GEODESICS ON AN ELLIPSOID – PITTMAN'S METHOD

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ABSTRACT

The direct and inverse problems of the geodesic on an ellipsoid are fundamental geodetic operations. This paper presents a detailed derivation of a set of recurrence relationships that can be used to obtain solutions to the direct and inverse problems with sub-millimetre accuracies for any length of line anywhere on an ellipsoid. These recurrence relationships were first described by Pittman (1986), but since then, little or nothing about them has appeared in the geodetic literature. This is unusual for such an elegant technique and it is hoped that this paper can redress this situation. Pittman's method has much to recommend it.

BIOGRAPHIES OF PRESENTERS

Rod Deakin and Max Hunter are lecturers in the School of Mathematical and Geospatial Sciences, RMIT University; Rod is a surveyor and Max is a mathematician, and both have extensive experience teaching undergraduate students.

INTRODUCTION

Twenty-one years ago (March 1986), Michael E. Pittman, an assistant professor of mathematical physics with the Department of Physics, University of New Orleans, Louisiana USA, published a paper titled 'Precision Direct and Inverse Solutions of the Geodesic' in *Surveying and Mapping* (the journal of the American Congress on Surveying & Mapping, now called *Surveying and Land Information Systems*). It was probably an unusual event – a physicist writing a technical article on geodetic computation – but even more unusual was Pittman's method; or as he put it in his paper, "The following method is rather different." And it certainly is.

Usual approaches could be roughly divided into two groups: (i) numerical integration schemes and (ii) series expansion of elliptic integrals. The first group could be further divided into integration schemes based on simple differential relationships of the ellipsoid (e.g., Kivioja 1971, Jank & Kivioja 1980, Thomas & Featherstone 2005), or numerical integration of elliptic integrals that are usually functions of elements of the ellipsoid and an auxiliary sphere (e.g., Saito 1970, 1979 and Sjöberg 2006). The second group includes the original method of F. W. Bessel (1826) that used an auxiliary sphere and various modifications to his method (e.g., Rainsford 1955, Vincenty 1975, 1976 and Bowring 1983, 1984).

Pittman developed simple recurrence relationships for the evaluation of elliptic integrals that yield distance and longitude difference between a point on a geodesic and the geodesic vertex. These equations can then be used to solve the direct and inverse problems. Pittman's technique is not limited by distance, does not involve any auxiliary surfaces, does not use arbitrarily truncated series and its accuracy is limited only by capacity of the computer used.

Pittman's paper was eight pages long and five of those contained a FORTRAN computer program. In the remaining three pages he presented a very concise development of two recurrence relationships and how they can be used to solve the direct and inverse problems of the geodesic on an ellipsoid (more about this later). His paper, a masterpiece of brevity, contained a single reference and an acknowledgement to Clifford J. Mugnier – then a lecturer in the Department of Civil Engineering, University of New Orleans – for numerous discussions. Unlike other published methods which have been discussed and developed in detail over the years, Pittman's method seems to have received no further treatment to our knowledge in the academic literature, excepting brief mentions in bibliographies and reference lists. Our purpose, in this paper, is to explain Pittman's elegant method as well as provide some useful information about the properties of the geodesic on an ellipsoid.

The Direct and Inverse problems of the geodesic on an ellipsoid

In geodesy, the *geodesic* is a unique curve on the surface of an ellipsoid defining the shortest distance between two points. A geodesic will cut meridians of an ellipsoid at angles α, known as *azimuths* and measured clockwise from north 0º to 360°. Figure 1 shows a geodesic curve C between two points A (ϕ_A, λ_A) and B (ϕ_B, λ_B) on an ellipsoid. ϕ, λ are geodetic latitude and longitude respectively and an ellipsoid is taken to mean a surface of revolution created by rotating an ellipse about its minor axis, *NS*.

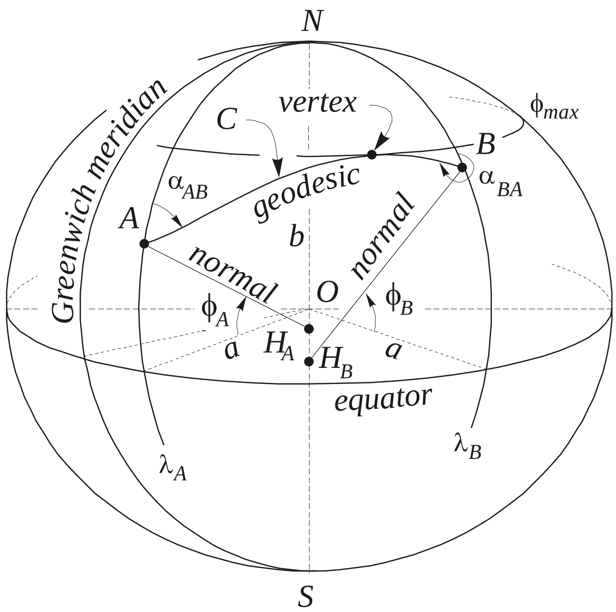

Fig. 1: Geodesic curve on an ellipsoid

The geodesic curve C of length s from A to B has a *forward azimuth* α_{AB} measured at A and a *reverse azimuth* α_{BA} measured at B and $\alpha_{AB} \neq \alpha_{BA}$. The *direct* problem on an ellipsoid is: given latitude and longitude of A and azimuth α_{AB} and geodesic distance s, compute the latitude and longitude of B and the reverse azimuth α_{BA}. The *inverse* problem is: given the latitudes and longitudes of A and B, compute the forward and reverse azimuths α_{AB}, α_{BA} and the geodesic distance s.

The geodesic is one of several curves of interest in geodesy. Other curves are: (i) normal section curves that are plane curves containing the normal at one of the terminal points; in Figure 1 there would be two normal section curves joining A and B and both would be of different lengths and also, both longer than the geodesic; (ii) curve of alignment that is the locus of all points P_k where the normal section plane through P_k contains the terminal points of the line; and (iii) great elliptic arcs that are plane curves containing the terminal points of the line and the centre of the ellipsoid. Normal section curves, curves of alignment and great elliptic arcs are all longer than the geodesic and Bowring (1972) gives equations for the differences in length between these curves and the geodesic.

Some ellipsoid relationships

The size and shape of an ellipsoid is defined by one of three pairs of parameters: (i) a, b where a and b are the *semi-major* and *semi-minor* axes lengths of an ellipsoid respectively, or (ii) a, f where f is the *flattening* of an ellipsoid, or (iii) a, e^2 where e^2 is the square of the first *eccentricity* of an ellipsoid. The ellipsoid parameters a, b, f, e^2 are related by the following equations

$$f = \frac{a-b}{a} = 1 - \frac{b}{a}; \quad b = a(1-f); \quad e^2 = \frac{a^2-b^2}{a^2} = 1 - \frac{b^2}{a^2} = f(2-f) \tag{1}$$

The second eccentricity e' of an ellipsoid is also of use and

$$(e')^2 = \frac{a^2-b^2}{b^2} = \frac{e^2}{1-e^2} = \frac{f(2-f)}{(1-f)^2} \tag{2}$$

In Figure 1, the normals to the surface at A and B intersect the rotational axis of the ellipsoid (NS line) at H_A and H_B making angles ϕ_A, ϕ_B with the equatorial plane of the ellipsoid. These are the latitudes of A and B respectively. The longitudes λ_A, λ_B are the angles between the Greenwich meridian plane and the meridian planes $ONAH_A$ and $ONBH_B$ containing the normals through A and B. ϕ and λ are *curvilinear* coordinates and meridians of longitude (curves of constant λ) and parallels of latitude (curves of constant ϕ) are parametric curves on the ellipsoidal surface. Planes containing the normal to the ellipsoid intersect the surface creating elliptical sections known as normal sections. Amongst the infinite number of possible normal sections at a point, each having a certain radius of curvature, two are of interest: (i) the *meridian* section, containing the axis of revolution of the ellipsoid and having the least radius of curvature, denoted by ρ (rho), and (ii) the *prime vertical* section, perpendicular to the meridian plane and having the greatest radius of curvature, denoted by ν (nu).

$$\rho = \frac{a\left(1-e^2\right)}{\left(1-e^2\sin^2\phi\right)^{\frac{3}{2}}} \quad \text{and} \quad \nu = \frac{a}{\left(1-e^2\sin^2\phi\right)^{\frac{1}{2}}} \tag{3}$$

In the development that follows, use will be made of relationships that can be obtained from the differential rectangle on the ellipsoid shown in Figure 2. Here *P* and *Q* are two points on the surface connected by a curve of length *ds* with azimuth α at *P*. The meridians λ and $\lambda + d\lambda$, and parallels ϕ and $\phi + d\phi$ form a differential rectangle on the surface of the ellipsoid.

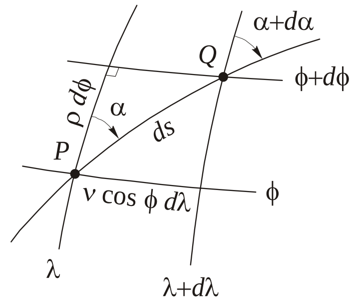

Fig. 2: Differential rectangle on ellipsoid

From Figure 2 the following relationships can be obtained

$$ds\sin\alpha = \nu\cos\phi \; d\lambda \quad \text{and} \quad ds\cos\alpha = \rho \; d\phi \tag{4}$$

Mathematical definition of a geodesic

A curve drawn on a surface so that its osculating plane at any point on the surface contains the normal to the surface is a geodesic (Lauf 1983). This definition, including a definition of the osculating plane, can be explained briefly by the following.

A point *P* on a curve (on a surface) has a position vector $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ where **i**,**j**,**k** are unit vectors in the directions of the *x*,*y*,*z* Cartesian coordinate axes and *t* is some scalar parameter. As *t* varies then the vector **r** sweeps out the curve *C* on the surface, hence the distance *s* along the curve is a function of *t*, given via $\frac{ds}{dt} = \frac{d}{dt}\mathbf{r}(t)$. Differentiating the vector **r** with respect to *s* gives a unit tangent vector **t** and differentiating **t** with respect to *s* gives the curvature vector $\kappa\mathbf{n}$, perpendicular to **t**. **n** is the principal normal vector, κ (kappa) is the curvature and $\rho = \frac{1}{\kappa}$ is the radius of curvature and also the radius of the *osculating* (kissing) circle touching *P*.

The osculating plane at *P* contains both **t** and **n** (and the osculating circle), and when this plane also contains the normal to the surface then the curvature κ is least and ρ is a maximum; this is *Meunier's theorem* (Lauf 1983), a fundamental theorem of surfaces. Therefore, if *P* and *Q* are very close and both lie on the surface and in the osculating plane, then the distance *ds* between them is the shortest possible distance on the suface.

The characteristic equation of a geodesic

The mathematical definition of a geodesic does little to help us develop solutions to the problem of computing distances of geodesics on an ellipsoid. It does lead to *the characteristic equation of a geodesic,* and this equation is the basis of all solutions to computing geodesic distances. This equation

$$\nu \cos\phi \sin\alpha = \text{constant} \tag{5}$$

is known as *Clairaut's equation* in honour of the French mathematical physicist Alexis-Claude Clairaut (1713-1765). In a paper in 1733 titled *Determination géométric de la perpendicular à la méridienne tracée par M. Cassini, ...* Clairaut made an elegant study of the geodesics of surfaces of revolution and stated his theorem embodied in the equation above (Struik 1933). His paper also included the property already pointed out by Johann Bernoulli (1667-1748): the osculating plane of the geodesic is normal to the surface (DSB 1971)

The characteristic equation of a geodesic shows that the geodesic on the ellipsoid has the intrinsic property that at any point, the product of the radius $r = \nu \cos\phi$ of the parallel of latitude and the sine of the azimuth, $\sin\alpha$, of the geodesic at that point is a constant. This means that as *r* decreases in higher latitudes, in both the northern and southern hemispheres, $\sin\alpha$ changes until it reaches a maximum or minimum of ± 1. Such a point is known as a *vertex* and the latitude ϕ will take maximum value ϕ_0.

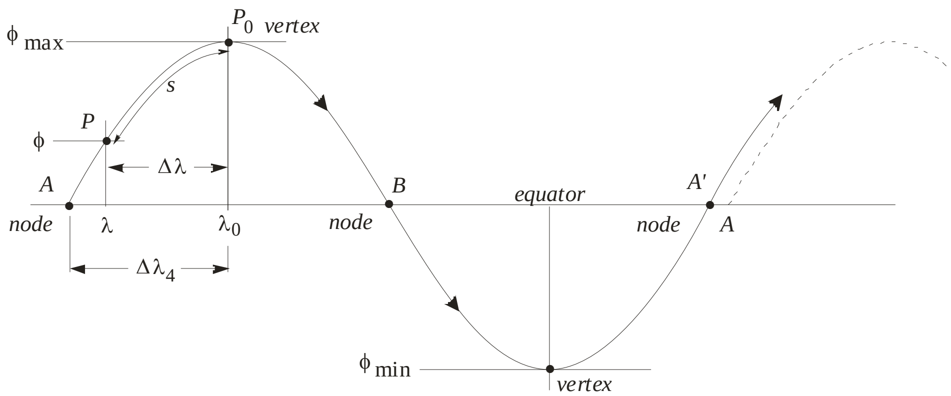

Fig. 3: Schematic diagram of the oscillation of a geodesic on an ellipsoid

Thus the geodesic oscillates over the surface of the ellipsoid between two parallels of latitude having a maximum in the Northern and Southern Hemispheres and crossing the equator at nodes. As we will demonstrate later, due to the eccentricity of the ellipsoid, the geodesic will not repeat after a complete revolution.

Figure 3 shows a schematic diagram of the oscillation of a geodesic on an ellipsoid. *P* is a point on a geodesic that crosses the equator at *A*, heading in a north-easterly direction reaching a maximum northerly latitude $\phi_{\max}$ at the vertex P_0 (north), then descends in a south-easterly direction crossing the equator at *B*, reaching a maximum southerly latitude $\phi_{\min}$ at P_0 (south), then ascends in a north-easterly direction crossing the equator again at A'. This is one complete revolution of the geodesic, but $\lambda_{A'}$ does not equal λ_A due to the eccentricity of the ellipsoid. Hence we say that the geodesic curve does not repeat after a complete revolution.

EQUATIONS FOR COMPUTATION ALONG GEODESICS

Using Clairaut's equation and simple differential relationships, expressions for distances s and longitude differences $\Delta\lambda$ (see Figure 3) between P on a geodesic and the vertex P_0 can be obtained. These expressions are in the form of elliptic integrals, which by their nature do not have exact (or closed) solutions.

Expanding the integrands into infinite series, integrating term-by-term and then truncating to a finite number of terms is the usual technique to obtain working solutions for s and $\Delta\lambda$ (e.g., Thomas 1970). In this section, we show how this method can be simplified by using recurrence relationships to generate solutions to the integrals in the series. Our relationships are slightly different from Pittman (1986) and our notation is a little different but in all other respects, we have followed his elegant approach.

Relationships between parametric latitude *ψ* and geodetic latitude *Φ*

Development of formulae is simplified if *parametric* latitude ψ is used rather than *geodetic* latitude ϕ. The connections between the two latitudes can be obtained from the following relationships.

Figure 4 shows a portion of a meridian NPE of an ellipsoid having semi-major axis $OE = a$ and semi-minor axis $ON = b$. P is a point on the ellipsoid and Q is a point on an auxiliary circle centred on O of radius a. P and Q have the same perpendicular distance from the axis of revolution ON. The normal to the ellipsoid at P cuts the major axis at an angle ϕ (the geodetic latitude) and intersects the rotational axis at H and the distance $PH = \nu$. The angle $QOE = \psi$ is the parametric latitude.

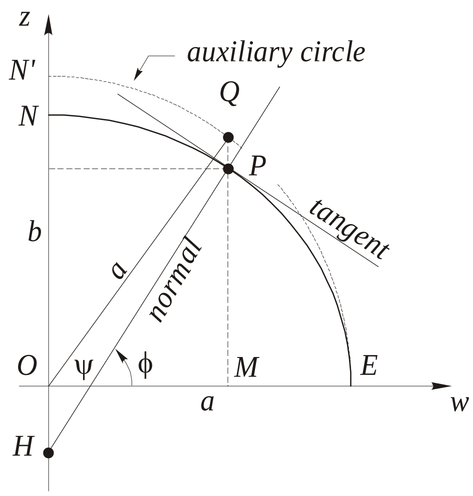

Fig. 4: Meridian section of ellipsoid

The Cartesian equation of the ellipse is $\frac{w^2}{a^2}+\frac{z^2}{b^2}=1$ and the Cartesian equation of the auxiliary circle is $w^2+z^2=a^2$. We may re-arrange both equations so that w^2 is on the left-hand side of the equals sign giving $w^2 = a^2 - \frac{a^2}{b^2}z^2$ (ellipse) and $w^2 = a^2 - z^2$ (circle). Now, since the w-coordinates of P and Q are the same then $a^2 - \frac{a^2}{b^2}z_P^2 = a^2 - z_Q^2$ which leads to $z_P = \frac{b}{a}z_Q$.

Using this relationship

$$w = OM = a\cos\psi \quad \text{and} \quad z = MP = b\sin\psi \tag{6}$$

and differentiating equations (6) with respect to ψ gives $\frac{dw}{d\psi} = -a\sin\psi$, $\frac{dz}{d\psi} = b\cos\psi$ and the chain rule gives $\frac{dz}{dw} = \frac{dz}{d\psi}\frac{d\psi}{dw} = -\frac{b}{a}\cot\psi$.

Now by definition, $\frac{dz}{dw}$ is the gradient of the tangent and from Figure 4 we may write $\frac{dz}{dw} = -\tan\left(90^\circ - \phi\right) = -\cot\phi$. Equating the two expressions for dz/dw gives a relationship between ψ and ϕ as

$$\tan\psi = \frac{b}{a}\tan\phi = \left(1 - f\right)\tan\phi \tag{7}$$

From equation (6) and Figure 4, $w = a\cos\psi = \nu\cos\phi$ and using equation (3) gives

$$\cos\psi = \frac{\cos\phi}{\left(1 - e^2\sin^2\phi\right)^{1/2}} \tag{8}$$

Alternatively, using the trigonometric identity $\sin^2 A + \cos^2 A = 1$, equation (8) can be written as

$$\sin\phi = \frac{\sin\psi}{\left(1 - e^2\cos^2\psi\right)^{1/2}} \tag{9}$$

The latitudes Φ_0 and ψ_0 of the geodesic vertex

Denoting the latitude of the vertex as ϕ_0 (a maximum), Clairaut's equation (5) gives

$$\nu_0\cos\phi_0 = \text{constant} = \nu\cos\phi\sin\alpha \tag{10}$$

Denoting the parametric latitude of the vertex as ψ_0 and using $a\cos\psi = \nu\cos\phi$ from before, equation (10) becomes $a\cos\psi_0 = a\cos\psi\sin\alpha$ and ψ_0 is defined as

$$\cos\psi_0 = \cos\psi\sin\alpha \tag{11}$$

Squaring both sides of equation (11) and using again the identity $\sin^2 A + \cos^2 A = 1$ we can obtain the azimuth α of a geodesic as

$$\cos\alpha = \frac{\sqrt{\cos^2\psi - \cos^2\psi_0}}{\cos\psi} \tag{12}$$

From equation (11) we see that if the azimuth α of a geodesic is known at *P* having parametric latitude ψ, the parametric latitude ψ_0 of the vertex P_0 can be computed. Conversely, given ψ and ψ_0 of points *P* and P_0 the azimuth of the geodesic between them may be computed from equation (12).

In the following sections, two differential equations; one for $\frac{ds}{d\psi}$ and the other for $\frac{d\lambda}{d\psi}$, will be developed that will enable solutions for the geodesic distance *s* and the longitude difference $\Delta\lambda$ between *P* and the vertex P_0.

Differential equations for distance $\frac{ds}{d\psi}$ and longitude difference $\frac{d\lambda}{d\psi}$

From equation (9) we may write $\sin^2\psi = \left(1-e^2\cos^2\psi\right)\sin^2\phi$ and differentiating implicitly and re-arranging gives

$$\frac{d\phi}{d\psi} = \frac{\left(1-e^2\sin^2\phi\right)\sin\psi\cos\psi}{\left(1-e^2\cos^2\psi\right)\sin\phi\cos\phi} \tag{13}$$

Using the chain rule and equation (4) gives an expression for the derivative $\frac{ds}{d\psi}$ as

$$\frac{ds}{d\psi} = \frac{ds}{d\phi}\frac{d\phi}{d\psi} = \frac{\rho}{\cos\alpha}\frac{\left(1-e^2\sin^2\phi\right)\sin\psi\cos\psi}{\left(1-e^2\cos^2\psi\right)\sin\phi\cos\phi} \tag{14}$$

Using equations (7), (8), (9) and the fact that $1-e^2 = \frac{b^2}{a^2}$, we may write

$$\frac{ds}{d\psi} = a\cos\psi\frac{\left(1-e^2\cos^2\psi\right)^{1/2}}{\left(\cos^2\psi-\cos^2\psi_0\right)^{1/2}} \tag{15}$$

Similarly, the chain rule and equations (4) and (15) gives

$$\frac{d\lambda}{d\psi} = \frac{d\lambda}{ds}\frac{ds}{d\psi} = \frac{\sin\alpha}{\nu\cos\phi}a\cos\psi\frac{\left(1-e^2\cos^2\psi\right)^{1/2}}{\left(\cos^2\psi-\cos^2\psi_0\right)^{1/2}} \tag{16}$$

Using equation (10) and the relationship $a\cos\psi = \nu\cos\phi$, we may write

$$\frac{d\lambda}{d\psi} = \frac{\cos\psi_0}{\cos\psi}\frac{\left(1-e^2\cos^2\psi\right)^{1/2}}{\left(\cos^2\psi-\cos^2\psi_0\right)^{1/2}} \tag{17}$$

Equations (15) and (17) are the basic differential equations that will yield solutions for distance s and longitude difference $\Delta\lambda$ along the geodesic curve between P and the vertex P_0.

Formula for computing geodesic distance *s* between *P* and the vertex P_0

Equation (15) can be simplified by letting $u = \sin\psi$ and $u_0 = \sin\psi_0$, so that $\frac{du}{d\psi} = \cos\psi$ and $\cos^2\psi - \cos^2\psi_0 = u_0^2 - u^2$, hence

$$\frac{ds}{d\psi} = a\frac{du}{d\psi}\frac{\left(1-e^2\cos^2\psi\right)^{1/2}}{\left(u_0^2-u^2\right)^{1/2}} \tag{18}$$

The chain rule gives $\frac{ds}{du} = \frac{ds}{d\psi} \Big/ \frac{du}{d\psi} = \frac{a\left(1-e^2\cos^2\psi\right)^{1/2}}{\left(u_0^2-u^2\right)^{1/2}}$ but using $\cos^2\psi = 1-\sin^2\psi$ and equations (1) and (2) we are able to obtain, after some manipulation

$$\frac{ds}{du} = \frac{b\left(1+\varepsilon u^2\right)^{1/2}}{\left(u_0^2-u^2\right)^{1/2}} \tag{19}$$

where $\varepsilon = \left(e'\right)^2$. The geodesic distance s between P and the vertex P_0 is given by

$$s = b\int_{p=u}^{p=u_0} \frac{\left(1+\varepsilon p^2\right)^{1/2}}{\left(u_0^2-p^2\right)^{1/2}}\,dp \tag{20}$$

where $\sin\psi \le p \le \sin\psi_0$. Equation (20) can be simplified by use of the *binomial series* and the numerator of the integrand is given by

$$\left(1+\varepsilon p^2\right)^{1/2} = \sum_{n=0}^{\infty} B_n^{\frac{1}{2}} \left(\varepsilon p^2\right)^n \tag{21}$$

where $B_n^{\frac{1}{2}}$ are binomial coefficients computed from the recurrence relationship

$$B_n^{\frac{1}{2}} = \frac{3-2n}{2n} B_{n-1}^{\frac{1}{2}}, \quad n \ge 1 \text{ and } B_0^{\frac{1}{2}} = 1 \tag{22}$$

Equation (20) can now be written as

$$s = b\int_u^{u_0} \frac{1}{\left(u_0^2-p^2\right)^{1/2}} \sum_{n=0}^{\infty} B_n^{\frac{1}{2}} \varepsilon^n p^{2n}\,dp = b\sum_{n=0}^{\infty} B_n^{\frac{1}{2}} \varepsilon^n \int_u^{u_0} \frac{p^{2n}}{\left(u_0^2-p^2\right)^{1/2}}\,dp = b\sum_{n=0}^{\infty} \varepsilon^n B_n^{\frac{1}{2}} I_n \tag{23}$$

where

$$I_n = \int_u^{u_0} \frac{p^{2n}}{\left(u_0^2-p^2\right)^{1/2}}\,dp, \quad \text{for } n \ge 0 \tag{24}$$

The solution of the integral I_n is fundamental to the computation of the distance s along the geodesic between P and P_0, and the usual technique is to find solutions for each integral I_n and expand equation (23) into a finite series; e.g. Thomas (1970, pp. 33-34). Pittman's (1986) approach, outlined below, was to developed the integral I_n as a recurrence equation having the general form $I_n = a_{n-1} + b_{n-1} I_{n-1}$ where the coefficients a_{n-1} and b_{n-1} are functions of n, ψ and ψ_0 and an initial value of I_0 is a function of ψ and ψ_0 only.

Now $$I_n = \int_u^{u_0} \frac{p^{2n}}{\left(u_0^2-p^2\right)^{1/2}}\,dp = -\int_u^{u_0} p^{2n-1} \frac{-p}{\left(u_0^2-p^2\right)^{1/2}}\,dp = -\int_u^{u_0} p^{2n-1} \frac{d}{dp}\left(u_0^2-p^2\right)^{1/2}\,dp$$

and using integration by parts (e.g., Ayres 1972) the integral I_n becomes

$$
\begin{aligned}
I_n &= -\left[p^{2n-1}\left(u_0^2 - p^2\right)^{1/2} - \int \left(u_0^2 - p^2\right)^{1/2}(2n-1)\, p^{2n-2} dp \right]_{p=u}^{p=u_0} \\
&= u^{2n-1}\left(u_0^2 - u^2\right)^{1/2} + (2n-1)\int_u^{u_0} \left(u_0^2 - p^2\right) \frac{p^{2n-2}}{\left(u_0^2 - p^2\right)^{1/2}} dp \\
&= u^{2n-1}\left(u_0^2 - u^2\right)^{1/2} + (2n-1)\left[u_0^2 I_{n-1} - I_n\right] \qquad (25)
\end{aligned}
$$

and

$$
2n\, I_n = u^{2n-1}\left(u_0^2 - u^2\right)^{1/2} + (2n-1)u_0^2 I_{n-1} \quad \text{for } n = 1, 2, 3, \ldots \qquad (26)
$$

Let $U = \dfrac{u}{u_0}$ so that $u = U u_0$, $u_0^2 - u^2 = u_0^2\left(1 - U^2\right)$ giving

$$
2n\, I_n = \left(U u_0\right)^{2n-1} u_0 \left(1 - U^2\right)^{1/2} + (2n-1)u_0^2 I_{n-1} \quad \text{for } n = 1, 2, 3, \ldots \qquad (27)
$$

Let $J_n = \dfrac{2n\, I_n}{u_0^{2n}}$ so that $J_{n-1} = \dfrac{2(n-1)u_0^2}{u_0^{2n}} I_{n-1}$ and the recurrence formula for I_n becomes a simpler recurrence formula for J_n

$$
J_n = U^{2n-1}\sqrt{1 - U^2} + \frac{2n-1}{2(n-1)} J_{n-1} \quad \text{for } n = 2, 3, \ldots \qquad (28)
$$

with initial condition

$$
J_1 = \frac{2I_1}{u_0^2} = U\sqrt{1 - U^2} + I_0 \qquad (29)
$$

I_0 has a simple result derived from equation (24) as follows:

$$
I_0 = \left(1/u_0\right)\int_u^{u_0} \left(1 - \left[p/u_0\right]^2\right)^{-1/2} dp \qquad (30)
$$

and with the transformation $p = u_0 \cos\theta$, $dp/d\theta = -u_0 \sin\theta$ and $1 - \left[p/u_0\right]^2 = 1 - \cos^2\theta$

$$
I_0 = \int_{\theta = \arccos\left(\frac{u}{u_0}\right)}^{0} (-1)\, d\theta = \arccos\left(\frac{u}{u_0}\right) = \arccos U \qquad (31)
$$

Using these results, the distance *s* along the geodesic between *P* and the vertex P_0 is

$$
\begin{aligned}
s &= b\left\{ I_0 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n} \varepsilon^n u_0^{2n} B_n^{\frac{1}{2}} J_n \right\} \\
&= bI_0 + \frac{b}{2}\varepsilon u_0^2 B_1^{\frac{1}{2}} J_1 + \frac{b}{4}\varepsilon^2 u_0^4 B_2^{\frac{1}{2}} J_2 + \frac{b}{6}\varepsilon^3 u_0^6 B_3^{\frac{1}{2}} J_3 + \cdots \\
&= D_0 + D_1 + D_2 + D_3 + \cdots \qquad (32)
\end{aligned}
$$

Formula for computing difference in longitude Δλ between *P* and *P*₀

Using the binomial series we may write equation (17) as

$$\frac{d\lambda}{d\psi}=\cos\psi_0\sum_{n=0}^{\infty}(-1)^n e^{2n}B_n^{\frac{1}{2}}\frac{\cos^{2n-1}\psi}{\left(\cos^2\psi-\cos^2\psi_0\right)^{1/2}} \tag{33}$$

and the difference in longitude between P and the vertex P_0 is

$$\Delta\lambda=\int_{\theta=\psi}^{\psi_0}\frac{d\lambda}{d\theta}d\theta=\cos\psi_0\sum_{n=0}^{\infty}(-1)^n e^{2n}B_n^{\frac{1}{2}}L_n \tag{34}$$

where the integral L_n is

$$L_n=\int_{\theta=\psi}^{\psi_0}\frac{\cos^{2n}\theta}{\cos\theta\left(\cos^2\theta-\cos^2\psi_0\right)^{1/2}}d\theta,\quad n\ge 0 \tag{35}$$

Again, let $u=\sin\psi$, $u_0=\sin\psi_0$ and put $p=\sin\theta$. Then $d\theta/dp=\sec\theta$, $\cos^2\theta=1-p^2$, and with

$$\frac{\cos^{2n}\theta}{\cos\theta}d\theta=\frac{\left(\cos^2\theta\right)^n}{\cos^2\theta}\cos\theta\,d\theta=\frac{\left(1-p^2\right)^n}{1-p^2}dp=\left(1-p^2\right)^{n-1}dp$$

and

$$\left(\cos^2\theta-\cos^2\psi_0\right)^{1/2}=\left(1-\sin^2\theta-\left(1-\sin^2\psi_0\right)\right)^{1/2}=\left(u_0^2-p^2\right)^{1/2}$$

giving

$$L_n=\int_u^{u_0}\frac{\left(1-p^2\right)^{n-1}}{\left(u_0^2-p^2\right)^{1/2}}dp,\quad n\ge 1 \tag{36}$$

Using the binomial series, the numerator of the integrand can be expanded into a polynomial $\left(1-p^2\right)^{n-1}=\sum_{m=0}^{n-1}(-1)^m B_m^{n-1}p^{2m}$, where the binomial coefficients B_m^{n-1} are given by

$$B_m^{n-1}=\frac{n-m}{m}B_{m-1}^{n-1}\qquad\text{for } m=2,3,4,\ldots \tag{37}$$

with an initial value $B_1^{n-1}=n-1$ and noting that $B_0^{n-1}=1$.

Using these results, equation (36) becomes

$$L_n=\sum_{m=0}^{n-1}(-1)^m B_m^{n-1}\int_u^{u_0}\frac{p^{2m}}{\left(u_0^2-p^2\right)^{1/2}}dp=\sum_{m=0}^{n-1}(-1)^m B_m^{n-1}I_m \tag{38}$$

where

$$I_m=\int_u^{u_0}\frac{p^{2m}}{\left(u_0^2-p^2\right)^{1/2}}dp,\quad\text{for}\quad m\ge 0 \tag{39}$$

and equation (39) is the same as equation (24) except for a change of index variable.

Using this similarity and the expressions above, the longitude difference given by equation (34) can be expressed as

$$\Delta\lambda = \cos\psi_0 \left\{ L_0 + \sum_{n=1}^{\infty} (-1)^n e^{2n} B_n^{\frac{1}{2}} \sum_{m=0}^{n-1} (-1)^m B_m^{n-1} I_m \right\} \tag{40}$$

Equation (40) can expanded as

$$\begin{aligned} \Delta\lambda = \cos\psi_0 \Bigg\{ L_0 &+ \left[-e^2 B_1^{\frac{1}{2}} + \sum_{n=2}^{\infty} (-1)^n e^{2n} B_n^{\frac{1}{2}} \right] I_0 \\ &+ \sum_{n=2}^{\infty} (-1)^n e^{2n} B_n^{\frac{1}{2}} \sum_{m=1}^{n-1} (-1)^m B_m^{n-1} I_m \Bigg\} \end{aligned} \tag{41}$$

and then simplified by use of the binomial series, where

$$\left(1-e^2\right)^{1/2} = \sum_{n=0}^{\infty} (-1)^n e^{2n} B_n^{\frac{1}{2}} = 1 + \sum_{n=1}^{\infty} (-1)^n e^{2n} B_n^{\frac{1}{2}} = 1 - e^2 B_1^{\frac{1}{2}} + \sum_{n=2}^{\infty} (-1)^n e^{2n} B_n^{\frac{1}{2}} \tag{42}$$

The terms in $[\cdots]$ of equation (41) are the last two terms on the right-hand side of equation (42) and using this equivalence gives

$$\begin{aligned} \Delta\lambda &= \cos\psi_0 \left\{ L_0 + \left(\sqrt{1-e^2} - 1 \right) I_0 + \sum_{n=2}^{\infty} (-1)^n e^{2n} B_n^{\frac{1}{2}} \sum_{m=1}^{n-1} (-1)^m B_m^{n-1} I_m \right\} \\ &= \cos\psi_0 \left\{ L_0 + \left(\sqrt{1-e^2} - 1 \right) I_0 + \tfrac{1}{2} \sum_{n=2}^{\infty} (-1)^n e^{2n} B_n^{\frac{1}{2}} \sum_{m=1}^{n-1} \frac{(-1)^m}{m} u_0^{2m} B_m^{n-1} J_m \right\} \end{aligned} \tag{43}$$

where I_0 is obtained from equation (31) and J_m are given by equation (28), noting that as before $J_m = \dfrac{2m}{u_0^{2m}} I_m$.

A simple expression for L_0 is obtained from equation (35) as follows

$$L_0 = \int_{\theta=\psi}^{\psi_0} \frac{1}{\cos\theta \left(\cos^2\theta - \cos^2\psi_0 \right)^{1/2}} d\theta = \int_{\theta=\psi}^{\psi_0} \frac{\sec^2\theta}{\left(\sin^2\psi_0 - \tan^2\theta \cos^2\psi_0 \right)^{1/2}} d\theta \tag{44}$$

Putting $x = \cot\psi_0 \tan\theta$ then $d\theta/dx = \tan\psi_0 \cos^2\theta$ and

$$\begin{aligned} \sin^2\psi_0 - \tan^2\theta \cos^2\psi_0 &= \sin^2\psi_0 \left(1 - \tan^2\theta \frac{\cos^2\psi_0}{\sin^2\psi_0} \right) \\ &= \sin^2\psi_0 \left(1 - \tan^2\theta \cot^2\psi_0 \right) \\ &= \sin^2\psi_0 \left(1 - x^2 \right) \end{aligned}$$

so that

$$L_0 = \frac{\tan\psi_0}{\sin\psi_0} \int_{x=\frac{\tan\psi}{\tan\psi_0}}^{1} \frac{dx}{\sqrt{1-x^2}} \tag{45}$$

since $\int \frac{dx}{\sqrt{1-x^2}} = \begin{cases} \arcsin x \\ \frac{\pi}{2} - \arccos x \end{cases}$, then using the second result gives

$$L_0 = \sec\psi_0 \int_{x=\frac{\tan\psi}{\tan\psi_0}}^{1} \frac{dx}{\sqrt{1-x^2}} = \sec\psi_0 \arccos\left(\frac{\tan\psi}{\tan\psi_0}\right) \tag{46}$$

Equation (40) can be simplified further to give the longitude difference $\Delta\lambda$ between P and the vertex P_0 as

$$\Delta\lambda = \cos\psi_0 \left\{M_0 + M_1 + M_2 + M_3 + \cdots\right\} \tag{47}$$

where

$$M_n = \begin{cases} L_0 & \text{for } n = 0 \\ \left(\sqrt{1-e^2} - 1\right) I_0 & \text{for } n = 1 \\ \frac{1}{2} B_n^{\frac{1}{2}} \left(-1\right)^n e^{2n} K_n & \text{for } n \geq 2 \end{cases} \tag{48}$$

and

$$K_n = \sum_{m=1}^{n-1} \frac{\left(-1\right)^m}{m} u_0^{2m} B_m^{n-1} J_m \quad \text{for } n = 2, 3, 4, \ldots \tag{49}$$

A GEODESIC ON AN ELLIPSOID DOES NOT REPEAT AFTER A SINGLE REVOLUTION

Earlier, it was mentioned that due to the eccentricity of the ellipsoid, the geodesic will not repeat after a complete revolution. Here is a demonstration of that fact.
When P is at the node A of Figure 3 then $\Delta\lambda = \Delta\lambda_4$ and using equation (17) we have

$$4\left(\Delta\lambda_4\right) = 4\cos\psi_0 \int_{\theta=0}^{\psi_0} \frac{\left(1-e^2\cos^2\theta\right)^{1/2}}{\cos\theta\left(\cos^2\theta - \cos^2\psi_0\right)^{1/2}} d\theta \tag{50}$$

Since this integral is difficult to evaluate, we instead determine upper and lower bounds for the quantity $4\left(\Delta\lambda_4\right)$ by using the bounds of the integration variable θ. This allows certain terms within the integral to be disposed of and a simplified integral evaluated.

For $0 \leq \theta \leq \psi_0$, the bounds on the numerator of the integrand are
$\left(1-e^2\right)^{1/2} \leq \left(1-e^2\cos^2\theta\right)^{1/2} \leq \left(1-e^2\cos^2\psi_0\right)^{1/2}$ so that on the one hand

$$\begin{aligned} 4\left(\Delta\lambda_4\right) &\leq 4\cos\psi_0 \int_{\theta=0}^{\psi_0} \frac{\left(1-e^2\cos^2\psi_0\right)^{1/2}}{\cos\theta\left(\cos^2\theta - \cos^2\psi_0\right)^{1/2}} d\theta \\ &= 4\cos\psi_0 \left(1-e^2\cos^2\psi_0\right)^{1/2} L_0\big|_{\psi=0} \\ &= 4\cos\psi_0 \left(1-e^2\cos^2\psi_0\right)^{1/2} \tfrac{1}{2}\pi\sec\psi_0 \\ &= 2\pi\left(1-e^2\cos^2\psi_0\right)^{1/2} \end{aligned} \tag{51}$$

while on the other hand

$$4\left(\Delta\lambda_4\right) \geq 4\cos\psi_0 \int_{\theta=0}^{\psi_0} \frac{\left(1-e^2\right)^{1/2}}{\cos\theta\left(\cos^2\theta - \cos^2\psi_0\right)^{1/2}}\,d\theta$$

$$= 2\pi\left(1-e^2\right)^{1/2} \tag{52}$$

Combining these inequalities gives the bounds for the quantity $4\left(\Delta\lambda_4\right)$ as

$$2\pi\left(1-e^2\right)^{1/2} \leq 4\left(\Delta\lambda_4\right) \leq 2\pi\left(1-e^2\cos^2\psi_0\right)^{1/2} \tag{53}$$

Therefore, after a single revolution, $4\left(\Delta\lambda_4\right) < 2\pi$ when $0^\circ < \psi_0 < 90^\circ$. Note that when $\psi_0 = 0^\circ$ the geodesic is an arc of the equator (a circle) and when $\psi_0 = 90^\circ$ the geodesic is an arc of the meridian (an ellipse).

NUMERICAL RESULTS FOR DISTANCE AND LONGITUDE EQUATIONS

Equations (32) and (47) for computing distance s and longitude difference $\Delta\lambda$ between P and the vertex P_0 are relatively simple summations of terms. To test the number of terms required for accurate answers, a geodesic was chosen with an azimuth $\alpha = 43^\circ 12' 36''$ at P having latitude $\phi = 9^\circ 35' 24''$ on the ellipsoid of the Geodetic Reference System 1980 (GRS80) (Moritz 1980), defined by $a = 6378137$ metres and $f = 1/298.257\,222101$.

Numerical constants for GRS80 ellipsoid and geodesic

$b \quad = a\left(1-f\right) \qquad = 6356752.314140356$ metres

$\psi \quad = \arctan\left[\left(1-f\right)\tan\phi\right] = 0.166826262923$ radians

$\psi_0 = \arccos\left[\cos\psi\sin\alpha\right] = 0.829602797993$ radians

$u \quad = \sin\psi \qquad = 0.166053515348; \quad u_0 = \sin\psi_0 \qquad = 0.737663250899$

$U \quad = \dfrac{u}{u_0} = \dfrac{\sin\psi}{\sin\psi_0} = 0.225107479796; \quad I_0 = \arccos U = 1.343742980976$ radians

$V \quad = \dfrac{\tan\psi}{\tan\psi_0} \qquad = 0.154125311675; \quad L_0 = \sec\psi_0 \arccos V = 2.097333540996$ radians

Table 1: Ellipsoid and geodesic constants and binomial coefficients for equations (32) and (47)

n	e^{2n}	ε^{n}	u_0^{2n}	$B_n^{\frac{1}{2}}$
1	6.694380022901e-003	6.739496775479e-003	0.544147071727	0.500000000000
2	4.481472389101e-005	4.542081678669e-005	0.296096035669	-0.125000000000
3	3.000067923478e-007	3.061134482735e-007	0.161119790759	0.062500000000
4	2.008359477428e-009	2.063050597570e-009	0.087672862339	-0.039062500000
5	1.344472156450e-011	1.390392284997e-011	0.047706931312	0.027343750000
6	9.000407545482e-014	9.370544321391e-014	0.025959586974	-0.020507812500
7	6.025214847044e-016	6.315275323850e-016	0.014125833235	0.016113281250
8	4.033507790574e-018	4.256177768135e-018	0.007686530791	-0.013092041016

Table 2: Recurrence formula values and distance components for equation (32)

n	J_n	D_n	
1	1.563072838216	8.541841303930e+006	8541841.303930 m
2	2.355723441968	9.109578467516e+003	9109.5784675
3	2.945217495733	-6.293571169346e+000	-6.2935712
4	3.436115617261	9.618619108010e-003	0.0096186
5	3.865631515581	-1.929070816523e-005	-0.0000193
6	4.252194740421	4.456897529564e-008	0.0000000
7	4.606544305836	-1.123696751599e-010	-0.0000000
8	4.935583185013	3.006580650377e-013	0.0000000
	sum	8.550944598425e+006	s = 8550944.598425 m

Table 3: Recurrence formula values and longitude components for equation (47)

n	J_n	M_n	
0		2.097333540996e+000	
1	1.563072838216	-4.505315819380e-003	
2	2.355723441968	2.382298926901e-006	
3	2.945217495733	1.267831357153e-008	
4	3.436115617261	6.525291638252e-011	
5	3.865631515581	3.431821056093e-013	
6	4.252194740421	1.852429353592e-015	$\Delta\lambda = \cos\psi_0\left(sum\right)$ $\cong 1.413013969112$ radians
7	4.606544305836	1.023576994037e-017	
8	4.935583185013	5.769507252421e-020	$= 80.959736823113$ degrees
	sum	2.092830620219e+000	$= 80^\circ\, 57'\, 35.052563''$

Inspection of these numerical values indicates than an upper limit of $N = 8$ in the summations is more than sufficient for accuracies of 0.000001 metre in distances and 0.000001 second of arc for longitude differences. [Results for s and $\Delta\lambda$ can be confirmed using Vincenty's equations (Vincenty 1975) that have been programmed in a Microsoft™ *Excel* workbook that can be downloaded from the website of Geoscience Australia at http://www.ga.gov.au/]

It should be noted here that the distance and longitude equations [equations (32) and (47)] are not themselves, solutions to the direct or inverse problems. Instead, they are the basic tools, which if used in certain ways, enable the solution to those problems.

In a computer program, equations (32) and (47) would be embedded in a function that returned s and $\Delta\lambda$ given the ellipsoid parameters (a, f), parametric latitudes (ψ, ψ_0) and the upper limit of summations (N). A brief explanation of how such a function might be used is given below.

USING THE DISTANCE AND LONGITUDE EQUATIONS TO COMPUTE THE DIRECT AND INVERSE PROBLEM

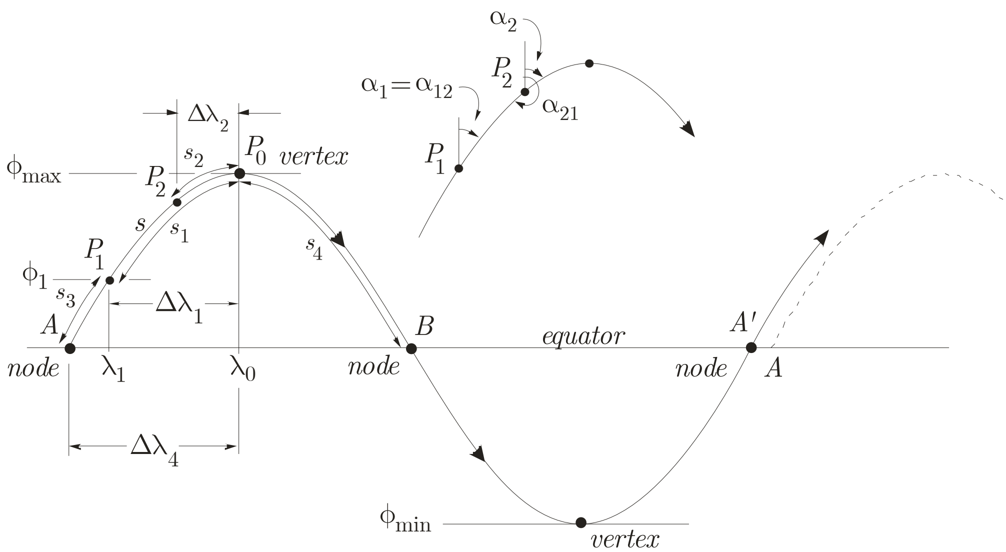

Fig. 5: Schematic diagram of a geodesic between P_1 and P_2 on an ellipsoid

Direct solution

The key here is to use the distance equation in an iterative computation of $\sin\psi_2$. Once this is known, then ϕ_2, λ_2 and α_{21} follow. The steps in the computation are:

1. Test the azimuth to determine whether the geodesic is heading towards or away from the <u>nearest</u> vertex P_0, noting that P_0 will be in the same hemisphere as P_1.
2. Compute ψ_1 and ψ_0; then use the distance and longitude equations to compute s_1 and $\Delta\lambda_1$ between P_1 and P_0, as well as λ_0. (see Fig. 5).
3. With $u = \sin\psi = 0$, compute s_4 and $\Delta\lambda_4$ between the node and P_0.

4. Compute $s_2 = \begin{cases} s - s_1 & \text{if geodesic is heading towards } P_0 \\ s + s_1 & \text{if geodesic is heading away from } P_0 \end{cases}$. If $s_2 > 0$ then P_2 is after P_0 and closer to another vertex P_0' in which case s_2 is reduced by multiples of $2s_4$ until $s_2 < s_4$ and the number of vertices *n* determined (vertices are $2s_4$ apart). If $s_2 < 0$ then P_2 is before P_0. (Note that in Fig. 5, $s_2 < 0$ and P_2 is before P_0)
5. Compute ψ_2 by iteration. An approximate value ψ_2' is found from equations (32) by taking the first term only; hence $\dfrac{s}{b} = I_0 = \arccos\left(\dfrac{\sin\psi}{\sin\psi_0}\right)$ and $\sin\psi_2' = \sin\psi_0 \cos\left(\dfrac{s_2}{b}\right)$.

   Now a re-arrangement of the differential equation (19) gives $du = \dfrac{ds}{b}\sqrt{\dfrac{u_0^2 - u^2}{1+\varepsilon u^2}}$ where $u = \sin\psi_2'$, $ds = s_2' - s_2$ and s_2' is computed from the distance equation with the approximate parametric latitude ψ_2'. Equation (19), linking *ds* and *du*, is the basis of the iterative solution for $\sin\psi_2$ (and hence ϕ_2).
6. After computing ψ_2 the longitude difference $\Delta\lambda_2$ is computed and depending on the number of vertices and the direction of the geodesic, λ_2 is determined. The azimuth α_2 follows from Clairaut's equation and the reverse azimuth α_{21} obtained.

Inverse solution

This is the more difficult of the two solutions since ψ_0 is unknown and must be determined by iteration, using approximations for s, α_1 and α_2 obtained by approximating the ellipsoid with a sphere and using spherical trigonometry. The steps in the computation are:

1. Convert longitudes of P_1 and P_2 to east longitudes in the range $0^\circ < \lambda_1, \lambda_2 < 360^\circ$ and determine a longitude difference $\Delta\lambda$ in the range $-180^\circ \le \Delta\lambda \le 180^\circ$. $\pm\Delta\lambda$ corresponding to east/west direction of the geodesic from P_1.
2. Compute parametric latitudes ψ_1 and ψ_2 then use these and $\Delta\lambda$ as latitudes and longitude difference on a sphere to compute spherical distance σ and spherical angles β_1 and β_2. These can be used to determine approximations of *s* and α_{12}.
3. Compute ψ_0 by iteration. Approximations $\Delta\lambda_1'$ and $\Delta\lambda_2'$ can be obtained from equation (47) noting that $M_0 = \sec\psi_0 \arccos\left(\dfrac{\tan\psi}{\tan\psi_0}\right)$ and ignoring terms $M_1, M_2, M_3, \ldots$

   This gives $\Delta\lambda_1' = \arccos\left(\dfrac{\tan\psi_1}{\tan\psi_0}\right)$ and $\Delta\lambda_2' = \arccos\left(\dfrac{\tan\psi_2}{\tan\psi_0}\right)$, and

$f\left(\psi_0\right)=\Delta\lambda'-\Delta\lambda=\left\{\pm\arccos\left(\frac{\tan\psi_1}{\tan\psi_0}\right)\pm\arccos\left(\frac{\tan\psi_2}{\tan\psi_0}\right)\pm\Delta\lambda_4'\right\}-\Delta\lambda$ where the ± signs are associated with the east/west direction of the geodesic.

ψ_0 can be found using Newton's iterative method (Williams 1972)

$$\left(\psi_0\right)_{n+1}=\left(\psi_0\right)_n-\frac{f\left(\psi_0\right)}{f'\left(\psi_0\right)} \tag{54}$$

where $f'\left(\psi_0\right)$ is the derivative of $f\left(\psi_0\right)$. An initial value of ψ_0 can be computed from equation (11).

4. Once ψ_0 is known then $s_1,\Delta\lambda_1;\, s_2,\Delta\lambda_2$ and $s_4,\Delta\lambda_4$ can be computed from the distance and longitude equations and *s* obtained. The forward and reverse azimuths can be found from Clairaut's equation (5).

CONCLUSION

Pittman's (1986) recurrence relationships for evaluating integrals allow beautifully compact equations for distance *s* and longitude difference $\Delta\lambda$ along a geodesic between *P* and the vertex P_0. These equations can be easily translated into a computer program function returning *s* and $\Delta\lambda$ given *a, f, u* and u_0. Using such a function, algorithms (as outlined above), can be constructed to solve the direct and inverse problems on the ellipsoid. Pittman's (1986) paper (which included FORTRAN computer code) has a concise development of the necessary equations and algorithms. The paper here has a more detailed development of the recurrence relationships (with a slightly different formulation) as well as additional information on the definition and properties of a geodesic.

Interestingly, Pittman's (1986) method is entirely different to other approaches that fall (roughly) into two groups: (i) numerical integration techniques and (ii) series expansion of integrals; the latter of these with a history of development extending back to Bessel's (1826) method. Numerical integration, a technique made practical with the arrival of computers in the mid to late 20th century, is relatively modern. So too is Pittman's method.

To our knowledge, this is the first paper (since the original) discussing his elegant method; a method that has much to recommend it, and one that we hope might become the object of study in undergraduate surveying courses and discussion in the geodetic literature.

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