \Big\{1 + \frac{1}{4}e^2\left(1 - \sin^2\psi_0 \sin^2 x\right) \\
& + \frac{1}{8}e^4\left(1 - 2\sin^2\psi_0 \sin^2 x + \sin^4\psi_0 \sin^4 x\right) \\
& + \frac{5}{64}e^6\left(1 - 3\sin^2\psi_0 \sin^2 x + 3\sin^4\psi_0 \sin^4 x - \sin^6\psi_0 \sin^6 x\right) \\
& + \frac{7}{128}e^8\Big(1 - 4\sin^2\psi_0 \sin^2 x + 6\sin^4\psi_0 \sin^4 x \\
& \qquad -4\sin^6\psi_0 \sin^6 x + \sin^8\psi_0 \sin^8 x\Big) \\
& + \cdots\Big\}\, dx
\end{aligned} \tag{105}$$

Now, expressions for $\sin^2 x$, $\sin^4 x$, … have been developed previously and are given in equations (78). These even powers of $\sin x$ may be substituted into equation (105) to give

$$\begin{aligned}
d\omega - d\lambda - = {} & \frac{e^2}{2}\cos\psi_0 \left\{1 + \frac{1}{4}e^2\left(1 - \sin^2\psi_0\left[\frac{1}{2} - \frac{1}{2}\cos 2x\right]\right)\right. \\
& + \frac{1}{8}e^4\Big(1 - 2\sin^2\psi_0\left[\frac{1}{2} - \frac{1}{2}\cos 2x\right] \\
& \qquad + \sin^4\psi_0\left[\frac{3}{8} + \frac{1}{8}\cos 4x - \frac{1}{2}\cos 2x\right]\Big) \\
& + \frac{5}{64}e^6\Big(1 - 3\sin^2\psi_0\left[\frac{1}{2} - \frac{1}{2}\cos 2x\right] \\
& \qquad + 3\sin^4\psi_0\left[\frac{3}{8} + \frac{1}{8}\cos 4x - \frac{1}{2}\cos 2x\right] \\
& \qquad - \sin^6\psi_0\left[\frac{5}{16} - \frac{1}{32}\cos 6x + \frac{3}{16}\cos 4x - \frac{15}{32}\cos 2x\right]\Big) \\
& + \frac{7}{128}e^8\Big(1 - 4\sin^2\psi_0\left[\frac{1}{2} - \frac{1}{2}\cos 2x\right] \\
& \qquad + 6\sin^4\psi_0\left[\frac{3}{8} + \frac{1}{8}\cos 4x - \frac{1}{2}\cos 2x\right] \\
& \qquad - 4\sin^6\psi_0\left[\frac{5}{16} - \frac{1}{32}\cos 6x + \frac{3}{16}\cos 4x - \frac{15}{32}\cos 2x\right] \\
& \qquad + \sin^8\psi_0\Big[\frac{35}{128} + \frac{1}{128}\cos 8x - \frac{1}{16}\cos 6x \\
& \qquad\qquad + \frac{7}{32}\cos 4x - \frac{7}{16}\cos 2x\Big] \\
& \left. + \cdots\right\} dx
\end{aligned} \tag{106}$$

Expanding the components of equation (106) associated with the even powers of e we have

$$\frac{1}{4}e^2\left(1-\frac{1}{2}\sin^2\psi_0+\frac{1}{2}\sin^2\psi_0\cos 2x\right) \tag{107}$$

$$\begin{aligned}\frac{1}{8}e^4\Big(1-\sin^2\psi_0+\sin^2\psi_0\cos 2x \\ +\frac{3}{8}\sin^4\psi_0+\frac{1}{8}\sin^4\psi_0\cos 4x-\frac{1}{2}\sin^4\psi_0\cos 2x\Big)\end{aligned} \tag{108}$$

$$\begin{aligned}\frac{5}{64}e^6\Big(1-\sin^2\psi_0+\sin^2\psi_0\cos 2x \\ +\frac{9}{8}\sin^4\psi_0+\frac{3}{8}\sin^4\psi_0\cos 4x-\frac{3}{2}\sin^4\psi_0\cos 2x \\ -\frac{5}{16}\sin^6\psi_0+\frac{1}{32}\sin^6\psi_0\cos 6x-\frac{3}{16}\sin^6\psi_0\cos 4x+\frac{15}{32}\sin^6\psi_0\cos 2x\Big)\end{aligned} \tag{109}$$

$$\begin{aligned}\frac{7}{128}e^8\Big(1-\sin^2\psi_0+\sin^2\psi_0\cos 2x \\ +\frac{9}{4}\sin^4\psi_0+\frac{3}{4}\sin^4\psi_0\cos 4x-3\sin^4\psi_0\cos 2x \\ -\frac{5}{4}\sin^6\psi_0+\frac{1}{8}\sin^6\psi_0\cos 6x-\frac{3}{4}\sin^6\psi_0\cos 4x \\ +\frac{15}{8}\sin^6\psi_0\cos 2x \\ +\frac{35}{128}\sin^8\psi_0+\frac{1}{128}\sin^8\psi_0\cos 8x-\frac{1}{16}\sin^8\psi_0\cos 6x \\ +\frac{7}{132}\sin^8\psi_0\cos 4x-\frac{7}{16}\sin^8\psi_0\cos 2x\Big)\end{aligned} \tag{110}$$

Gathering together the constant terms and the coefficients of $\cos 2x$, $\cos 4x$, $\cos 6x$, etc. in equations (107) to (110), we can write equation (106) as

$$d\omega-d\lambda=\frac{e^2}{2}\cos\psi_0\left\{C_0+C_2\cos 2x+C_4\cos 4x+C_6\cos 6x+C_8\cos 8x+\cdots\right\}dx \tag{111}$$

where the coefficients C_0, C_2, C_4, etc. are

$$
\begin{aligned}
C_0 = 1 &+ \frac{1}{4}e^2 + \frac{1}{8}e^4 + \frac{5}{64}e^6 + \frac{7}{128}e^8 + \cdots \\
&- \left(\frac{1}{8}e^2 + \frac{1}{8}e^4 + \frac{15}{128}e^6 + \frac{7}{64}e^8 + \cdots\right)\sin^2\psi_0 \\
&+ \left(\frac{3}{64}e^4 + \frac{45}{512}e^6 + \frac{63}{512}e^8 + \cdots\right)\sin^4\psi_0 \\
&- \left(\frac{25}{1024}e^6 + \frac{35}{512}e^8 + \cdots\right)\sin^6\psi_0 \\
&+ \left(\frac{245}{16384}e^8 + \cdots\right)\sin^8\psi_0 \\
&- \cdots
\end{aligned} \tag{112}
$$

$$
\begin{aligned}
C_2 = &\left(\frac{1}{8}e^2 + \frac{1}{8}e^4 + \frac{15}{128}e^6 + \frac{7}{64}e^8 + \cdots\right)\sin^2\psi_0 \\
&- \left(\frac{1}{16}e^4 + \frac{15}{128}e^6 + \frac{21}{128}e^8 + \cdots\right)\sin^4\psi_0 \\
&+ \left(\frac{75}{2048}e^6 + \frac{105}{1024}e^8 + \cdots\right)\sin^6\psi_0 \\
&+ \left(\frac{49}{2048}e^8 + \cdots\right)\sin^8\psi_0 \\
&- \cdots
\end{aligned} \tag{113}
$$

$$
\begin{aligned}
C_4 = &\left(\frac{1}{64}e^4 + \frac{15}{512}e^6 + \frac{21}{512}e^8 + \cdots\right)\sin^4\psi_0 \\
&- \left(\frac{15}{1024}e^6 + \frac{21}{512}e^8 + \cdots\right)\sin^6\psi_0 \\
&+ \left(\frac{49}{1096}e^8 + \cdots\right)\sin^8\psi_0 \\
&- \cdots
\end{aligned} \tag{114}
$$

$$
\begin{aligned}
C_6 = &\left(\frac{5}{2048}e^6 + \frac{7}{1024}e^8 + \cdots\right)\sin^6\psi_0 \\
&- \left(\frac{7}{2048}e^8 + \cdots\right)\sin^8\psi_0 \\
&+ \cdots
\end{aligned} \tag{115}
$$

$$
C_8 = \left(\frac{7}{16384}e^8 + \cdots\right)\sin^8\psi_0 - \cdots \tag{116}
$$

The longitude differences (spherical ω minus geodetic λ) are given by the integral

$$
\Delta\omega - \Delta\lambda = \frac{e^2}{2}\cos\psi_0 \int_{x=\sigma_1}^{x=\sigma_1+\sigma} \left\{C_0 + C_2\cos 2x + C_4\cos 4x + C_6\cos 6x + C_8\cos 8x + \cdots\right\} dx \tag{117}
$$

where $\Delta\omega = \omega_2 - \omega_1$ is the difference in longitudes of P_1' and P_2' on the auxiliary sphere and $\Delta\lambda = \lambda_2 - \lambda_1$ is the difference in longitudes of P_1 and P_2 on the ellipsoid.

Equation (117) has a similar form to equation (81) and the solution of the integral in equation (117) can be achieved by the same method used to solve the integral in equation (81). Hence, similarly to equation (88) and also Rapp (1981 equation (55), p. 13)

$$\Delta\omega - \Delta\lambda = \frac{e^2}{2}\cos\psi_0 \left\{ C_0\sigma + C_2\cos 2\sigma_m \sin\sigma + \frac{C_4}{2}\cos 4\sigma_m \sin 2\sigma \right. \\ \left. + \frac{C_6}{3}\cos 6\sigma_m \sin 3\sigma + \frac{C_8}{4}\cos 8\sigma_m \sin 4\sigma + \cdots \right\} \tag{118}$$

Rainsford (1955, p. 14, equations 10 and 11) has the differences in longitudes $\Delta\omega - \Delta\lambda$ as a function of the flattening f and the azimuth of the geodesic at the equator α_E; noting that from either equations (61) or (69) we may obtain the relationships

$$\sin\alpha_E = \cos\psi_0 \tag{119}$$

$$1 - \sin^2\alpha_E = \sin^2\psi_0 \tag{120}$$

Also, since $e^2 = f(2-f) = 2f - f^2$, even powers of the eccentricity e can be expressed as functions of the flattening f

$$\begin{aligned} e^2 &= 2f - f^2 \\ e^4 &= 4f^2 - 4f^3 + f^4 \\ e^6 &= 8f^3 - 12f^4 + 6f^5 - f^6 \\ e^8 &= 16f^4 - 32f^5 + 24f^6 - 8f^7 + f^8 \end{aligned} \tag{121}$$

Re-arranging equation (118) and using equation (119) gives

$$\Delta\omega - \Delta\lambda = \sin\alpha_E \left\{ \frac{e^2}{2}C_0\sigma + \frac{e^2}{2}C_2\cos 2\sigma_m \sin\sigma + \frac{e^2}{4}C_4\cos 4\sigma_m \sin 2\sigma \right. \\ \left. + \frac{e^2}{6}C_6\cos 6\sigma_m \sin 3\sigma + \frac{e^2}{8}C_8\cos 8\sigma_m \sin 4\sigma + \cdots \right\} \tag{122}$$

Now, with equations (112) and (120) the coefficient $\frac{e^2}{2}C_0$ can be written as

$$\begin{aligned} \frac{e^2}{2}C_0 = {} & \frac{e^2}{2} + \frac{1}{8}e^4 + \frac{1}{16}e^6 + \frac{5}{128}e^8 + \cdots \\ & -\left(\frac{1}{16}e^4 + \frac{1}{16}e^6 + \frac{15}{256}e^8 \cdots\right)\left(1 - \sin^2\alpha_E\right) \\ & +\left(\frac{3}{128}e^6 + \frac{45}{1024}e^8 + \cdots\right)\left(1 - \sin^2\alpha_E\right)^2 \\ & -\left(\frac{25}{2048}e^8 + \cdots\right)\left(1 - \sin^2\alpha_E\right)^3 \\ & + \cdots \end{aligned} \tag{123}$$

noting here that terms greater than e^8 have been ignored.

Using equations (121) in equation (123) with the trigonometric identity $\cos^2\alpha_E + \sin^2\alpha_E = 1$ gives

$$\begin{aligned}\frac{e^2}{2}C_0 = f - \frac{7}{8}f^5 + \cdots \\ &-\left(\frac{1}{4}f^2 + \frac{1}{4}f^3 + \frac{1}{4}f^4 - \frac{3}{2}f^5 + \cdots\right)\cos^2\alpha_E \\ &+\left(\frac{3}{16}f^3 + \frac{27}{64}f^4 - \frac{81}{64}f^5 + \cdots\right)\cos^4\alpha_E \\ &-\left(\frac{25}{128}f^4 - \frac{25}{64}f^5 + \cdots\right)\cos^6\alpha_E \\ &+\cdots\end{aligned} \tag{124}$$

Now for any geodetic ellipsoid $e^8 \simeq 2.01\text{e-}009$ and $f^4 \simeq 1.26\text{e-}010$, and since terms greater than e^8 have been ignored in the development of equation (123) then no additional errors will be induced by ignoring terms greater than f^4 in equation (124). Hence we define

$$\begin{aligned}\frac{e^2}{2}C_0 \equiv f\Big\{1 - \frac{1}{4}f\left(1 + f + f^2\right)\cos^2\alpha_E \\ &+\frac{3}{16}f^2\left(1 + \frac{9}{4}f\right)\cos^4\alpha_E \\ &-\frac{25}{128}f^3\cos^6\alpha_E\Big\}\end{aligned} \tag{125}$$

Using similar reasoning we also define

$$\frac{e^2}{2}C_2 \equiv f\left\{\frac{1}{4}f\left(1 + f + f^2\right)\cos^2\alpha_E - \frac{1}{4}f^2\left(1 + \frac{9}{4}f\right)\cos^4\alpha_E + \frac{75}{256}f^3\cos^6\alpha_E\right\} \tag{126}$$

$$\frac{e^2}{4}C_4 \equiv f\left\{\frac{1}{32}f^2\left(1 + \frac{9}{4}f\right)\cos^4\alpha_E - \frac{15}{256}f^3\cos^6\alpha_E\right\} \tag{127}$$

$$\frac{e^2}{6}C_6 \equiv f\left\{\frac{5}{768}f^3\cos^6\alpha_E\right\} \tag{128}$$

Using equations (125) to (128) enables equation (122) to be approximated by

$$\boxed{\Delta\omega - \Delta\lambda = f\sin\alpha_E\left\{A_0\sigma + A_2\cos 2\sigma_m\sin\sigma + A_4\cos 4\sigma_m\sin 2\sigma + A_6\cos 6\sigma_m\sin 3\sigma\right\}} \tag{129}$$

where $\Delta\omega = \omega_2 - \omega_1$ is the difference in longitudes of P_1' and P_2' on the auxiliary sphere and $\Delta\lambda = \lambda_2 - \lambda_1$ is the difference in longitudes of P_1 and P_2 on the ellipsoid, and the coefficients are

$$
\begin{aligned}
A_0 &= 1-\frac{1}{4}f\left(1+f+f^2\right)\cos^2\alpha_E+\frac{3}{16}f^2\left(1+\frac{9}{4}f\right)\cos^4\alpha_E-\frac{25}{128}f^3\cos^6\alpha_E \\
A_2 &= \frac{1}{4}f\left(1+f+f^2\right)\cos^2\alpha_E-\frac{1}{4}f^2\left(1+\frac{9}{4}f\right)\cos^4\alpha_E+\frac{75}{256}f^3\cos^6\alpha_E \\
A_4 &= \frac{1}{32}f^2\left(1+\frac{9}{4}f\right)\cos^4\alpha_E-\frac{15}{256}f^3\cos^6\alpha_E \\
A_6 &= \frac{5}{768}f^3\cos^6\alpha_E
\end{aligned}
\tag{130}
$$

The approximation (129) and the coefficients (130) are the same as Rainsford (1955, equations 10 and 11, p. 14) and also Rapp (1981, equation 56, p. 13).

Equation (129) can be used in two ways which will be discussed in detail later. Briefly, however, the first way is in the direct problem – after σ (and σ_m from $2\sigma_m = 2\sigma_1 + \sigma$) has been solved iteratively – to compute the difference $\Delta\omega - \Delta\lambda$. And in the inverse problem to compute the longitude difference iteratively.

VINCENTY'S MODIFICATIONS OF RAINSFORD'S EQUATIONS

In 1975, T. Vincenty (1975) produced other forms of equations (91) and (129) more suited to computer evaluation and requiring a minimum of trigonometric function evaluations. These equations may be obtained in the following manner.

Vincenty's modification of Rainsford's equation for distance

The starting point here is equation (91) [Rainsford's equation for distance] that can be re-arranged as

$$
\begin{aligned}
\sigma &= \frac{s}{bB_0}-\frac{B_2}{B_0}\cos 2\sigma_m\sin\sigma-\frac{B_4}{B_0}\cos 4\sigma_m\sin 2\sigma-\frac{B_6}{B_0}\cos 6\sigma_m\sin 3\sigma \\
&\quad -\frac{B_8}{B_0}\cos 8\sigma_m\sin 4\sigma
\end{aligned}
\tag{131}
$$

or

$$
\sigma = \frac{s}{bB_0}+\Delta\sigma \tag{132}
$$

where

$$\Delta\sigma = -\frac{B_2}{B_0}\cos 2\sigma_m \sin\sigma - \frac{B_4}{B_0}\cos 4\sigma_m \sin 2\sigma - \frac{B_6}{B_0}\cos 6\sigma_m \sin 3\sigma$$
$$-\frac{B_8}{B_0}\cos 8\sigma_m \sin 4\sigma \tag{133}$$

Now, from equations (92) $B_0 = 1 + \frac{1}{4}u^2 - \frac{3}{64}u^4 + \frac{5}{256}u^6 - \frac{175}{16384}u^8 = 1 + x$ and $\frac{1}{B_0} = (1+x)^{-1}$. Using a special case of the binomial series [equation (72) with $\beta = -1$ and with $|x| < 1$]

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots$$

allows us to write

$$\begin{aligned}\frac{1}{B_0} &= 1 - \left(\frac{1}{4}u^2 - \frac{3}{64}u^4 + \cdots\right) + \left(\frac{1}{4}u^2 - \frac{3}{64}u^4 + \cdots\right)^2 - \left(\frac{1}{4}u^2 - \frac{3}{64}u^4 + \cdots\right)^3 \\ &\quad + \left(\frac{1}{4}u^2 - \frac{3}{64}u^4 + \cdots\right)^4 - \cdots \\ &= 1 - \frac{1}{4}u^2 + \frac{7}{64}u^4 - \frac{15}{256}u^6 + \frac{579}{16384}u^8 - \cdots\end{aligned}$$

and using this result gives

$$\begin{aligned}\frac{B_2}{B_0} &= \left(-\frac{1}{4}u^2 + \frac{1}{16}u^4 - \frac{15}{512}u^6 + \frac{35}{2048}u^8 - \cdots\right)\left(1 - \frac{1}{4}u^2 + \frac{7}{64}u^4 - \frac{15}{256}u^6 + \frac{579}{16384}u^8 - \cdots\right) \\ &= -\frac{1}{4}u^2 + \frac{1}{8}u^4 - \frac{37}{512}u^6 + \frac{47}{1024}u^8 - \cdots\end{aligned}$$

Similarly, the other ratios are obtained and

$$\begin{aligned}\frac{B_2}{B_0} &= -\frac{1}{4}u^2 + \frac{1}{8}u^4 - \frac{37}{512}u^6 + \frac{47}{1024}u^8 - \cdots \\ \frac{B_4}{B_0} &= -\frac{1}{128}u^4 + \frac{1}{128}u^6 - \frac{27}{4096}u^8 + \cdots \\ \frac{B_6}{B_0} &= -\frac{1}{1536}u^6 + \frac{1}{1024}u^8 - \cdots \\ \frac{B_8}{B_0} &= -\frac{5}{65536}u^8 + \cdots\end{aligned} \tag{134}$$

For a geodesic on the GRS80 ellipsoid, having $\alpha_E = 0^\circ$ (which makes u^2 a maximum) and with $\sigma = 22.5^\circ$, $\sigma_m = 22.5^\circ$ (which makes $\cos 8\sigma_m \sin 4\sigma = 1$) the maximum value of the last term in equations (131) and (133) is $\frac{B_8}{B_0}\cos 8\sigma_m \sin 4\sigma = 1.5739827\text{e-}013$ radians.

This is equivalent to an arc length of 0.000001 m on a sphere of radius 6378137 m. This term will be ignored and $\Delta\sigma$ is defined as

$$\Delta\sigma \equiv -\frac{B_2}{B_0}\cos 2\sigma_m \sin\sigma - \frac{B_4}{B_0}\cos 4\sigma_m \sin 2\sigma - \frac{B_6}{B_0}\cos 6\sigma_m \sin 3\sigma \tag{135}$$

Now, using the trigonometric identities

$$\sin 2A = 2\sin A\cos A \qquad \cos 2A = 2\cos^2 A - 1$$
$$\sin 3A = 3\sin A - 4\sin^3 A \qquad \cos 3A = 4\cos^3 A - 3\cos A$$

then

$$\cos 4A = 2\cos^2 2A - 1$$
$$\cos 6A = 4\cos^3 2A - 3\cos 2A$$

and using these identities in equation (135) gives

$$\begin{aligned}\Delta\sigma = &-\frac{B_2}{B_0}\cos 2\sigma_m \sin\sigma - \frac{B_4}{B_0}\left(2\cos^2 2\sigma_m - 1\right)\left(2\sin\sigma\cos\sigma\right)\\ &-\frac{B_6}{B_0}\left(4\cos^3 2\sigma_m - 3\cos 2\sigma_m\right)\left(3\sin\sigma - 4\sin^3\sigma\right)\end{aligned}$$

which may be written as

$$\begin{aligned}\Delta\sigma = \sin\sigma\Big\{&-\frac{B_2}{B_0}\cos 2\sigma_m - 2\frac{B_4}{B_0}\cos\sigma\left(2\cos^2 2\sigma_m - 1\right)\\ &-\frac{B_6}{B_0}\cos 2\sigma_m\left(3 - 4\sin^2\sigma\right)\left(4\cos^2 2\sigma_m - 3\right)\Big\}\end{aligned} \tag{136}$$

Now

$$\begin{aligned}\left(\frac{-B_2}{B_0}\right)^2 &= \frac{1}{16}u^4 - \frac{1}{16}u^6 + \frac{53}{1024}u^8 - \cdots\\ \left(\frac{-B_2}{B_0}\right)^3 &= \frac{1}{64}u^6 - \frac{3}{128}u^8 + \cdots\end{aligned} \tag{137}$$

Comparing equations (137) with equations (134) we have

$$\begin{aligned}-2\left(\frac{B_4}{B_0}\right) &= \frac{1}{64}u^4 - \frac{1}{64}u^6 + \frac{54}{4096}u^8\\ \frac{1}{4}\left(\frac{-B_2}{B_0}\right)^2 &= \frac{1}{64}u^4 - \frac{1}{64}u^6 + \frac{53}{4096}u^8\end{aligned}$$

and these two equations differ by $\frac{1}{4096}u^8$ which would be equivalent to a maximum error of 5.0367e-013 radians or 0.000003 m on a sphere of radius 6378137 m. Ignoring this small difference, we define

$$-2\left(\frac{B_4}{B_0}\right) \equiv \frac{1}{4}\left(\frac{-B_2}{B_0}\right)^2 \tag{138}$$

Again, comparing equations (137) with equations (134) we have

$$\begin{aligned} -\left(\frac{B_6}{B_0}\right) &= \frac{1}{1536}u^6 + \frac{1}{1024}u^8 \\ \frac{1}{24}\left(\frac{-B_2}{B_0}\right)^2 &= \frac{1}{1536}u^6 + \frac{3}{3072}u^8 \end{aligned}$$

and noting that $\frac{1}{1024}u^8 = \frac{3}{3072}u^8$ we may say

$$-\left(\frac{B_6}{B_0}\right) = \frac{1}{24}\left(\frac{-B_2}{B_0}\right)^3 \tag{139}$$

Using equations (138) and (139) we may write equation (136) as

$$\begin{aligned} \Delta\sigma = \sin\sigma \Bigg\{ & \left(\frac{-B_2}{B_0}\right)\cos 2\sigma_m + \frac{1}{4}\left(\frac{-B_2}{B_0}\right)^2 \cos\sigma\left(2\cos^2 2\sigma_m - 1\right) \\ & + \frac{1}{24}\left(\frac{-B_2}{B_0}\right)^3 \cos 2\sigma_m\left(3 - 4\sin^2\sigma\right)\left(4\cos^2 2\sigma_m - 3\right)\Bigg\} \end{aligned}$$

We may now express the great circle arc length σ as

$$\boxed{\sigma = \frac{s}{bA'} + \Delta\sigma} \tag{140}$$

where

$$\boxed{\begin{aligned} \Delta\sigma = B'\sin\sigma \Big\{ & \cos 2\sigma_m + \frac{1}{4}B'\Big[\cos\sigma\left(2\cos^2 2\sigma_m - 1\right) \\ & - \frac{1}{6}B'\cos 2\sigma_m\left(-3 + 4\sin^2\sigma\right)\left(-3 + 4\cos^2 2\sigma_m\right)\Big]\Big\} \end{aligned}} \tag{141}$$

and

$$
\begin{aligned}
A' = B_0 &= 1 + \frac{1}{4}u^2 - \frac{3}{64}u^4 + \frac{5}{256}u^6 - \frac{175}{16384}u^8 \\
&= 1 + \frac{4096}{16384}u^2 - \frac{768}{16384}u^4 + \frac{320}{16384}u^6 - \frac{175}{16384}u^8 \\
&= 1 + \frac{u^2}{16384}\Big(4096 + u^2\left(-768 + u^2\left(320 - 175u^2\right)\right)\Big) \qquad (142)
\end{aligned}
$$

$$
\begin{aligned}
B' = \frac{-B_2}{B_0} &= \frac{1}{4}u^2 - \frac{1}{8}u^4 + \frac{37}{512}u^6 - \frac{47}{1024}u^8 \\
&= \frac{256}{1024}u^2 - \frac{128}{1024}u^4 + \frac{74}{1024}u^6 - \frac{47}{1024}u^8 \\
&= \frac{u^2}{1024}\Big(256 + u^2\left(-128 + u^2\left(74 - 47u^2\right)\right)\Big) \qquad (143)
\end{aligned}
$$

Equations (140) to (143) are the same as those given by Vincenty (1975, equations 7, 6, 3 and 4, p. 89). Vincenty notes in his paper that these equations were derived from Rainsford's inverse formula and that most significant terms in u^8 were retained, but he gave no outline of his method.

Vincenty's modification of Rainsford's equation for longitude difference

The starting point here is equation (129) [Rainsford's equation for longitude differences] with coefficients A_0, A_2, A_4 and A_6. Referring to this equation, Rainsford (1955, p. 14) states:

"The A coefficients are given as functions of f since they converge more rapidly than when given as functions of e^2. The maximum value of any term in f^4 (i.e. f^3 in the $A's$) is less than $0''.00001$ even for a line half round the world. Thus the A_6 term may be omitted altogether and the following simplified forms used even for precise results:"

Rainsford's simplified formula is

$$
\Delta\omega - \Delta\lambda = f \sin\alpha_E \left\{A_0'\sigma + A_2'\cos 2\sigma_m \sin\sigma + A_4'\cos 4\sigma_m \sin 2\sigma\right\} \qquad (144)
$$

where $\Delta\omega = \omega_2 - \omega_1$ is the difference in longitudes of P_1' and P_2' on the auxiliary sphere and $\Delta\lambda = \lambda_2 - \lambda_1$ is the difference in longitudes of P_1 and P_2 on the ellipsoid, and the coefficients are

$$
\begin{aligned}
A_0' &= 1 - \frac{1}{4} f(1+f)\cos^2\alpha_E - \frac{3}{16} f^2 \cos^4 \alpha_E \\
A_2' &= \frac{1}{4} f(1+f)\cos^2\alpha_E - \frac{1}{4} f^2 \cos^4 \alpha_E \\
A_4' &= \frac{1}{32} f^2 \cos^4 \alpha_E
\end{aligned} \tag{145}
$$

Equation (144) can be written as

$$
\Delta\omega - \Delta\lambda = A_0' f \sin\alpha_E \left\{ \sigma + \frac{A_2'}{A_0'} \cos 2\sigma_m \sin\sigma + \frac{A_4'}{A_0'} \cos 4\sigma_m \sin 2\sigma \right\} \tag{146}
$$

Using the trigonometric double angle formulas $\sin 2A = 2\sin A \cos A$, $\cos 2A = 2\cos^2 A - 1$ we can write

$$
\begin{aligned}
\sin 2\sigma &= 2\sin\sigma\cos\sigma \\
\cos 4\sigma_m &= 2\cos^2 2\sigma_m - 1
\end{aligned}
$$

and equation (146) becomes

$$
\begin{aligned}
\Delta\omega - \Delta\lambda &= A_0' f \sin\alpha_E \left\{ \sigma + \frac{A_2'}{A_0'} \cos 2\sigma_m \sin\sigma + \frac{A_4'}{A_0'} \left(2\cos^2 2\sigma_m - 1\right)\left(2\sin\sigma\cos\sigma\right) \right\} \\
&= A_0' f \sin\alpha_E \left\{ \sigma + \sin\sigma \left[\frac{A_2'}{A_0'} \cos 2\sigma_m + 2\frac{A_4'}{A_0'} \cos\sigma \left(2\cos^2 2\sigma_m - 1\right) \right] \right\}
\end{aligned} \tag{147}
$$

Now the coefficient A_0' may be re-arranged as follows

$$
\begin{aligned}
A_0' &= 1 - \frac{1}{4} f(1+f)\cos^2\alpha_E + \frac{3}{16} f^2 \cos^4 \alpha_E \\
&= 1 - \left(\frac{4}{16} f(1+f)\cos^2\alpha_E - \frac{3}{16} f^2 \cos^4 \alpha_E \right) \\
&= 1 - \frac{f}{16} \cos^2\alpha_E \left(4(1+f) - 3f\cos^2\alpha_E\right) \\
&= 1 - \frac{f}{16} \cos^2\alpha_E \left(4 + f\left(4 - 3\cos^2\alpha_E\right)\right)
\end{aligned}
$$

or

$$
A_0' = 1 - C
$$

where

$$
C = \frac{f}{16} \cos^2\alpha_E \left(4 + f\left(4 - 3\cos^2\alpha_E\right)\right)
$$

Now using these relationships and a special result of the binomial series [equation (72) with $x = -C$ and $\beta = -1$] we may write

$$\frac{1}{A_0'} = \frac{1}{1-C} = (1-C)^{-1} = 1 + C + C^2 + C^3 + \cdots$$

and

$$\frac{A_2'}{A_0'} = \frac{1}{4} f \cos^2 \alpha_E + \frac{1}{4} f^2 \cos^2 \alpha_E - \frac{3}{16} f^2 \cos^4 \alpha_E + \frac{1}{8} f^3 \cos^4 \alpha_E + \cdots$$

Ignoring terms greater than f^3 (greater than f^2 in $\frac{A_2'}{A_0'}$) we have

$$\begin{aligned} \frac{A_2'}{A_0'} &\equiv \frac{1}{4} f \cos^2 \alpha_E + \frac{1}{4} f^2 \cos^2 \alpha_E - \frac{3}{16} f^2 \cos^4 \alpha_E \\ &= \frac{f}{16} \cos^2 \alpha_E \left(4 + f\left(4 - 3\cos^2 \alpha_E\right)\right) \\ &= C \end{aligned}$$

Also

$$\frac{A_4'}{A_0'} = \frac{1}{32} f^2 \cos^4 \alpha_E + \frac{1}{128} f^3 \cos^6 \alpha_E + \cdots$$

and ignoring terms greater than f^3 (greater than f^2 in $\frac{A_4'}{A_0'}$) we have

$$\frac{A_4'}{A_0'} \equiv \frac{1}{32} f^2 \cos^4 \alpha_E \quad \text{and} \quad 2\frac{A_4'}{A_0'} = \frac{1}{16} f^2 \cos^4 \alpha_E$$

Now

$$C^2 = \frac{1}{16} f^2 \cos^4 \alpha_E + \frac{1}{8} f^3 \cos^4 \alpha_E - \frac{3}{32} f^3 \cos^6 \alpha_E + \cdots$$

and ignoring terms greater than f^3 (greater than f^2 in C^2) we have

$$C^2 \equiv \frac{1}{16} f^2 \cos^4 \alpha_E = 2\frac{A_4'}{A_0'}$$

Using these results we may write equation (147) as

$$\boxed{\Delta\lambda = \Delta\omega - (1-C) f \sin\alpha_E \left\{\sigma + C \sin\sigma\left[\cos 2\sigma_m + C\cos\sigma\left(-1 + 2\cos^2 2\sigma_m\right)\right]\right\} \qquad (148)}$$

where $\Delta\omega = \omega_2 - \omega_1$ is the difference in longitudes of P_1' and P_2' on the auxiliary sphere and $\Delta\lambda = \lambda_2 - \lambda_1$ is the difference in longitudes of P_1 and P_2 on the ellipsoid, and

$$C = \frac{f}{16} \cos^2 \alpha_E \left(4 + f\left(4 - 3\cos^2 \alpha_E\right)\right) \tag{149}$$

Equations (148) and (149) are essentially the same as Vincenty (1975, equations 11 and 10, p.89) – Vincenty uses L and λ where we have used $\Delta\lambda$ and $\Delta\omega$ respectively – although he gives no outline of his method of deriving his equations from Rainsford's.

SOLVING THE DIRECT AND INVERSE PROBLEMS ON THE ELLIPSOID USING VINCENTY'S EQUATIONS

Vincenty (1975) set out methods of solving the direct and inverse problems on the ellipsoid. His methods were different from those proposed by Rainsford (1955) even though his equations (140) to (143) for spherical arc length σ and (148) and (149) for longitude λ were simplifications of Rainsford's equations. His approach was to develop solutions more applicable to computer programming rather than the mechanical methods used by Rainsford. Vincenty's method relies upon the auxiliary sphere and there are several equations using spherical trigonometry. Since distances are often small when compared with the Earth's circumference, resulting spherical triangles can have very small sides and angles. In such cases, usual spherical trigonometry formula, e.g., sine rule and cosine rule, may not furnish accurate results and other, less common formula, are used. Vincenty's equations and his methods are now widely used in geodetic computations.

In the solutions of the direct and inverse problems set out in subsequent sections, the following notation and relationships are used.

a, f semi-major axis length and flattening of ellipsoid.

b semi-minor axis length of the ellipsoid, $b = a(1-f)$

e^2 eccentricity of ellipsoid squared, $e^2 = f(2-f)$

e'^2 2nd-eccentricity of ellipsoid squared, $e'^2 = \dfrac{e^2}{1-e^2}$

ϕ, λ latitude and longitude on ellipsoid: ϕ measured 0° to $\pm 90^\circ$ (north latitudes positive and south latitudes negative) and λ measured 0° to $\pm 180^\circ$ (east longitudes positive and west longitudes negative).

s length of the geodesic on the ellipsoid.

α_1, α_2 azimuths of the geodesic, clockwise from north 0° to 360°; α_2 in the direction P_1P_2 produced.

α_{12} azimuth of geodesic P_1P_2; $\alpha_{12} = \alpha_1$

α_{21} reverse azimuth; azimuth of geodesic P_2P_1; $\alpha_{21} = \alpha_2 \pm 180^\circ$

α_E azimuth of geodesic at the equator, $\sin\alpha_E = \cos\psi_0$

u^2 $= e'^2 \sin^2\psi_0$

ψ parametric latitude, $\tan\psi = (1-f)\tan\phi$

ψ_0 parametric latitude of geodesic vertex, $\cos\psi_0 = \cos\psi\sin\alpha = \sin\alpha_E$

ψ,ω latitude and longitude on auxiliary sphere: ψ measured 0° to $\pm 90^\circ$ (north latitudes positive and south latitudes negative) and ω measured 0° to $\pm 180^\circ$ (east longitudes positive and west longitudes negative).

$\Delta\lambda,\Delta\omega$ longitude differences; $\Delta\lambda = \lambda_2 - \lambda_1$ (ellipsoid) and $\Delta\omega = \omega_2 - \omega_1$ (spherical)

σ angular distance (great circle arc) $P_1'P_2'$ on the auxiliary sphere.

σ_1 angular distance from equator to P_1' on the auxiliary sphere, $\tan\sigma_1 = \dfrac{\tan\psi_1}{\cos\alpha_1}$

σ_m angular distance from equator to mid-point of great circle arc $P_1'P_2'$ on the auxiliary sphere, $2\sigma_m = 2\sigma_1 + \sigma$

THE DIRECT PROBLEM ON THE ELLIPSOID USING VINCENTY'S EQUATIONS

Using Vincenty's equations the direct problem on the ellipsoid

> [given latitude and longitude of P_1 on the ellipsoid and azimuth α_{12} and geodesic distance s to P_2 on the ellipsoid, compute the latitude and longitude of P_2 and the reverse azimuth α_{21}]

may be solved by the following sequence.

With the ellipsoid constants a, f, $b = a(1-f)$, $e^2 = f(2-f)$ and $e'^2 = \dfrac{e^2}{1-e^2}$ and given $\phi_1, \lambda_1, \alpha_1 = \alpha_{12}$ and s

1. Compute parametric latitude ψ_1 of P_1 from

$$\tan\psi_1 = (1-f)\tan\phi_1$$

2. Compute the parametric latitude of the geodesic vertex ψ_0 from

$$\cos\psi_0 = \cos\psi_1 \sin\alpha_1$$

3. Compute the geodesic constant u^2 from

$$u^2 = e'^2 \sin^2\psi_0$$

4. Compute angular distance σ_1 on the auxiliary sphere from the equator to P_1' from

$$\tan\sigma_1 = \frac{\tan\psi_1}{\cos\alpha_1}$$

5. Compute the azimuth of the geodesic at the equator α_E from

$$\sin\alpha_E = \cos\psi_0 = \cos\psi_1 \sin\alpha_1$$

6. Compute Vincenty's constants A' and B' from

$$\begin{aligned}
A' &= 1 + \frac{u^2}{16384}\left(4096 + u^2\left(-768 + u^2\left(320 - 175u^2\right)\right)\right) \\
B' &= \frac{u^2}{1024}\left(256 + u^2\left(-128 + u^2\left(74 - 47u^2\right)\right)\right)
\end{aligned}$$

7. Compute angular distance σ on the auxiliary sphere from P_1' to P_2' by iteration using the following sequence of equations until there is negligible change in σ

$$\begin{aligned}
2\sigma_m &= 2\sigma_1 + \sigma \\
\Delta\sigma &= B'\sin\sigma\left\{\cos 2\sigma_m + \frac{1}{4}B'\left[\cos\sigma\left(2\cos^2 2\sigma_m - 1\right)\right.\right. \\
&\qquad \left.\left. - \frac{1}{6}B'\cos 2\sigma_m\left(-3 + 4\sin^2\sigma\right)\left(-3 + 4\cos^2 2\sigma_m\right)\right]\right\} \\
\sigma &= \frac{s}{bA'} + \Delta\sigma
\end{aligned}$$

The first approximation for σ in this iterative solution can be taken as $\sigma \simeq \dfrac{s}{bA'}$

8. After computing the spherical arc length σ the latitude of P_2 can be computed using spherical trigonometry and the relationship $\tan\phi_2 = \dfrac{\tan\psi_2}{(1-f)}$

$$\tan\phi_2 = \frac{\sin\psi_1\cos\sigma + \cos\psi_1\sin\sigma\cos\alpha_1}{(1-f)\sqrt{\sin^2\alpha_E + \left(\sin\psi_1\sin\sigma - \cos\psi_1\cos\sigma\cos\alpha_1\right)^2}}$$

9. Compute the longitude difference $\Delta\omega$ on the auxiliary sphere from

$$\tan\Delta\omega = \frac{\sin\sigma\sin\alpha_1}{\cos\psi_1\cos\sigma - \sin\psi_1\sin\sigma\cos\alpha_1}$$

10. Compute Vincenty's constant C from

$$C = \frac{f}{16}\cos^2\alpha_E\left(4 + f\left(4 - 3\cos^2\alpha_E\right)\right)$$

11. Compute the longitude difference $\Delta\lambda$ on the ellipsoid from

$$\Delta\lambda = \Delta\omega - (1 - C)\, f\sin\alpha_E\left\{\sigma + C\sin\sigma\left[\cos 2\sigma_m + C\cos\sigma\left(-1 + 2\cos^2 2\sigma_m\right)\right]\right\}$$

12. Compute azimuth α_2 from

$$\tan\alpha_2 = \frac{\sin\alpha_E}{\cos\psi_1\cos\sigma\cos\alpha_1 - \sin\psi_1\sin\sigma}$$

13. Compute reverse azimuth α_{21}

$$\alpha_{21} = \alpha_2 \pm 180^\circ$$

Shown below is the output of a MATLAB function *Vincenty_Direct.m* that solves the direct problem on the ellipsoid.

The ellipsoid is the GRS80 ellipsoid and ϕ,λ for P_1 are -45° and 132° respectively with $\alpha_{12} = 1^\circ\, 43'\, 25.876544''$ and $s = 3880\,275.684153$ m . ϕ,λ <u>computed</u> for P_2 are -10° and 133° respectively with the reverse azimuth $\alpha_{21} = 181^\circ\, 14'\, 22.613213''$

```
>> Vincenty_Direct

//////////////////////////////////////////////////
// DIRECT CASE on ellipsoid: Vincenty's method //
//////////////////////////////////////////////////

ellipsoid parameters
a    =  6378137.000000000
f    = 1/298.257222101000
b    =  6356752.314140356100
e2   =  6.694380022901e-003
ep2  =  6.739496775479e-003

Latitude & Longitude of P1
latP1 = -45  0  0.000000 (D M S)
lonP1 = 132  0  0.000000 (D M S)

Azimuth & Distance P1-P2
az12 =    1 43 25.876544 (D M S)
s    =    3880275.684153

Parametric Latitude of P1
psiP1 = -44 54 13.636256 (D M S)

Parametric Latitude of vertex P0
psiP0 =  88 46 44.750547 (D M S)

Geodesic constant u2 (u-squared)
u2 =  6.736437077728e-003
```

```
angular distance on auxiliary sphere from equator to P1'
sigma1 = -7.839452835875e-001 radians

Vincenty's constants A and B
A =  1.001681988050e+000
B =  1.678458818215e-003

angular distance sigma on auxiliary sphere from P1' to P2'
sigma =  6.099458753810e-001 radians
iterations =  5

Latitude of P2
latP2 = -10  0  0.000000 (D M S)

Vincenty's constant C
C =  8.385253517062e-004

Longitude difference P1-P2
dlon =   1  0  0.000000 (D M S)

Longitude of P2
lon2 = 133  0  0.000000 (D M S)

Reverse azimuth
alpha21 = 181 14 22.613213 (D M S)

>>
```

THE INVERSE PROBLEM ON THE ELLIPSOID USING VINCENTY'S EQUATIONS

Using Vincenty's equations the inverse problem on the ellipsoid

> [given latitudes and longitudes of P_1 and P_2 on the ellipsoid compute the forward and reverse azimuths α_{12} and α_{21} and the geodesic distance s]

may be solved by the following sequence.

With the ellipsoid constants a, f, $b = a(1-f)$, $e^2 = f(2-f)$ and $e'^2 = \dfrac{e^2}{1-e^2}$ and given ϕ_1, λ_1 and ϕ_2, λ_2

1. Compute parametric latitudes ψ_1 and ψ_2 of P_1 and P_2 from

$$\tan\psi = (1-f)\tan\phi$$

2. Compute the longitude difference $\Delta\lambda$ on the ellipsoid

$$\Delta\lambda = \lambda_2 - \lambda_1$$

3. Compute the longitude difference $\Delta\omega$ on the auxiliary sphere between P_1' to P_2' by iteration using the following sequence of equations until there is negligible change in $\Delta\omega$. Note that σ should be computed using the atan2 function after evaluating $\sin\sigma = \sqrt{\sin^2\sigma}$ and $\cos\sigma$. This will give $-180^\circ < \sigma \le 180^\circ$.

$$\sin^2\sigma = \left(\cos\psi_2 \sin\Delta\omega\right)^2 + \left(\cos\psi_1 \sin\psi_2 - \sin\psi_1 \cos\psi_2 \cos\Delta\omega\right)^2$$
$$\cos\sigma = \sin\psi_1 \sin\psi_2 + \cos\psi_1 \cos\psi_2 \cos\Delta\omega$$
$$\tan\sigma = \frac{\sin\sigma}{\cos\sigma}$$
$$\sin\alpha_E = \frac{\cos\psi_1 \cos\psi_2 \sin\Delta\omega}{\sin\sigma}$$
$$\cos 2\sigma_m = \cos\sigma - \frac{2\sin\psi_1 \sin\psi_2}{\cos^2\alpha_E}$$
$$C = \frac{f}{16}\cos^2\alpha_E\left(4 + f\left(4 - 3\cos^2\alpha_E\right)\right)$$
$$\Delta\omega = \Delta\lambda + (1 - C) f \sin\alpha_E \left\{\sigma + C\sin\sigma\left[\cos 2\sigma_m + C\cos\sigma\left(-1 + 2\cos^2 2\sigma_m\right)\right]\right\}$$

The first approximation for $\Delta\omega$ in this iterative solution can be taken as $\Delta\omega \simeq \Delta\lambda$

4. Compute the parametric latitude of the geodesic vertex ψ_0 from

$$\cos\psi_0 = \sin\alpha_E$$

5. Compute the geodesic constant u^2 from

$$u^2 = e'^2 \sin^2\psi_0$$

6. Compute Vincenty's constants A' and B' from

$$A' = 1 + \frac{u^2}{16384}\left(4096 + u^2\left(-768 + u^2\left(320 - 175u^2\right)\right)\right)$$
$$B' = \frac{u^2}{1024}\left(256 + u^2\left(-128 + u^2\left(74 - 47u^2\right)\right)\right)$$

7. Compute geodesic distance s from

$$\Delta\sigma = B'\sin\sigma\left\{\cos 2\sigma_m + \frac{1}{4}B'\left[\cos\sigma\left(2\cos^2 2\sigma_m - 1\right)\right.\right.$$
$$\left.\left. - \frac{1}{6}B'\cos 2\sigma_m\left(-3 + 4\sin^2\sigma\right)\left(-3 + 4\cos^2 2\sigma_m\right)\right]\right\}$$
$$s = bA(\sigma - \Delta\sigma)$$

8. Compute the forward azimuth $\alpha_{12} = \alpha_1$ from

$$\tan\alpha_1 = \frac{\cos\psi_2 \sin\Delta\omega}{\cos\psi_1 \sin\psi_2 - \sin\psi_1 \cos\psi_2 \cos\Delta\omega}$$

9. Compute azimuth α_2 from

$$\tan\alpha_2 = \frac{\cos\psi_1 \sin\Delta\omega}{-\sin\psi_1\cos\psi_2 + \cos\psi_1\sin\psi_2\cos\Delta\omega}$$

10. Compute reverse azimuth α_{21}

$$\alpha_{21} = \alpha_2 \pm 180^\circ$$

Shown below is the output of a MATLAB function *Vincenty_Inverse.m* that solves the inverse problem on the ellipsoid.

The ellipsoid is the GRS80 ellipsoid. ϕ,λ for P_1 are -10° and 110° respectively and ϕ,λ for P_2 are -45° and 155° respectively. Computed azimuths are $\alpha_{12} = 140^\circ\, 30'\, 03.017703''$ and $\alpha_{21} = 297^\circ\, 48'\, 47.310738''$, and geodesic distance $s = 5783\,228.548429$ m.

```
>> Vincenty_Inverse

///////////////////////////////////////////////////
// INVERSE CASE on ellipsoid: Vincenty's method //
///////////////////////////////////////////////////

ellipsoid parameters
a    =  6378137.000000000
f    = 1/298.257222101000
b    =  6356752.314140356100
e2   =  6.694380022901e-003
ep2  =  6.739496775479e-003

Latitude & Longitude of P1
latP1 = -10  0  0.000000 (D M S)
lonP1 = 110  0  0.000000 (D M S)

Latitude & Longitude of P2
latP2 = -45  0  0.000000 (D M S)
lonP2 = 155  0  0.000000 (D M S)

Parametric Latitudes of P1 and P2
psiP1 =  -9 58  1.723159 (D M S)
psiP2 = -44 54 13.636256 (D M S)

Longitude difference on ellipsoid P1-P2
dlon =  45  0  0.000000 (D M S)

Longitude difference on auxiliary sphere P1'-P2'
domega =  9.090186019005e-001 radians
iterations =  5

Parametric Latitude of vertex P0
psiP0 =  51 12 36.239192 (D M S)

Geodesic constant u2 (u-squared)
u2 =  4.094508823114e-003

Vincenty's constants A and B
A =  1.001022842684e+000
B =  1.021536528199e-003
```

```
Azimuth & Distance P1-P2
az12 =  140 30  3.017703 (D M S)
s    =    5783228.548429

Reverse azimuth
alpha21 = 297 48 47.310738 (D M S)

>>
```

EXCEL WORKBOOK vincenty.xls FROM GEOSCIENCE AUSTRALIA

Geoscience Australia has made available an Excel workbook **vincenty.xls** containing four spreadsheets labelled Ellipsoids, Direct Solution, Inverse Solution and Test Data. The Direct Solution and Inverse Solution spreadsheets are implementations of Vincenty's equations. The Excel workbook **vincenty.xls** can be downloaded via the Internet at the Geoscience Australia website (http://www.ga.gov.au/) following the links to Geodetic Calculations then Calculate Bearing Distance from Latitude Longitude. At this web page the spreadsheet **vincenty.xls** is available for use or downloading. Alternatively, the Intergovernmental Committee on Surveying and Mapping (ICSM) has produced an on-line publication *Geocentric Datum of Australia Technical Manual* Version 2.2 (GDA Technical Manual, ICSM 2002) with a link to **vincenty.xls**.

The operation of **vincenty.xls** is relatively simple, but since the spreadsheets use the Excel *solver* for the iterative solutions of certain equations then the Iteration box must be checked on the Calculation sheet. The Calculation sheet is found under Tools/Options on the Excel toolbar. Also, on the Calculation sheet make sure the Maximum change box has a value of 0.000000000001.

The Direct Solution and Inverse Solution spreadsheets have statements that the spreadsheets have been tested in the Australian region but not exhaustively tested worldwide.

To test **vincenty.xls**, direct and inverse solutions between points on a geographic rectangle $ABCD$ covering Australia were computed using **vincenty.xls** and MATLAB functions *Vincenty_Direct.m* and *Vincenty_Inverse.m*. Figure 16 shows the geographic rectangle $ABCD$ whose sides are the meridians of longitude 110° and 155° and parallels of latitude -10° and -45°. Several lines were chosen on and across this rectangle.

Figure 16: Geographic rectangle covering Australia

P_1	P_2	azimuth α	distance s
$\phi = -10^\circ$ $\lambda = 110^\circ$	$\phi = -10^\circ$ $\lambda = 155^\circ$	$\alpha_{12} = 94^\circ\ 06'\ 55.752\,182''$ $\alpha_{21} = 265^\circ\ 53'\ 04.247\,818''$	$s = 4\,929\,703.675\,416$ m
$\phi = -10^\circ$ $\lambda = 110^\circ$	$\phi = -45^\circ$ $\lambda = 155^\circ$	$\alpha_{12} = 140^\circ\ 30'\ 03.017\,703''$ $\alpha_{21} = 297^\circ\ 48'\ 47.310\,738''$	$s = 5\,783\,228.548\,429$ m
$\phi = -10^\circ$ $\lambda = 110^\circ$	$\phi = -45^\circ$ $\lambda = 110^\circ$	$\alpha_{12} = 180^\circ\ 00'\ 00.000\,000''$ $\alpha_{21} = 0^\circ\ 00'\ 00.000\,000''$	$s = 3\,879\,089.544\,659$ m
$\phi = -10^\circ$ $\lambda = 155^\circ$	$\phi = -45^\circ$ $\lambda = 110^\circ$	$\alpha_{12} = 219^\circ\ 29'\ 56.982\,297''$ $\alpha_{21} = 62^\circ\ 11'\ 12.689\,262''$	$s = 5\,783\,228.548\,429$ m
$\phi = -45^\circ$ $\lambda = 132^\circ$	$\phi = -10^\circ$ $\lambda = 133^\circ$	$\alpha_{12} = 1^\circ\ 43'\ 25.876\,544''$ $\alpha_{21} = 181^\circ\ 14'\ 22.613\,213''$	$s = 3\,880\,275.684\,153$ m
$\phi = -35^\circ$ $\lambda = 110^\circ$	$\phi = -36^\circ$ $\lambda = 155^\circ$	$\alpha_{12} = 105^\circ\ 00'\ 10.107\,712''$ $\alpha_{21} = 257^\circ\ 56'\ 53.869\,209''$	$s = 4\,047\,421.887\,193$ m

Table 1: Geodesic curves between P_1 and P_2 on the GRS80 ellipsoid

Table 1 shows a number of long geodesics that are either bounding meridians of the rectangle or geodesics crossing the rectangle. All of these results have been computed using the MATLAB function *Vincenty_Inverse.m* and verified by using the MATLAB function *Vincenty_Direct.m.* Each of the lines were then computed using the Inverse Solution spreadsheet of the Excel workbook **vincenty.xls**; all azimuths were identical and the differences between distances were 0.000002 m on one line and 0.000001 m on two other lines. Each of the lines were then verified by using the Direct Solution spreadsheet (all computed latitudes and longitudes we in exact agreement). It could be concluded that the Excel workbook **vincenty.xls** gives results accurate to at least the 5th decimal of distance and the 6th decimal of seconds of azimuth for any geodesic in Australia.

Vincenty (1975) verifies his equations by comparing his results with Rainsford's over five test lines (Rainsford 1955). On one of these lines – line (a) $\phi_1 = 55^\circ\, 45'$, $\lambda_1 = 0^\circ\, 00'$, $\alpha_{12} = 96^\circ\, 36'\, 08.79960''$, $s = 14\,110\,526.170$ m on Bessel's ellipsoid $a = 6377397.155$ m $1/f = 299.1528128$ – Vincenty finds his direct solution gives $\phi_2 = -33^\circ\, 26'\, 00.000012''$, $\lambda_2 = 108^\circ\, 13'\, 00.000\,007''$ and $\alpha_{21} = 137^\circ\, 52'\, 22.014\,528''$. We can confirm that the MATLAB function *Vincenty_Direct.m* also gives these results, but it is interesting to note that the Direct Solution spreadsheet of the Excel workbook **vincenty.xls** does not give these results. This is due to the Excel *solver* – used to determine a value by iteration – returning an incorrect value. Whilst the error in the Excel *solver* result is small, it is, nonetheless, significant and users should be aware of the likelihood or erroneous results over very long geodesics using **vincenty.xls**.

MATLAB FUNCTIONS

Shown below are two MATLAB functions *Vincenty_Direct.m* and *Vincenty_Inverse.m* that have been written to test Vincenty's equations and his direct and inverse methods of solution. Both functions call another function *DMS.m* that is also shown.

MATLAB function *Vincenty_ Direct.m*

```
function Vincenty_Direct
%  Vincenty_Direct computes the "direct case" on the ellipsoid using
%  Vinventy's method.
%  Given the size and shape of the ellipsoid and the latitude and
%  longitude of P1 and the azimuth and geodesic distance of P1 to P2,
%  this function computes the latitude and longitude of P2 and the
%  reverse azimuth P2 to P1.

%============================================================================
% Function:  Vincenty_Direct
%
% Useage:    Vincenty_Direct;
%
% Author:
%  Rod Deakin,
%  Department of Mathematical and Geospatial Sciences,
%  RMIT University,
%  GPO Box 2476V, MELBOURNE VIC 3001
%  AUSTRALIA
%  email: rod.deakin@rmit.edu.au
%
% Date:
%  Version 1.0   2 March 2008
%
% Functions Required:
%      [D,M,S] = DMS(DecDeg)
%
% Remarks:
%  This function computes the DIRECT CASE on the ellipsoid.  Given the size
%  and shape of an ellipsoid (defined by parameters a and f, semi-major
%  axis and flattening respectively) and the latitude and longitude of P1
%  and the azimuth (az12) P1 to P2 and the geodesic distance (s) P1 to P2,
%  the function computes the latitude and longitude of P2 and the reverse
%  azimuth (az21) P2 to P1.  Latitudes and longitudes of the geodesic
%  vertices P0 and P0' are also output as well as distances and longitude
%  difference from P1 and P2 to the relevant vertices.
%
% References:
%  [1] Deakin, R.E, and Hunter, M.N., 2007. 'Geodesics on an Ellipsoid -
%         Bessels' Method', School of Mathematical and Geospatial Sciences,
%         RMIT University, January 2007.
%  [2] Vincenty, T., 1975. 'Direct and Inverse solutions of geodesics on
%         the ellipsoid with application of nested equations', Survey
%         Review, Vol. 23, No. 176, pp.88-93, April 1975.
%
% Variables:
%  a            - semi-major axis of ellipsoid
%  A            - Vincenty's constant for computation of sigma
%  alpha1       - azimuth P1-P2 (radians)
%  az12         - azimuth P1-P2 (degrees)
%  az21         - azimuth P2-P1 (degrees)
%  b            - semi-minor axis of ellipsoid
%  A            - Vincenty's constant for computation of sigma
%  cos_alpha1   - cosine of azimuth of geodesic P1-P2 at P1
%  dlambda      - longitude difference P1 to P2 (radians)
%  domega       - longitude difference P1' to P2' (radians)
%  d2r          - degree to radian conversion factor
%  e2           - eccentricity of ellipsoid squared
%  ep2          - 2nd eccentricity squared
%  f            - flattening of ellipsoid
%  flat         - denominator of flattening, f = 1/flat
%  lambda1      - longitude of P1 (radians)
%  lambda2      - longitude of P2 (radians)
%  lat1         - latitude of P1 (degrees)
```

```
%  lat2         - latitude of P2 (degrees)
%  lon1         - longitude of P1 (degrees)
%  lon2         - longitude of P2 (degrees)
%  phi1         - latitude of P1 (radians)
%  phi2         - latitude of P2 (radians)
%  pion2        - pi/2
%  psi0         - parametric latitude of P0 (radians)
%  psi1         - parametric latitude of P1 (radians)
%  psi2         - parametric latitude of P2 (radians)
%  s            - geodesic distance P1 to P2
%  sigma1       - angular distance (radians) on auxiliary sphere from
%                 equator to P1'
%  sin_alpha1   - sine of azimuth of geodesic P1-P2 at P1
%  twopi        - 2*pi
%  u2           - geodesic constant u-squared
%
%
%==============================================================================

% Define some constants
d2r   = 180/pi;
twopi = 2*pi;
pion2 = pi/2;

% Set defining ellipsoid parameters
a    = 6378137;           % GRS80
flat = 298.257222101;
% a    = 6377397.155;         % Bessel (see Ref [2], p.91)
% flat = 299.1528128;

% Compute derived ellipsoid constants
f   = 1/flat;
b   = a*(1-f);
e2  = f*(2-f);
ep2 = e2/(1-e2);

%-----------------------------------------
% latitude and longitude of P1 (degrees)
%-----------------------------------------
 lat1 =  -45;
 lon1 =  132;

% lat and lon of P1 (radians)
phi1    = lat1/d2r;
lambda1 = lon1/d2r;

%--------------------------------------
% azimuth of geodesic P1-P2 (degrees)
%--------------------------------------
az12 = 1 + 43/60 + 25.876544/3600;
%
% azimuth of geodesic P1-P2 (radians)
alpha1 = az12/d2r;

% sine and cosine of azimuth P1-P2
sin_alpha1 = sin(alpha1);
cos_alpha1 = cos(alpha1);

%-------------------
% geodesic distance
%-------------------
s = 3880275.684153;

% [1] Compute parametric latitude psi1 of P1
psi1 = atan((1-f)*tan(phi1));

% [2] Compute parametric latitude of vertex
psi0 = acos(cos(psi1)*sin_alpha1);
```

```
% [3] Compute geodesic constant u2 (u-squared)
u2 = ep2*(sin(psi0)^2);

% [4] Compute angular distance sigma1 on the auxiliary sphere from equator
%     to P1'
sigma1 = atan2(tan(psi1),cos_alpha1);

% [5] Compute the sine of the azimuth of the geodesic at the equator
sin_alphaE = cos(psi0);

% [6] Compute Vincenty's constants A and B
A = 1 + u2/16384*(4096 + u2*(-768 + u2*(320-175*u2)));
B = u2/1024*(256 + u2*(-128 + u2*(74-47*u2)));

% [7] Compute sigma by iteration
sigma = s/(b*A);
iter = 1;
while 1
    two_sigma_m = 2*sigma1 + sigma;
    s1 = sin(sigma);
    s2 = s1*s1;
    c1 = cos(sigma);
    c1_2m = cos(two_sigma_m);
    c2_2m = c1_2m*c1_2m;
    t1 = 2*c2_2m-1;
    t2 = -3+4*s2;
    t3 = -3+4*c2_2m;
    delta_sigma = B*s1*(c1_2m+B/4*(c1*t1-B/6*c1_2m*t2*t3));
    sigma_new = s/(b*A)+delta_sigma;
    if abs(sigma_new-sigma)<1e-12
        break;
    end;
    sigma = sigma_new;
    iter = iter + 1;
end;
s1 = sin(sigma);
c1 = cos(sigma);

% [8] Compute latitude of P2
y = sin(psi1)*c1+cos(psi1)*s1*cos_alpha1;
x = (1-f)*sqrt(sin_alphaE^2+(sin(psi1)*s1-cos(psi1)*c1*cos_alpha1)^2);
phi2 = atan2(y,x);
lat2 = phi2*d2r;

% [9] Compute longitude difference domega on the auxiliary sphere
y = s1*sin_alpha1;
x = cos(psi1)*c1-sin(psi1)*s1*cos_alpha1;
domega = atan2(y,x);

% [10] Compute Vincenty's constant C
x = 1-sin_alphaE^2;
C = f/16*x*(4+f*(4-3*x));

% [11] Compute longitude difference on ellipsoid
two_sigma_m = 2*sigma1 + sigma;
c1_2m = cos(two_sigma_m);
c2_2m = c1_2m*c1_2m;
dlambda = domega-(1-C)*f*sin_alphaE*(sigma+C*s1*(c1_2m+C*c1*(-1+2*c2_2m)));
dlon = dlambda*d2r;
lon2 = lon1+dlon;

% [12] Compute azimuth alpha2
y = sin_alphaE;
x = cos(psi1)*c1*cos_alpha1-sin(psi1)*s1;
alpha2 = atan2(y,x);

% [13] Compute reverse azimuth az21
```

```
az21 = alpha2*d2r + 180;
if az21 > 360
    az21 = az21-360;
end;



%-----------------------------------------------
% Print computed quantities, latitudes and azimuth
%-----------------------------------------------


fprintf('\n////////////////////////////////////////////////');
fprintf('\n// DIRECT CASE on ellipsoid: Vincenty''s method //');
fprintf('\n////////////////////////////////////////////////');
fprintf('\n\nellipsoid parameters');
fprintf('\na    = %18.9f',a);
fprintf('\nf    = 1/%16.12f',flat);
fprintf('\nb    = %21.12f',b);
fprintf('\ne2   = %20.12e',e2);
fprintf('\nep2  = %20.12e',ep2);

fprintf('\n\nLatitude & Longitude of P1');
[D,M,S] = DMS(lat1);
if D==0 && lat1<0
    fprintf('\nlatP1 =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nlatP1 = %3d %2d %9.6f (D M S)',D,M,S);
end;
[D,M,S] = DMS(lon1);
if D==0 && lon1<0
    fprintf('\nlonP1 =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nlonP1 = %3d %2d %9.6f (D M S)',D,M,S);
end;
fprintf('\n\nAzimuth & Distance P1-P2');
[D,M,S] = DMS(az12);
fprintf('\naz12 = %4d %2d %9.6f (D M S)',D,M,S);
fprintf('\ns    = %17.6f',s);

fprintf('\n\nParametric Latitude of P1');
[D,M,S] = DMS(psi1*d2r);
if D==0 && psi1<0
    fprintf('\npsiP1 =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\npsiP1 = %3d %2d %9.6f (D M S)',D,M,S);
end;

fprintf('\n\nParametric Latitude of vertex P0');
[D,M,S] = DMS(psi0*d2r);
if D==0 && psi0<0
    fprintf('\npsiP0 =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\npsiP0 = %3d %2d %9.6f (D M S)',D,M,S);
end;

fprintf('\n\nGeodesic constant u2 (u-squared)');
fprintf('\nu2 = %20.12e',u2);

fprintf('\n\nangular distance on auxiliary sphere from equator to P1''');
fprintf('\nsigma1 = %20.12e radians',sigma1);

fprintf('\n\nVincenty''s constants A and B');
fprintf('\nA = %20.12e',A);
fprintf('\nB = %20.12e',B);

fprintf('\n\nangular distance sigma on auxiliary sphere from P1'' to P2''');
fprintf('\nsigma = %20.12e radians',sigma);
```

```
fprintf('\niterations = %2d',iter);

fprintf('\n\nLatitude of P2');
[D,M,S] = DMS(lat2);
if D==0 && lat2<0
    fprintf('\nlatP2 =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nlatP2 = %3d %2d %9.6f (D M S)',D,M,S);
end;

fprintf('\n\nVincenty''s constant C');
fprintf('\nC = %20.12e',C);

fprintf('\n\nLongitude difference P1-P2');
[D,M,S] = DMS(dlon);
if D==0 && dlon<0
    fprintf('\ndlon =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\ndlon = %3d %2d %9.6f (D M S)',D,M,S);
end;

fprintf('\n\nLongitude of P2');
[D,M,S] = DMS(lon2);
if D==0 && lon2<0
    fprintf('\nlon2 =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nlon2 = %3d %2d %9.6f (D M S)',D,M,S);
end;

fprintf('\n\nReverse azimuth');
[D,M,S] = DMS(az21);
fprintf('\nalpha21 = %3d %2d %9.6f (D M S)',D,M,S);

fprintf('\n\n');
```

MATLAB function *Vincenty_ Inverse.m*

```
function Vincenty_Inverse
%  Vincenty_Inverse computes the "inverse case" on the ellipsoid using
%  Vinventy's method.
%  Given the size and shape of the ellipsoid and the latitudes and
%  longitudes of P1 and P2 this function computes the geodesic distance
%  P1 to P2 and the forward and reverse azimuths

%============================================================================
% Function:  Vincenty_Inverse
%
% Useage:    Vincenty_Inverse;
%
% Author:
%  Rod Deakin,
%  Department of Mathematical and Geospatial Sciences,
%  RMIT University,
%  GPO Box 2476V, MELBOURNE VIC 3001
%  AUSTRALIA
%  email: rod.deakin@rmit.edu.au
%
% Date:
%  Version 1.0   7 March 2008
%
% Functions Required:
%      [D,M,S] = DMS(DecDeg)
%
% Remarks:
%  This function computes the INVERSE CASE on the ellipsoid.  Given the size
```

```
%  and shape of an ellipsoid (defined by parameters a and f, semi-major
%  axis and flattening respectively) and the latitudes and longitudes of P1
%  this function computes the forward azimuth (az12) P1 to P2, the reverse
%  azimuth (az21) P2 to P1 and the geodesic distance (s) P1 to P2.
%
% References:
%  [1] Deakin, R.E, and Hunter, M.N., 2007. 'Geodesics on an Ellipsoid -
%        Bessels' Method', School of Mathematical and Geospatial Sciences,
%        RMIT University, January 2007.
%  [2] Vincenty, T., 1975. 'Direct and Inverse solutions of geodesics on
%        the ellipsoid with application of nested equations', Survey
%        Review, Vol. 23, No. 176, pp.88-93, April 1975.
%
% Variables:
%  A            - Vincenty's constant for computation of sigma
%  a            - semi-major axis of ellipsoid
%  alpha1       - azimuth at P1 for the line P1-P2 (radians)
%  alpha2       - azimuth at P2 for the line P1-P2 extended (radians)
%  az12         - azimuth P1-P2 (degrees)
%  az21         - azimuth P2-P1 (degrees)
%  B            - Vincenty's constant for computation of sigma
%  b            - semi-minor axis of ellipsoid
%  C            - Vincenty's constant for computation of longitude
%                 difference
%  cdo          - cos(domega)
%  cos_sigma    - cos(sigma)
%  delta_sigma  - small change in sigma
%  dlambda      - longitude difference P1 to P2 (radians)
%  domega       - longitude difference P1' to P2' (radians)
%  d2r          - degree to radian conversion factor
%  e2           - eccentricity of ellipsoid squared
%  ep2          - 2nd eccentricity squared
%  f            - flattening of ellipsoid
%  flat         - denominator of flattening, f = 1/flat
%  lambda1      - longitude of P1 (radians)
%  lambda2      - longitude of P2 (radians)
%  lat1         - latitude of P1 (degrees)
%  lat2         - latitude of P2 (degrees)
%  lon1         - longitude of P1 (degrees)
%  lon2         - longitude of P2 (degrees)
%  phi1         - latitude of P1 (radians)
%  phi2         - latitude of P2 (radians)
%  pion2        - pi/2
%  psi0         - parametric latitude of P0 (radians)
%  psi1         - parametric latitude of P1 (radians)
%  psi2         - parametric latitude of P2 (radians)
%  s            - geodesic distance P1 to P2
%  sdo          - sin(domega)
%  sigma        - angular distance (radians) on auxiliary sphere from P1'
%                 to P2'
%  sin_alphaE   - sine of azimuth of geodesic P1-P2 at equator
%  sin_sigma    - sin(sigma)
%  twopi        - 2*pi
%  u2           - geodesic constant u-squared
%
%
%============================================================================

% Define some constants
d2r   = 180/pi;
twopi = 2*pi;
pion2 = pi/2;

% Set defining ellipsoid parameters
 a    = 6378137;          % GRS80
 flat = 298.257222101;
% a    = 6377397.155;        % Bessel (see Ref [2], p.91)
% flat = 299.1528128;
```

```
% Compute derived ellipsoid constants
f   = 1/flat;
b   = a*(1-f);
e2  = f*(2-f);
ep2 = e2/(1-e2);

%--------------------------------------
% latitude and longitude of P1 (degrees)
%--------------------------------------
lat1 =  -10;
lon1 =  110;

% lat and lon of P1 (radians)
phi1    = lat1/d2r;
lambda1 = lon1/d2r;

%--------------------------------------
% latitude and longitude of P2 (degrees)
%--------------------------------------
lat2 =  -45;
lon2 =  155;

% lat and lon of P2 (radians)
phi2    = lat2/d2r;
lambda2 = lon2/d2r;

% [1] Compute parametric latitudes psi1 and psi2 of P1 and P2
psi1 = atan((1-f)*tan(phi1));
psi2 = atan((1-f)*tan(phi2));

s1 = sin(psi1);
s2 = sin(psi2);
c1 = cos(psi1);
c2 = cos(psi2);

% [2] Compute longitude difference dlambda on the ellipsoid
dlambda = lambda2-lambda1; % (radians)
dlon = lon2-lon1;          % (degrees)

% [3] Compute longitude difference domega on the auxiliary sphere by
%     iteration
domega = dlambda;
iter = 1;
while 1
    sdo = sin(domega);
    cdo = cos(domega);
    x = c2*sdo;
    y = c1*s2 - s1*c2*cdo;
    sin_sigma = sqrt(x*x + y*y);
    cos_sigma = s1*s2 + c1*c2*cdo;
    sigma = atan2(sin_sigma,cos_sigma);
    sin_alphaE = c1*c2*sdo/sin_sigma;
    % Compute c1_2m = cos(2*sigma_m)
    x = 1-(sin_alphaE*sin_alphaE);
    c1_2m = cos_sigma - (2*s1*s2/x);
    % Compute Vincenty's constant C
    C = f/16*x*(4+f*(4-3*x));
    % Compute domega
    c2_2m = c1_2m*c1_2m;
    domega_new = dlambda+(1-C)*f*sin_alphaE*(sigma+C*sin_sigma*(c1_2m+C*cos_sigma*(-
1+2*c2_2m)));
    if abs(domega-domega_new)<1e-12
        break;
    end;
    domega = domega_new;
    iter = iter + 1;
end;
```

```
% [4] Compute parametric latitude of vertex
psi0 = acos(sin_alphaE);

% [5] Compute geodesic constant u2 (u-squared)
u2 = ep2*(sin(psi0)^2);

% [6] Compute Vincenty's constants A and B
A = 1 + u2/16384*(4096 + u2*(-768 + u2*(320-175*u2)));
B = u2/1024*(256 + u2*(-128 + u2*(74-47*u2)));

% [7] Compute geodesic distance s
t1 = 2*c2_2m-1;
t2 = -3+4*sin_sigma*sin_sigma;
t3 = -3+4*c2_2m;
delta_sigma = B*sin_sigma*(c1_2m+B/4*(cos_sigma*t1-B/6*c1_2m*t2*t3));
s = b*A*(sigma-delta_sigma);

% [8] Compute forward azimuth alpha1
y = c2*sdo;
x = c1*s2 - s1*c2*cdo;
alpha1 = atan2(y,x);
if alpha1<0
    alpha1 = alpha1+twopi;
end;
az12 = alpha1*d2r;

% [9] Compute azimuth alpha2
y = c1*sdo;
x = -s1*c2 + c1*s2*cdo;
alpha2 = atan2(y,x);

% [10] Compute reverse azimuth az21
az21 = alpha2*d2r + 180;
if az21 > 360
    az21 = az21-360;
end;

%--------------------------------------------------
% Print computed quantities, latitudes and azimuth
%--------------------------------------------------

fprintf('\n///////////////////////////////////////////////////');
fprintf('\n// INVERSE CASE on ellipsoid: Vincenty''s method //');
fprintf('\n///////////////////////////////////////////////////');
fprintf('\n\nellipsoid parameters');
fprintf('\na    = %18.9f',a);
fprintf('\nf    = 1/%16.12f',flat);
fprintf('\nb    = %21.12f',b);
fprintf('\ne2   = %20.12e',e2);
fprintf('\nep2  = %20.12e',ep2);

fprintf('\n\nLatitude & Longitude of P1');
[D,M,S] = DMS(lat1);
if D==0 && lat1<0
    fprintf('\nlatP1 =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nlatP1 = %3d %2d %9.6f (D M S)',D,M,S);
end;
[D,M,S] = DMS(lon1);
if D==0 && lon1<0
    fprintf('\nlonP1 =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nlonP1 = %3d %2d %9.6f (D M S)',D,M,S);
end;
fprintf('\n\nLatitude & Longitude of P2');
[D,M,S] = DMS(lat2);
if D==0 && lat2<0
```

```
    fprintf('\nlatP2 =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nlatP2 = %3d %2d %9.6f (D M S)',D,M,S);
end;
[D,M,S] = DMS(lon2);
if D==0 && lon2<0
    fprintf('\nlonP2 =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nlonP2 = %3d %2d %9.6f (D M S)',D,M,S);
end;

fprintf('\n\nParametric Latitudes of P1 and P2');
[D,M,S] = DMS(psi1*d2r);
if D==0 && psi1<0
    fprintf('\npsiP1 =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\npsiP1 = %3d %2d %9.6f (D M S)',D,M,S);
end;
[D,M,S] = DMS(psi2*d2r);
if D==0 && psi2<0
    fprintf('\npsiP2 =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\npsiP2 = %3d %2d %9.6f (D M S)',D,M,S);
end;

fprintf('\n\nLongitude difference on ellipsoid P1-P2');
[D,M,S] = DMS(dlon);
if D==0 && dlon<0
    fprintf('\ndlon =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\ndlon = %3d %2d %9.6f (D M S)',D,M,S);
end;

fprintf('\n\nLongitude difference on auxiliary sphere P1''-P2''');
fprintf('\ndomega = %20.12e radians',sigma);
fprintf('\niterations = %2d',iter);

fprintf('\n\nParametric Latitude of vertex P0');
[D,M,S] = DMS(psi0*d2r);
if D==0 && psi0<0
    fprintf('\npsiP0 =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\npsiP0 = %3d %2d %9.6f (D M S)',D,M,S);
end;

fprintf('\n\nGeodesic constant u2 (u-squared)');
fprintf('\nu2 = %20.12e',u2);

fprintf('\n\nVincenty''s constants A and B');
fprintf('\nA = %20.12e',A);
fprintf('\nB = %20.12e',B);

fprintf('\n\nAzimuth & Distance P1-P2');
[D,M,S] = DMS(az12);
fprintf('\naz12 = %4d %2d %9.6f (D M S)',D,M,S);
fprintf('\ns    = %17.6f',s);

fprintf('\n\nReverse azimuth');
[D,M,S] = DMS(az21);
fprintf('\nalpha21 = %3d %2d %9.6f (D M S)',D,M,S);

fprintf('\n\n');
```

MATLAB function *DMS.m*

```
function [D,M,S] = DMS(DecDeg)
% [D,M,S] = DMS(DecDeg)  This function takes an angle in decimal degrees and returns
%   Degrees, Minutes and Seconds

val = abs(DecDeg);
D = fix(val);
M = fix((val-D)*60);
S = (val-D-M/60)*3600;

if abs(S-60) < 5.0e-10
    M = M + 1;
    S = 0.0;
end
if M == 60
    D = D + 1;
    M = 0.0;
end
if D >=360
   D = D - 360;
end

if(DecDeg<=0)
    D = -D;
end
return
```

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