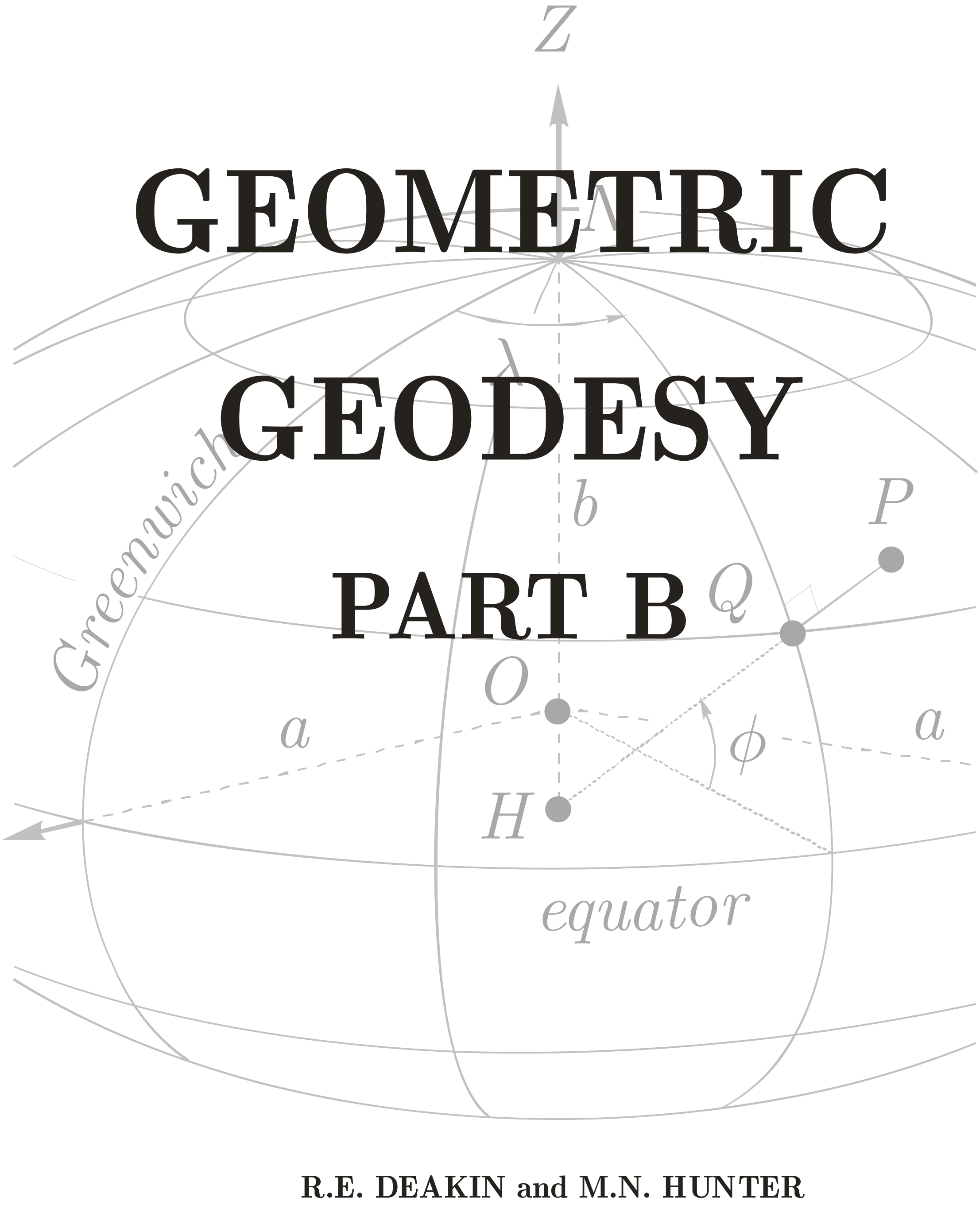
GEOMETRIC GEODESY

PART B



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FOREWORD

These notes are the second part of an introduction to ellipsoidal geometry related to geodesy. They are mainly concerned with the computation of distance and direction between points on a reference ellipsoid. The Earth's terrestrial surface is highly irregular and unsuitable for any mathematical computations, instead an ellipsoid – a surface of revolution created by rotating an ellipse about its minor axis – is adopted and points on the Earth's surface are projected onto the ellipsoid, via a normal to the ellipsoid. All computations are made using these projected points on this reference ellipsoid.

These notes are intended for undergraduate students studying courses in surveying, geodesy and map projections. The derivations of equations given herein are detailed, and in some cases elementary, but they do convey the vital connection between geodesy and the mathematics taught to undergraduate students.

These notes are a collection of papers written by the authors on the topic of computation of distance and azimuth between points on the reference ellipsoid. There are five lines or curves of interest in geodesy: the geodesic which is the curve of shortest length; the normal section curve; the curve of alignment; the great elliptic arc; and the loxodrome. The most important is of course the geodesic since it is the shortest distance between two points, but the other curves have their uses in navigation (the loxodrome) and in field surveying (normal section and curve of alignment).

The methods of computation outlined in these papers have been developed with the computer in mind – perhaps with the exception of F. W. Bessel's paper of 1826 – and most have MATLAB functions that demonstrate the application of the methods.

There is a certain amount of repetition in the papers as they are separate documents intended to give the reader an overview of the particular geodetic problem and then a detailed solution with computer examples of algorithms. So the student will see repeated treatments of the ellipsoid and associated formula as well as various solutions of the direct and inverse problems of geodesy. But, there may be something useful within the detail for the interested reader.

TABLE OF CONTENTS

pages

1. **THE CALCULATION OF LONGITUDE AND LATITUDE FROM GEODETIC MEASUREMENTS** 11

This is an English translation of *Über die Berechnung der geographischen Längen und Brieten aus geodätischen Vermessungen,* Astronomicische Nachrichten (Astronomical Notes) **4**(86), 241-254, 1826 by F. W. Bessel. This paper could be regarded as the first practical (and modern) treatment of computation using the geodesic. Bessel's method, with slight variation is still used today.

2. **GEODESICS ON AN ELLIPSOID – BESSEL'S METHOD** 66

This paper gives a detailed derivation of equations for the solution of the direct and inverses problems on the ellipsoid. Bessel's method as presented here is slightly different from the original but leads to Rainsford's equations and Vincenty's modifications. Vincenty's equations are commonly used for geodesic computation.

3. **GEODESICS ON AN ELLIPSOID – PITTMAN'S METHOD** 19

An alternative to Bessel's method (Vincenty's equations) that uses recurrence relationships to solve the direct and inverse problems. These methods are unusual but they give extremely good accuracies over any length of line anywhere on the ellipsoid.

TABLE OF CONTENTS (continued)

		pages
4.	**THE NORMAL SECTION CURVE ON AN ELLIPSOID**	53
5.	**THE CURVE OF ALIGNMENT ON AN ELLIPSOID**	20
6.	**THE GREAT ELLIPTIC ARC ON AN ELLIPSOID**	19
7.	**THE LOXODROME ON AN ELLIPSOID**	20

The calculation of longitude and latitude from geodesic measurements*

F. W. Bessel

Königsberg Observatory

(Originally published: October 1825; translated: August 13, 2009)

1. INTRODUCTION

Consider a geodesic line between two points A and B on the surface of the Earth. Given the position of A, the length of the line and its azimuth at A, we wish to determine the position of B and the azimuth of the line there. This problem occurs so frequently that I undertook to construct tables to simplify the computation. In order to explain the method clearly, I start by deriving the fundamental properties of geodesic lines on a spheroid of revolution. Even though aspects of this derivation may already be well known, the benefit of having the entire development presented together outweighs the cost of repeating it.[1]

2. THE CHARACTERISTIC EQUATION FOR A GEODESIC

Take two points A and B on the surface on a spheroid[2] of revolution joined by some specified curve. Consider two neighboring points on the curve with latitudes ϕ and $\phi + d\phi$ and longitudes relative to A of w and $w + dw$ (measuring east positive). Let the distance between them be ds, the azimuth of line directed toward A be α (measured clockwise from north), the radius of the circle of latitude be r, and the meridional radius of curvature by R; then we find[3]

$$\begin{aligned} ds \cos\alpha &= -R\, d\phi = \frac{dr}{\sin\phi}, \\ ds \sin\alpha &= -r\, dw, \end{aligned} \tag{1}$$

which gives

$$ds = \sqrt{R^2\, d\phi^2 + r^2\, dw^2}.$$

If we write p for $d\phi/dw$ and U for $\sqrt{R^2 p^2 + r^2}$, this becomes

$$ds = U\, dw.$$

The distance along the curve between the two points A and B is therefore

$$s = \int U\, dw,$$

where the integration is from A to B. If the curve is the geodesic or *shortest* path, then the relation between ϕ and w must be such that the integral is a minimum. If we perturb this relation so that ϕ is replaced by $\phi + z$ where z is an arbitrary function of w which vanishes at the end points (because these points lie on both curves), then the perturbed length,

$$s' = \int U'\, dw,$$

must be larger than s for all z.

Expanding $U(\phi, p)$ in a Taylor series, we obtain[4]

$$U' = U + \frac{\partial U}{\partial \phi} z + \frac{\partial U}{\partial p}\frac{dz}{dw} + \ldots$$

and therefore we have

$$s' = s + \int \left(\frac{\partial U}{\partial \phi} z + \frac{\partial U}{\partial p}\frac{dz}{dw} \right) dw + \ldots,$$

where we have explicitly included terms only up to first order in z. For s to be a minimum, we require that

$$\int \left(\frac{\partial U}{\partial \phi} z + \frac{\partial U}{\partial p}\frac{dz}{dw} \right) dw + \ldots \ge 0$$

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*This is an English translation of *Über die Berechnung der geographischen Längen und Breiten aus geodätischen Vermessungen*, Astronomische Nachrichten **4**(86), 241–254 (1826), doi:10.1002/asna.18260041601. The paper also appears in *Abhandlungen von Friedrich Wilhelm Bessel*, Vol. 3, pp. 5–14 (W. Engelmann, Leipzig, 1876). The translation has been prepared and edited by Charles F. F. Karney ⟨ckarney@sarnoff.com⟩ and Rodney E. Deakin ⟨rod.deakin@rmit.edu.au⟩, with the assistance of Max Hunter and Stephan Brunner. The mathematical notation has been updated to conform to current conventions and, in a few places, the equations have been rearranged for clarity. Several errors have been corrected, a figure has been included, and the tables have been recomputed. A transcription of the original paper with the updated mathematical notation and with the corrections is available at arXiv:0908.1823. A contemporary, but partial, translation into English appeared in Quart. Jour. Roy. Inst. **21**(41), 138–152 (1826).

[1] In Secs. 2–4, Bessel gives a concise summary of the work of several other authors, notably, Clairaut, du Séjour, Legendre, and Oriani. Bessel's contributions, which start in Sec. 5, consist of his methods for expanding the distance and longitude integrals and his compilation of tables to provide a practical method for computing geodesics. Two sentences have been omitted from this translation of the introduction. In one, Bessel refers to two letters he published earlier in the *Astronomische Nachrichten* which do not, however, have a direct bearing on the present work. In the other, he criticizes "du Séjour's method," but without providing details; in any case, such criticism is misplaced because du Séjour had died over 30 years earlier and Bessel does not cite more recent work.

[2] "Spheroid" here is used in the sense of a shape approximating a sphere. Sections 2 and 3 treat the case of a rotationally symmetric earth. In Sec. 4, Bessel specializes to a rotationally symmetric ellipsoid.

[3] The minus signs appear in (1) because α is the back azimuth, pointing to A, while ds advances the geodesic away from A. In this section, Bessel assumes an easterly geodesic so that $ds/dw > 0$. However the final result, Eq. (2), is general.

[4] The notation here employs partial derivatives instead of Bessel's less formal use of differentials.

for all z. Since this must also hold if z is replaced by $-z$ and since we can take z so small that the first order terms are bigger that the sum of all the higher order terms (except if the first order terms vanish), it follows that the condition that s be minimum is

$$\int \left(\frac{\partial U}{\partial \phi} z + \frac{\partial U}{\partial p} \frac{dz}{dw} \right) dw = 0.$$

Integrating the second term by parts to give $z(\partial U/\partial p) - \int z[d(\partial U/\partial p)/dw]\, dw$ and remembering that z vanishes at the end points, we obtain

$$\int z \left\{ \frac{\partial U}{\partial \phi} - \frac{d}{dw} \left(\frac{\partial U}{\partial p} \right) \right\} dw = 0.$$

Since this integral must vanish for *arbitrary* z, we find[5]

$$\frac{\partial U}{\partial \phi} - \frac{d}{dw} \left(\frac{\partial U}{\partial p} \right) = 0$$

or, multiplying by $d\phi/dw = p$,

$$\frac{\partial U}{\partial \phi} \frac{d\phi}{dw} + \frac{\partial U}{\partial p} \frac{dp}{dw} - \frac{dp}{dw} \frac{\partial U}{\partial p} - p \frac{d}{dw} \left(\frac{\partial U}{\partial p} \right) = 0,$$

which on integrating with respect to w becomes[6]

$$U - p \left(\frac{dU}{dp} \right) = \mathrm{const.}$$

Substituting $\sqrt{r^2 + R^2 p^2}$ for U, we obtain[7]

$$\frac{r}{\sqrt{1 + (R^2/r^2) p^2}} = -r \sin\alpha = \mathrm{const.},$$

which is the well known characteristic equation of the geodesic.

If the azimuth of the geodesic at A (in the direction of B) is α' and the distance of A from the rotation axis is r', we have

$$r' \sin(\alpha' + 180°) = r \sin\alpha,$$

or

$$r' \sin\alpha' = -r \sin\alpha. \qquad (2)$$

3. THE AUXILIARY SPHERE

Let the maximum distance of the spheroid to the rotation axis be a, so that r and r' are less than or equal to a; we can then write[8]

$$r' = a \cos u', \quad r = a \cos u,$$

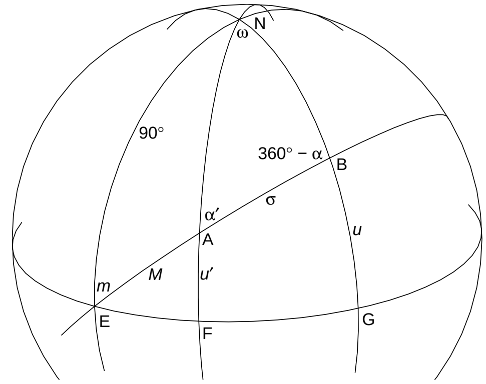

Figure 1 Spherical triangles on the auxiliary sphere. EAB is the geodesic, N is the pole; EFG is the equator; and NE, NAF, and NBG are meridians.

and equation (2) becomes

$$\cos u' \sin\alpha' = -\cos u \sin\alpha. \qquad (3)$$

This equation relates two sides of a spherical triangle,[9] $90° - u'$ and $90° - u$, and their opposite angles, $360° - \alpha$ and α'. The third side σ and its opposite angle ω will appear in the following calculations giving elegant expressions for the joint variations of s, u and w. In particular, using the well known differential formulas of spherical trigonometry, we find[10]

$$\begin{aligned} du &= -\cos\alpha\, d\sigma, \\ \cos u\, d\omega &= -\sin\alpha\, d\sigma. \end{aligned}$$

Substituting these in equations (1) and expressing r in terms of u gives

$$\begin{aligned} ds &= a \frac{\sin u}{\sin\phi} d\sigma, \\ dw &= \frac{\sin u}{\sin\phi} d\omega. \end{aligned} \qquad (4)$$

4. THE EQUATIONS FOR A GEODESIC ON AN ELLIPSOID

I now assume that the meridian is an ellipse with equatorial semi-axis a, polar semi-axis b, and eccentricity $e = \sqrt{a^2 - b^2}/a$.[11] The equation for an ellipse expressed in terms

[5] This is the Euler-Lagrange equation of the calculus of variations.

[6] This is now known as the Beltrami identity.

[7] A. C. Clairaut gives a geometric derivation of this result in Mém. de l'Acad. Roy. des Sciences de Paris, 1733, 406–416 (1735). The equation also follows from conservation of angular momentum for a mass sliding without friction on a spheroid of revolution.

[8] The quantity u is the *reduced* or *parametric* latitude.

[9] See the triangle ABN on the "auxiliary sphere" in Fig. 1; Equation (3) is the sine rule applied to angles A and B of the triangle.

[10] Here and in the rest of the paper, the differentials give the movement of point B along the geodesic defined with point A and α' held fixed.

[11] In Bessel's time, it was known that the earth could be approximated by an oblate ellipsoid, $a > b$, but the eccentricity had not been determined accurately. Therefore, Bessel computes tables which are applicable to oblate ellipsoids with a range of eccentricities. However, the series expansions that Bessel obtains, (11) and (12), can also to applied to prolate ellipsoids, $a < b$, by allowing $e^2 < 0$.

of cartesian coordinates is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Differentiating this and setting $dy/dx = -\cot\phi$, we obtain

$$\frac{x\sin\phi}{a^2} - \frac{y\cos\phi}{b^2} = 0;$$

eliminating y between these equations then gives

$$x = \frac{a\cos\phi}{\sqrt{1 - e^2\sin^2\phi}}.$$

The quantity x is the same as $r = a\cos u$, which gives the relationships between ϕ and u,

$$\begin{aligned}
\cos u &= \frac{\cos\phi}{\sqrt{1 - e^2\sin^2\phi}}, & \cos\phi &= \frac{\cos u\sqrt{1-e^2}}{\sqrt{1 - e^2\cos^2 u}},\\
\sin u &= \frac{\sin\phi\sqrt{1-e^2}}{\sqrt{1 - e^2\sin^2\phi}}, & \sin\phi &= \frac{\sin u}{\sqrt{1 - e^2\cos^2 u}},\\
\tan u &= \tan\phi\sqrt{1-e^2}, & \tan\phi &= \frac{\tan u}{\sqrt{1-e^2}},
\end{aligned}$$

and

$$\frac{\sin u}{\sin\phi} = \sqrt{1 - e^2\cos^2 u}.$$

Substituting this into (4), we obtain the differential equations for a geodesic on an ellipsoid

$$\begin{aligned}
ds &= a\sqrt{1 - e^2\cos^2 u}\,d\sigma,\\
dw &= \sqrt{1 - e^2\cos^2 u}\,d\omega.
\end{aligned} \tag{5}$$

5. THE DISTANCE INTEGRAL

To integrate the first of these differential equations, I use the three relations between u', u, α', α and σ,[12]

$$\begin{aligned}
\sin u &= \sin u'\cos\sigma + \cos u'\cos\alpha'\sin\sigma,\\
-\cos u\cos\alpha &= -\sin u'\sin\sigma + \cos u'\cos\alpha'\cos\sigma,\\
-\cos u\sin\alpha &= \cos u'\sin\alpha'.
\end{aligned} \tag{6}$$

It is convenient to write these in terms of the auxiliary angles m and M defined by[13]

$$\begin{aligned}
\sin u' &= \cos m\sin M,\\
\cos u'\cos\alpha' &= \cos m\cos M,\\
\cos u'\sin\alpha' &= \sin m.
\end{aligned} \tag{7}$$

[12] Referring to Fig. 1, consider two central cartesian coordinate systems with the xy plane containing the geodesic EAB, and either A or B lying on the x axis. Equations (6) give the transformation between the coordinates of N in the two systems, $[\sin u', \cos u'\cos\alpha', \cos u'\sin\alpha']$ and $[\sin u, -\cos u\cos\alpha, -\cos u\sin\alpha]$, namely a rotation by σ about the z axis.

[13] The auxiliary angles m and M are an angle and a side of the spherical triangle EAN shown in Fig. 1. Equations (7) are the sine rule on angles E and F of triangle AEF, the cosine rule on angle F of triangle AEF, and the sine rule on angles A and E of triangle ANE.

Equations (6) then become[14]

$$\begin{aligned}
\sin u &= \cos m\sin(M+\sigma),\\
\cos u\cos\alpha &= -\cos m\cos(M+\sigma),\\
\cos u\sin\alpha &= -\sin m.
\end{aligned} \tag{8}$$

This gives

$$\cos^2 u = 1 - \cos^2 m\sin^2(M+\sigma),$$

and the equation for ds becomes

$$ds = a\sqrt{1-e^2}\sqrt{1 + k^2\sin^2(M+\sigma)}\,d\sigma, \tag{9}$$

where

$$k = \frac{e\cos m}{\sqrt{1-e^2}}.$$

This differential equation may be integrated in terms of the elliptic integrals introduced by Legendre.[15] Because the tools to compute these special functions are not yet sufficiently versatile,[16] we instead develop a series solution which converges rapidly because e^2 is so small. We readily achieve this by decomposing the term under the square root into two complex factors, namely[17]

$$\begin{aligned}
ds = a\frac{\sqrt{1-e^2}}{1-\epsilon}\,d\sigma\times&\\
\sqrt{1 - \epsilon\exp\bigl(2i(M+\sigma)\bigr)}&\sqrt{1 - \epsilon\exp\bigl(-2i(M+\sigma)\bigr)}\,,
\end{aligned}$$

where

$$\epsilon = \frac{\sqrt{1+k^2}-1}{\sqrt{1+k^2}+1}, \quad k = \frac{2\sqrt{\epsilon}}{1-\epsilon}.$$

Expanding the two factors in the radicals in infinite series and multiplying the results gives[18]

$$\begin{aligned}
ds = a\frac{\sqrt{1-e^2}}{1-\epsilon}\,d\sigma\bigl[&A - 2B\cos 2(M+\sigma)\\
&- 2C\cos 4(M+\sigma) - 2D\cos 6(M+\sigma) - \ldots\bigr],
\end{aligned}$$

[14] These are analogs of Eqs. (7) with meridian NAF replaced by NBG.

[15] A. M. Legendre, *Exercices du calcul intégral*, Vol. 1 (Courcier, Paris, 1811).

[16] Even though good numerical algorithms for elliptic integrals are available, these usually require linking to an additional library and, for that reason, computations of geodesics are still usually in terms of a series.

[17] The notation has been simplified here compared to Bessel's original formulation in which k and ϵ are expressed in terms of E through $k = \tan E$ and $\epsilon = \tan^2\frac{1}{2}E$. By using ϵ as the expansion parameter and by dividing out the factor $1-\epsilon$, Bessel has ensured that the terms that he is expanding are invariant under the transformation $\epsilon \to -\epsilon$, $M+\sigma \to \pi/2-(M+\sigma)$. This symmetry causes half the terms in the expansions in ϵ to vanish.

[18] The use of complex exponentials facilitates the series expansions by avoiding the need to employ awkward trigonometric identities. If we write $\sqrt{1-x} = 1 - \frac{1}{2}x - \frac{1\cdot1}{2\cdot4}x^2 - \frac{1\cdot1\cdot3}{2\cdot4\cdot6}x^3 - \frac{1\cdot1\cdot3\cdot5}{2\cdot4\cdot6\cdot8}x^4 - \ldots = \sum_j a_j x^j$, then the coefficient of $\cos\bigl(2l(M+\sigma)\bigr)\epsilon^{l+2j}$ is a_j^2 for $l = 0$ and $2a_j a_{j+l}$ for $l > 0$.

where A, B, C, … are given by

$$A = 1 + \left(\frac{1}{2}\right)^2 \epsilon^2 + \left(\frac{1\cdot 1}{2\cdot 4}\right)^2 \epsilon^4 + \left(\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\right)^2 \epsilon^6 + \dots,$$
$$B = \frac{1}{2}\epsilon - \frac{1\cdot 1}{2\cdot 4}\,\frac{1}{2}\epsilon^3 - \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\,\frac{1\cdot 1}{2\cdot 4}\epsilon^5 - \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}\,\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\epsilon^7 - \dots,$$
$$C = \frac{1\cdot 1}{2\cdot 4}\epsilon^2 - \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\,\frac{1}{2}\epsilon^4 - \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}\,\frac{1\cdot 1}{2\cdot 4}\epsilon^6 - \frac{1\cdot 1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 10}\,\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\epsilon^8 - \dots,$$
$$D = \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\epsilon^3 - \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}\,\frac{1}{2}\epsilon^5 - \frac{1\cdot 1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 10}\,\frac{1\cdot 1}{2\cdot 4}\epsilon^7 - \frac{1\cdot 1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12}\,\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\epsilon^9 - \dots,$$

etc.

Integrating the equation for ds starting at $\sigma = 0$, we obtain

$$s = \frac{b}{1-\epsilon}\bigl[A\sigma - \tfrac{2}{1}B\cos(2M+\sigma)\sin\sigma - \tfrac{2}{2}C\cos(4M+2\sigma)\sin 2\sigma - \tfrac{2}{3}D\cos(6M+3\sigma)\sin 3\sigma - \dots\bigr]. \tag{10}$$

6. SOLVING THE DISTANCE EQUATION

The series (10) gives the distance s between A and B in terms of u', α', and σ; if, however, s and α' have been measured and u' is known from the latitude at A, then σ is obtained by solving (10). The latitude of B and the azimuth of the geodesic there are found from (8). Equation (10) can be solved either by reverting the series or by successive approximation—the latter way is however the simplest if the tables I have compiled are used.

I write[19]

$$\sigma = \frac{\alpha}{b}s + \beta\cos(2M+\sigma)\sin\sigma + \gamma\cos(4M+2\sigma)\sin 2\sigma + \delta\cos(6M+3\sigma)\sin 3\sigma + \dots, \tag{11}$$

where

$$\alpha = \frac{648\,000}{\pi}\,\frac{1-\epsilon}{A},$$
$$\beta = \frac{648\,000}{\pi}\,\frac{2B}{A},$$
$$\gamma = \frac{648\,000}{\pi}\,\frac{C}{A},$$
$$\delta = \frac{648\,000}{\pi}\,\frac{2D}{3A},$$

etc.

[19] The units for σ, α, β, … are arc seconds. Bessel here adopts a conflicting notation for the coefficient α which should not be confused with the azimuth.

The tables give the logarithms[20] of α, β, and γ as a function of the argument

$$\log k = \log\frac{e\cos m}{\sqrt{1-e^2}}.$$

By this choice, the variation of $\log\beta$ and $\log\gamma$ are very close to two and four times that of the argument, which simplifies interpolation into the table.[21]

We take $\alpha s/b$ as the first approximation of σ, substitute this into the second term to obtain a second approximation, with which we recalculate the second term and add the third. The convergence of the series is sufficiently fast that, even if the argument is $\bar{1}.1$ (which is only possible if the flattening of the ellipsoid, $1 - b/a$, exceeds $\frac{1}{128}$), the approximation never needs to be carried further in order to keep the errors in σ under $0.001''$. The term involving δ does not exceed $0.0005''$ at this value of the argument.

7. ACCURACY OF THE TABLES

The values of $\log\alpha$ in the table are given to 8 decimal places.[22] An error of half a unit of the last place results in an error of only $0.0005''$ or $0.008\,\mathrm{toise}$ over a distance corresponding to $\sigma = 12°4'$ or 700 000 toises.[23] Similarly, I retain only sufficient digits in the tabulation of $\log\beta$ to ensure that the error in this term is less than $0.0005''$; for this purpose, I use 6 digits at the end of the table and fewer digits for smaller values of the argument. The third term never exceeds $0.17''$, even at the end of the table; therefore I include only 3 decimal places for $\log\gamma$. Thus the errors are $0.001''$ for distances up to 700 000 toises; even if the distance is of the order of a quarter meridian (i.e., $\sigma = 90°$), the error is less than $0.01''$.

8. AN EXAMPLE

In order to illustrate the use of the tables, I consider the results from the great survey by von Müffling.[24] Relative to

[20] In this paper, $\log x$ denotes the common logarithm (base 10) and we use $\operatorname{colog} x = \log(1/x)$. The tables in the original paper contained a number of errors of one unit in the last place. These errors do not, for the most part, affect the results obtained from the tables when rounded to $0.001''$. In addition, there were systematic errors in the tabulated values of $\log\beta$ equivalent to a relative error of order ϵ^2 in β which result in discrepancies from 1 to 17 units in the last place on the final page (the 6-figure portion) of the tables. In calculations involving logarithms, a bar over a numeral indicates that that numeral should be negated, e.g., $\log 0.02 \approx \bar{2}.3 = (-2) + 0.3$. In the original paper, logarithms are written modulo 10, e.g., $\log 0.02 \approx 8.3$. The notation "$(-)$" in these calculations indicates that the quantity whose logarithm is being taken is negative.

[21] The columns headed Δ give the first differences of the immediately preceding columns and aid in interpolating the data. Bessel would have used a table of "proportional parts" to compute the interpolated values.

[22] Working with 8-figure logarithms provides about 2 bits more precision than IEEE single precision floating point numbers.

[23] The toise was a French unit of length. It can be converted to meters by $1\,\mathrm{toise} = 864\,\mathrm{ligne}$, $443.296\,\mathrm{ligne} = 1\,\mathrm{m}$, or $1\,\mathrm{toise} \approx 1.949\,\mathrm{m}$.

[24] F. K. F. von Müffling, Astron. Nachr. **2**(27), 33–38 (1824).

Seeberg (point A), the distance and azimuth to Dunkirk (point B) are[25]

$$\log s = 5.478\,303\,14,$$
$$\alpha' = 274^\circ\,21'\,3.18''.$$

I assume the latitude of the Observatory at Seeberg to be $\phi' = 50^\circ\,56'\,6.7''$ and the ellipsoid parameters to be $\log b = 6.513\,354\,64$, $\log e = \bar{2}.905\,4355$.[26]

From $\tan u' = \sqrt{1-e^2}\tan\phi'$, we find

$$\begin{aligned}
\log\tan\phi' &= 0.090\,626\,65\\
\log\sqrt{1-e^2} &= \underline{\bar{1}.998\,590\,60}\\
\log\tan u' &= 0.089\,217\,25; \qquad u' = 50^\circ\,50'\,39.057''.
\end{aligned}$$

Given u' and α', we can compute M, $\cos m$ and $\sin m$ from equations (7):[27]

$$\begin{aligned}
\log\sin u' &= \bar{1}.889\,543\,51\\
\log\cos u' &= \bar{1}.800\,326\,27\\
\log\cos\alpha' &= \bar{2}.880\,037\,33\\
\log\sin\alpha' &= \underline{\bar{1}.998\,746\,62}(-)\\
\log(\cos m\sin M) &= \bar{1}.889\,543\,51\\
\log(\cos m\cos M) &= \bar{2}.680\,363\,60\\
\log\,\sin m &= \underline{\bar{1}.799\,072\,89}(-)\\
M &= 86^\circ\,27'\,53.949''; \quad 2M = 172^\circ\,55'\,47.9''\\
\log\cos m &= \bar{1}.890\,370\,63 \qquad 4M = 345^\circ\,51'\,36''.
\end{aligned}$$

The argument in the tables, $\log\bigl((e/\sqrt{1-e^2})\cos m\bigr)$, is

$$\begin{aligned}
\log\frac{e}{\sqrt{1-e^2}} &= \bar{2}.906\,845\\
\log\cos m &= \underline{\bar{1}.890\,371}\\
\text{Argument} &= \bar{2}.797\,216.
\end{aligned}$$

Looking up $\log\alpha$ in the tables, and calculating $\alpha s/b$ gives[28]

$$\begin{aligned}
\log\alpha &= 5.313\,998\,92\\
\operatorname{colog} b &= \bar{7}.486\,645\,36\\
\log s &= \underline{5.478\,303\,14}\\
\log\frac{\alpha s}{b} &= 4.278\,947\,42; \qquad \frac{\alpha}{b}s = 5^\circ\,16'\,48.481''.
\end{aligned}$$

Adopting this as the first approximation to the value of σ, we obtain the second by adding the first term in the series (11),

$$\begin{aligned}
\log\beta &= 2.305\,94\\
\log\cos(2M+\sigma) &= \bar{1}.999\,79(-)\\
\log\sin\sigma &= \underline{\bar{2}.963\,91}\\
& \quad 1.269\,64(-) = -18.61''.
\end{aligned}$$

We now update the value of this term with the second approximation of $\sigma = 5^\circ\,16'\,48.5'' - 18.6'' = 5^\circ\,16'\,29.9''$ and so obtain as the third approximation:

$$\begin{aligned}
\log\beta &= 2.305\,94\\
\log\cos(2M+\sigma) &= \bar{1}.999\,79(-)\\
\log\sin\sigma &= \underline{\bar{2}.963\,48}\\
& \quad 1.269\,21(-) = -18.587'',
\end{aligned}$$

$$\begin{aligned}
\log\gamma &= \bar{2}.394\\
\log\cos(4M+2\sigma) &= \bar{1}.999\\
\log\sin 2\sigma &= \underline{\bar{1}.263}\\
& \quad \bar{3}.656 = +0.005''.
\end{aligned}$$

Gathering the terms in (11) gives $\sigma = 5^\circ\,16'\,48.481'' - 18.587'' + 0.005'' = 5^\circ\,16'\,29.899''$ and so, finally, we determine α, u and ϕ from equations (8),

$$\begin{aligned}
M+\sigma &= 91^\circ\,44'\,23.848''\\
\log\sin(M+\sigma) &= \bar{1}.999\,799\,71\\
\log\bigl(-\cos(M+\sigma)\bigr) &= \bar{2}.482\,349\,32\\
\log\cos m &= \bar{1}.890\,370\,63\\
\log(-\sin m) &= \underline{\bar{1}.799\,072\,89}\\
\log\,\sin u &= \bar{1}.890\,170\,34\\
\log(\cos u\cos\alpha) &= \bar{2}.372\,719\,95\\
\log(\cos u\sin\alpha) &= \underline{\bar{1}.799\,072\,89}\\
\log\cot\alpha &= \bar{2}.573\,647\,06; \qquad \alpha = 87^\circ\,51'\,15.523''\\
\log\cos u &= \bar{1}.799\,377\,50\\
\log\tan u &= 0.090\,792\,84\\
\operatorname{colog}\sqrt{1-e^2} &= \underline{0.001\,409\,40}\\
\log\tan\phi &= 0.092\,202\,24; \qquad \phi = 51^\circ\,2'\,12.719''.
\end{aligned}$$

In this example, I carried out the trigonometric calculations to 8 decimals; however the tables of $\log\alpha$, $\log\beta$, and $\log\gamma$ in fact allow α and ϕ to be determined slightly more accurately than this. If only standard 7-figure logarithm tables are available, the last digits in the tabulated values of $\log\alpha$, $\log\beta$, and $\log\gamma$ may be neglected.

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[25] Seeberg: $50^\circ 56'$N $10^\circ 44'$E; Dunkirk: $51^\circ 2'$N $2^\circ 23'$E.

[26] In present-day units, this is $a \approx 6377\,\mathrm{km}$, flattening $f \approx 1/308.6$, $s \approx 586\,\mathrm{km}$. In this example, Bessel uses the toise as the unit of length and the second as the unit of arc.

[27] Bessel solves 3 equations (7) for 2 unknowns M and m. The redundancy serves as a check for the hand calculation and can also improve the accuracy of the calculation, for example, in the case where $\sin m \approx 1$.

[28] It is necessary to use second differences when interpolating in the table for $\log\alpha$. The argument, $\bar{2}.797\,216$, lies $q = 0.7216$ of the way between $\bar{2}.79$ and $\bar{2}.80$. Bessel's central 2nd-order interpolation formula for the last 6 digits of $\log\alpha$ gives $401\,284 + q(-1941) + \frac{1}{4}q(q-1)(1853 - 1004 - 1028) = 399\,892$. For the other table look-ups, linear interpolation using first differences suffices.

9. THE LONGITUDE INTEGRAL

We turn now to the determination of the longitude difference w by integrating (5),

$$dw = \sqrt{1 - e^2 \cos^2 u}\, d\omega.$$

This integral contains two separate constants m and e, which cannot be combined. Thus it not possible to construct tables to allow a rigorous solution of this problem which are valid for arbitrary e.[29] However, we can achieve our goal by sacrificing strict rigor and by making an approximation which results in errors which are inconsequential in our application.

We start by writing

$$dw = d\omega - \bigl(1 - \sqrt{1 - e^2 \cos^2 u}\bigr) d\omega,$$

and substitute in the second term

$$d\omega = \frac{\sin\alpha' \cos u'}{\cos^2 u}\, d\sigma.$$

On integrating, we obtain

$$w = \omega - \sin\alpha' \cos u' \int \frac{1 - \sqrt{1 - e^2 \cos^2 u}}{\cos^2 u}\, d\sigma.$$

Let us write

$$\frac{1 - \sqrt{1 - e^2 \cos^2 u}}{\cos^2 u} = \frac{e^2}{2}(1 + e^2 p \cos^2 u)^q (1 + y);$$

in other words, we set

$$\begin{aligned} 1 + y &= \frac{2(1 - \sqrt{1 - e^2 \cos^2 u})}{e^2 \cos^2 u (1 + e^2 p \cos^2 u)^q} \\ &= \frac{1 + \frac{1}{4} e^2 \cos^2 u + \frac{1}{8} e^4 \cos^4 u + \frac{5}{64} e^6 \cos^6 u + \dots}{\begin{pmatrix} 1 + qpe^2 \cos^2 u + \frac{q(q-1)}{1\cdot 2} p^2 e^4 \cos^4 u \\ \qquad + \frac{q(q-1)(q-2)}{1\cdot 2\cdot 3} p^3 e^6 \cos^6 u + \dots \end{pmatrix}}. \end{aligned}$$

The first three terms in the denominator and in the numerator are equal, provided that

$$p = -\tfrac{3}{4}, \qquad q = -\tfrac{1}{3},$$

which gives

$$\begin{aligned} 1 + y &= \frac{1 + \frac{1}{4} e^2 \cos^2 u + \frac{1}{8} e^4 \cos^4 u + \frac{5}{64} e^6 \cos^6 u + \dots}{1 + \frac{1}{4} e^2 \cos^2 u + \frac{1}{8} e^4 \cos^4 u + \frac{7}{96} e^6 \cos^6 u + \dots} \\ &= 1 + \tfrac{1}{192} e^6 \cos^6 u + \dots \end{aligned}$$

From this, we see that neglecting y results in an error of order e^8 or an error in w of $\frac{1}{384} e^8 \sigma$. This would not be discernible even in the calculation of long geodesics to 10 decimal places.[30]

Thus, for the present purposes, we may take $y \approx 0$ enabling us to tabulate the integral in a way that is valid for all e.

10. SERIES EXPANSION FOR LONGITUDE

Introducing this approximation, we have

$$\begin{aligned} w &\approx \omega - \frac{e^2}{2} \sin m \int \frac{d\sigma}{\sqrt[3]{1 - \frac{3}{4} e^2 \cos^2 u}} \\ &= \omega - \frac{e^2}{2} \sin m \int \frac{d\sigma}{\sqrt[3]{1 - \frac{3}{4} e^2 + \frac{3}{4} e^2 \cos^2 m \sin^2(M + \sigma)}}. \end{aligned}$$

If we set

$$k' = \frac{\sqrt{\frac{3}{4}} e \cos m}{\sqrt{1 - \frac{3}{4} e^2}},$$

we can express the integral in the second term as

$$\int \frac{d\sigma}{\sqrt[3]{1 - \frac{3}{4} e^2}\, \sqrt[3]{1 + k'^2 \sin^2(M + \sigma)}}.$$

Following the same procedure used in expanding the integral for ds in Sec. 5, we introduce ϵ' defined by[31]

$$\epsilon' = \frac{\sqrt{1 + k'^2} - 1}{\sqrt{1 + k'^2} + 1}, \quad k' = \frac{2\sqrt{\epsilon'}}{1 - \epsilon'},$$

and separate the integrand into two complex factors,

$$\int \frac{\sqrt[3]{(1 - \epsilon')^2 / \bigl(1 - \frac{3}{4} e^2\bigr)}\, d\sigma}{\sqrt[3]{1 - \epsilon' \exp\bigl(2i(M + \sigma)\bigr)}\, \sqrt[3]{1 - \epsilon' \exp\bigl(-2i(M + \sigma)\bigr)}}.$$

If we expand these in infinite series, the product becomes[32]

$$\begin{aligned} \frac{2}{\sqrt[3]{1 - \frac{3}{4} e^2}} \int \Bigl(&\alpha' + \beta' \cos 2(M + \sigma) + 2\gamma' \cos 4(M + \sigma) \\ &+ 3\delta' \cos 6(M + \sigma) + \dots \Bigr) d\sigma, \end{aligned}$$

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[29] As a practical matter, it would have been impossible for Bessel to provide a complete tabulation of a function of two parameters. He could have tabulated the function for a fixed value of e, which would greatly reduced the utility of his method, especially given the uncertainties in the measurements of e. Instead, Bessel manipulates the expression for dw to move the dependence on the second parameter into a small term that may be neglected.

[30] For a flattening of $\frac{1}{128}$, the error in the longitude difference over a distance equivalent to a quarter meridian, i.e., $10\,000$ km, is less than $0.000\,05''$.

[31] Bessel gives the relationship between k' and ϵ' in terms of E', where $k' = \tan E'$ and $\epsilon' = \tan^2 \frac{1}{2} E'$.

[32] There are a series of errors in the original paper leading up to (12). Here we assume that the original Eq. (12) defines α', β', γ', …, which makes this equation analogous to (11), and correct the preceding equations to be consistent.

where[33]

$$\alpha' = \tfrac{1}{2}\sqrt[3]{(1-\epsilon')^2}\left[1 + \left(\frac{1}{3}\right)^2 \epsilon'^2 + \left(\frac{1\cdot 4}{3\cdot 6}\right)^2 \epsilon'^4 + \dots\right],$$

$$\beta' = \tfrac{1}{1}\sqrt[3]{(1-\epsilon')^2}\left[\frac{1}{3}\epsilon' + \frac{1\cdot 4}{3\cdot 6}\,\frac{1}{3}\epsilon'^3 + \frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}\,\frac{1\cdot 4}{3\cdot 6}\epsilon'^5 + \dots\right],$$

$$\gamma' = \tfrac{1}{2}\sqrt[3]{(1-\epsilon')^2}\left[\frac{1\cdot 4}{3\cdot 6}\epsilon'^2 + \frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}\,\frac{1}{3}\epsilon'^4 + \frac{1\cdot 4\cdot 7\cdot 10}{3\cdot 6\cdot 9\cdot 12}\,\frac{1\cdot 4}{3\cdot 6}\epsilon'^6 + \dots\right],$$

$$\delta' = \tfrac{1}{3}\sqrt[3]{(1-\epsilon')^2}\left[\frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}\epsilon'^3 + \frac{1\cdot 4\cdot 7\cdot 10}{3\cdot 6\cdot 9\cdot 12}\,\frac{1}{3}\epsilon'^5 + \frac{1\cdot 4\cdot 7\cdot 10\cdot 13}{3\cdot 6\cdot 9\cdot 12\cdot 15}\,\frac{1\cdot 4}{3\cdot 6}\epsilon'^7 + \dots\right],$$

etc.

Integrating from $\sigma = 0$ then gives

$$w \approx \omega - \frac{e^2 \sin m}{\sqrt[3]{1 - \frac{3}{4}e^2}}\Bigl(\alpha'\sigma + \beta'\cos(2M+\sigma)\sin\sigma + \gamma'\cos(4M+2\sigma)\sin 2\sigma + \delta'\cos(6M+3\sigma)\sin 3\sigma + \dots\Bigr). \quad (12)$$

11. COMPUTING THE LONGITUDE DIFFERENCE

The first two coefficients of this series are given in the 4th and 5th columns of the tables[34] as functions of the argument

$$\log k' = \log\left(\frac{\sqrt{\frac{3}{4}}e}{\sqrt{1-\frac{3}{4}e^2}}\cos m\right).$$

The convergence is commensurate with the 3 first columns of the tables. We calculate ω using one of the formulas for spherical triangles (Sec. 3), either[35]

$$\sin\omega = \frac{\sin\sigma\sin\alpha'}{\cos u} = \frac{-\sin\sigma\sin\alpha}{\cos u'} = \frac{\sin\sigma\sin m}{\cos u\cos u'},$$

or[36]

$$\tan\tfrac{1}{2}\omega = \frac{\sin\frac{1}{2}(u'-u)}{\cos\frac{1}{2}(u'+u)}\cot\tfrac{1}{2}(\alpha'+\alpha) = \frac{\cos\frac{1}{2}(u'-u)}{\sin\frac{1}{2}(u'+u)}\cot\tfrac{1}{2}(\alpha'-\alpha).$$

and evaluate w by means of the tables.

I will continue with the example in Sec. 8 and calculate the longitude difference between Dunkirk and Seeberg using this prescription. Solving the spherical triangle for ω gives

$$\begin{aligned} \log\sin\sigma &= \bar{2}.963\,483\,83 \\ \log(-\sin\alpha) &= \bar{1}.999\,695\,39(-) \\ \operatorname{colog}\cos u' &= 0.199\,673\,73 \\ \log\sin\omega &= \bar{1}.162\,852\,95(-); \quad \omega = -8^\circ\,21'\,57.741''. \end{aligned}$$

The argument for the last two columns of the tables is $\log\bigl((\sqrt{\frac{3}{4}}e/\sqrt{1-\frac{3}{4}e^2})\cos m\bigr)$, giving

$$\begin{aligned} \log\frac{\sqrt{\frac{3}{4}}e}{\sqrt{1-\frac{3}{4}e^2}} &= \bar{2}.844\,022 \\ \log\cos m &= \bar{1}.890\,371 \\ \text{Argument} &= \bar{2}.734\,393. \end{aligned}$$

Computing the terms in the series (12) gives

$$\begin{aligned} \log\alpha' &= \bar{1}.698\,758 \\ \log(-\sin m) &= \bar{1}.799\,073 \\ \log\frac{e^2}{\sqrt[3]{1-\frac{3}{4}e^2}} &= \bar{3}.811\,575 \\ \log\sigma &= 4.278\,523 \\ & \quad 1.587\,929 = +38.719'', \end{aligned}$$

and

$$\begin{aligned} \log\beta' &= 1.703 \\ \log(-\sin m) &= \bar{1}.799 \\ \log\frac{e^2}{\sqrt[3]{1-\frac{3}{4}e^2}} &= \bar{3}.812 \\ \log\bigl(\cos(2M+\sigma)\sin\sigma\bigr) &= \bar{2}.963(-) \\ & \quad \bar{2}.277(-) = -0.019''. \end{aligned}$$

The sum of both terms is $+38.700''$, and adding this to ω, we find the longitude difference,

$$w = -8^\circ\,21'\,19.041''.$$

12. CONCLUSION

This illustration of the use of these tables shows that the accuracy of the calculation is limited not by the neglect of terms of high order in the eccentricity, but by the number of decimal places included. The steps in the calculation are, for the most part, the same as for a spherical earth; in order to account for the earth's ellipticity one needs, in addition, only to solve equation (11) and to evaluate the series (12). Since this approach is sufficiently convenient even for routine use, it is unnecessary to use an approximate method which is valid only for small distances.

(The tables are shown on the following pages.)

[33] See footnote 18 and set $(1-x)^{-1/3} = 1 + \frac{1}{3}x + \frac{1\cdot 4}{3\cdot 6}x^2 + \frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}x^3 + \frac{1\cdot 4\cdot 7\cdot 10}{3\cdot 6\cdot 9\cdot 12}x^4 + \dots$

[34] The value of β' in the tables includes the factor of $648\,000/\pi$ necessary to convert from radians to arc seconds.

[35] The first two relations are the sine rule for angle N of triangle ABN of Fig. 1. The last relation is obtained, for example, by substituting for $\sin\alpha'$ from (7).

[36] These are Napier's analogies for angle N of triangle ABN.

TABLES for computing geodesics 1.

Arg	$\log\alpha$	$-\Delta$	$\log\beta$	Δ	$\log\gamma$	Δ	$\log\alpha'$	$-\Delta$	$\log\beta'$	Δ
$\bar{4}.4$	5.314 425 13	1	$\bar{3}.5124$	2000			$\bar{1}.698\,970$	0	$\bar{3}.035$	200
$\bar{4}.5$	5.314 425 12	0	$\bar{3}.7124$	2000			$\bar{1}.698\,970$	0	$\bar{3}.235$	200
$\bar{4}.6$	5.314 425 12	1	$\bar{3}.9124$	2000			$\bar{1}.698\,970$	0	$\bar{3}.435$	200
$\bar{4}.7$	5.314 425 11	2	$\bar{2}.1124$	2000			$\bar{1}.698\,970$	0	$\bar{3}.635$	200
$\bar{4}.8$	5.314 425 09	3	$\bar{2}.3124$	2000			$\bar{1}.698\,970$	0	$\bar{3}.835$	200
$\bar{4}.9$	5.314 425 06	4	$\bar{2}.5124$	2000			$\bar{1}.698\,970$	0	$\bar{2}.035$	200
$\bar{3}.0$	5.314 425 02	6	$\bar{2}.7124$	2000			$\bar{1}.698\,970$	0	$\bar{2}.235$	200
$\bar{3}.1$	5.314 424 96	10	$\bar{2}.9124$	2000			$\bar{1}.698\,970$	0	$\bar{2}.435$	200
$\bar{3}.2$	5.314 424 86	16	$\bar{1}.1124$	2000			$\bar{1}.698\,970$	0	$\bar{2}.635$	200
$\bar{3}.3$	5.314 424 70	25	$\bar{1}.3124$	2000			$\bar{1}.698\,970$	0	$\bar{2}.835$	200
$\bar{3}.4$	5.314 424 45	40	$\bar{1}.5124$	2000			$\bar{1}.698\,970$	1	$\bar{1}.035$	200
$\bar{3}.50$	5.314 424 05	5	$\bar{1}.7124$	200			$\bar{1}.698\,969$	0	$\bar{1}.235$	20
$\bar{3}.51$	5.314 424 00	6	$\bar{1}.7324$	200			$\bar{1}.698\,969$	0	$\bar{1}.255$	20
$\bar{3}.52$	5.314 423 94	5	$\bar{1}.7524$	200			$\bar{1}.698\,969$	0	$\bar{1}.275$	20
$\bar{3}.53$	5.314 423 89	6	$\bar{1}.7724$	200			$\bar{1}.698\,969$	0	$\bar{1}.295$	20
$\bar{3}.54$	5.314 423 83	6	$\bar{1}.7924$	200			$\bar{1}.698\,969$	0	$\bar{1}.315$	20
$\bar{3}.55$	5.314 423 77	7	$\bar{1}.8124$	200			$\bar{1}.698\,969$	0	$\bar{1}.335$	20
$\bar{3}.56$	5.314 423 70	7	$\bar{1}.8324$	200			$\bar{1}.698\,969$	0	$\bar{1}.355$	20
$\bar{3}.57$	5.314 423 63	7	$\bar{1}.8524$	200			$\bar{1}.698\,969$	0	$\bar{1}.375$	20
$\bar{3}.58$	5.314 423 56	7	$\bar{1}.8724$	200			$\bar{1}.698\,969$	0	$\bar{1}.395$	20
$\bar{3}.59$	5.314 423 49	8	$\bar{1}.8924$	200			$\bar{1}.698\,969$	0	$\bar{1}.415$	20
$\bar{3}.60$	5.314 423 41	8	$\bar{1}.9124$	200			$\bar{1}.698\,969$	0	$\bar{1}.435$	20
$\bar{3}.61$	5.314 423 33	8	$\bar{1}.9324$	200			$\bar{1}.698\,969$	0	$\bar{1}.455$	20
$\bar{3}.62$	5.314 423 25	9	$\bar{1}.9524$	200			$\bar{1}.698\,969$	0	$\bar{1}.475$	20
$\bar{3}.63$	5.314 423 16	10	$\bar{1}.9724$	200			$\bar{1}.698\,969$	0	$\bar{1}.495$	20
$\bar{3}.64$	5.314 423 06	9	$\bar{1}.9924$	200			$\bar{1}.698\,969$	0	$\bar{1}.515$	20
$\bar{3}.65$	5.314 422 97	11	0.0124	200			$\bar{1}.698\,969$	1	$\bar{1}.535$	20
$\bar{3}.66$	5.314 422 86	10	0.0324	200			$\bar{1}.698\,968$	0	$\bar{1}.555$	20
$\bar{3}.67$	5.314 422 76	11	0.0524	200			$\bar{1}.698\,968$	0	$\bar{1}.575$	20
$\bar{3}.68$	5.314 422 65	12	0.0724	200			$\bar{1}.698\,968$	0	$\bar{1}.595$	20
$\bar{3}.69$	5.314 422 53	12	0.0924	200			$\bar{1}.698\,968$	0	$\bar{1}.615$	20
$\bar{3}.70$	5.314 422 41	13	0.1124	200			$\bar{1}.698\,968$	0	$\bar{1}.635$	20
$\bar{3}.71$	5.314 422 28	14	0.1324	200			$\bar{1}.698\,968$	0	$\bar{1}.655$	20
$\bar{3}.72$	5.314 422 14	14	0.1524	200			$\bar{1}.698\,968$	0	$\bar{1}.675$	20
$\bar{3}.73$	5.314 422 00	15	0.1724	200			$\bar{1}.698\,968$	0	$\bar{1}.695$	20
$\bar{3}.74$	5.314 421 85	15	0.1924	200			$\bar{1}.698\,968$	0	$\bar{1}.715$	20
$\bar{3}.75$	5.314 421 70	16	0.2124	200			$\bar{1}.698\,968$	0	$\bar{1}.735$	20
$\bar{3}.76$	5.314 421 54	17	0.2324	200			$\bar{1}.698\,968$	1	$\bar{1}.755$	20
$\bar{3}.77$	5.314 421 37	18	0.2524	200			$\bar{1}.698\,967$	0	$\bar{1}.775$	20
$\bar{3}.78$	5.314 421 19	18	0.2724	200			$\bar{1}.698\,967$	0	$\bar{1}.795$	20
$\bar{3}.79$	5.314 421 01	20	0.2924	200			$\bar{1}.698\,967$	0	$\bar{1}.815$	20
$\bar{3}.80$	5.314 420 81	20	0.3124	200			$\bar{1}.698\,967$	0	$\bar{1}.835$	20
$\bar{3}.81$	5.314 420 61	22	0.3324	200			$\bar{1}.698\,967$	0	$\bar{1}.855$	20
$\bar{3}.82$	5.314 420 39	22	0.3524	200			$\bar{1}.698\,967$	0	$\bar{1}.875$	20
$\bar{3}.83$	5.314 420 17	23	0.3724	200			$\bar{1}.698\,967$	0	$\bar{1}.895$	20
$\bar{3}.84$	5.314 419 94	25	0.3924	200			$\bar{1}.698\,967$	1	$\bar{1}.915$	20
$\bar{3}.85$	5.314 419 69	25	0.4124	200			$\bar{1}.698\,966$	0	$\bar{1}.935$	20
$\bar{3}.86$	5.314 419 44	27	0.4324	200			$\bar{1}.698\,966$	0	$\bar{1}.955$	20
$\bar{3}.87$	5.314 419 17	28	0.4524	200			$\bar{1}.698\,966$	0	$\bar{1}.975$	20
$\bar{3}.88$	5.314 418 89	30	0.4724	200			$\bar{1}.698\,966$	0	$\bar{1}.995$	20
$\bar{3}.89$	5.314 418 59	31	0.4924	200			$\bar{1}.698\,966$	1	0.015	20
$\bar{3}.90$	5.314 418 28		0.5124				$\bar{1}.698\,965$		0.035	

TABLES for computing geodesics 2.

Arg	$\log\alpha$	$-\Delta$	$\log\beta$	Δ	$\log\gamma$	Δ	$\log\alpha'$	$-\Delta$	$\log\beta'$	Δ
$\bar{3}.90$	5.314 418 28	32	0.512 35	2000			$\bar{1}.698\,965$	0	0.035	20
$\bar{3}.91$	5.314 417 96	34	0.532 35	2000			$\bar{1}.698\,965$	0	0.055	20
$\bar{3}.92$	5.314 417 62	35	0.552 35	2000			$\bar{1}.698\,965$	0	0.075	20
$\bar{3}.93$	5.314 417 27	37	0.572 35	2000			$\bar{1}.698\,965$	0	0.095	20
$\bar{3}.94$	5.314 416 90	39	0.592 35	2000			$\bar{1}.698\,965$	1	0.115	20
$\bar{3}.95$	5.314 416 51	41	0.612 35	2000			$\bar{1}.698\,964$	0	0.135	20
$\bar{3}.96$	5.314 416 10	42	0.632 35	2000			$\bar{1}.698\,964$	0	0.155	20
$\bar{3}.97$	5.314 415 68	45	0.652 35	2000			$\bar{1}.698\,964$	1	0.175	20
$\bar{3}.98$	5.314 415 23	47	0.672 35	1999			$\bar{1}.698\,963$	0	0.195	20
$\bar{3}.99$	5.314 414 76	48	0.692 34	2000			$\bar{1}.698\,963$	0	0.215	20
$\bar{2}.00$	5.314 414 28	52	0.712 34	2000			$\bar{1}.698\,963$	1	0.235	20
$\bar{2}.01$	5.314 413 76	53	0.732 34	2000			$\bar{1}.698\,962$	0	0.255	20
$\bar{2}.02$	5.314 413 23	56	0.752 34	2000			$\bar{1}.698\,962$	0	0.275	20
$\bar{2}.03$	5.314 412 67	59	0.772 34	2000			$\bar{1}.698\,962$	1	0.295	20
$\bar{2}.04$	5.314 412 08	61	0.792 34	2000			$\bar{1}.698\,961$	0	0.315	20
$\bar{2}.05$	5.314 411 47	65	0.812 34	2000			$\bar{1}.698\,961$	1	0.335	20
$\bar{2}.06$	5.314 410 82	67	0.832 34	2000			$\bar{1}.698\,960$	0	0.355	20
$\bar{2}.07$	5.314 410 15	71	0.852 34	1999			$\bar{1}.698\,960$	0	0.375	20
$\bar{2}.08$	5.314 409 44	74	0.872 33	2000			$\bar{1}.698\,960$	1	0.395	20
$\bar{2}.09$	5.314 408 70	77	0.892 33	2000			$\bar{1}.698\,959$	0	0.415	20
$\bar{2}.10$	5.314 407 93	81	0.912 33	2000			$\bar{1}.698\,959$	1	0.435	20
$\bar{2}.11$	5.314 407 12	85	0.932 33	2000			$\bar{1}.698\,958$	1	0.455	20
$\bar{2}.12$	5.314 406 27	89	0.952 33	2000			$\bar{1}.698\,957$	0	0.475	20
$\bar{2}.13$	5.314 405 38	93	0.972 33	1999			$\bar{1}.698\,957$	1	0.495	20
$\bar{2}.14$	5.314 404 45	98	0.992 32	2000			$\bar{1}.698\,956$	0	0.515	20
$\bar{2}.15$	5.314 403 47	102	1.012 32	2000			$\bar{1}.698\,956$	1	0.535	20
$\bar{2}.16$	5.314 402 45	107	1.032 32	2000			$\bar{1}.698\,955$	1	0.555	20
$\bar{2}.17$	5.314 401 38	112	1.052 32	2000			$\bar{1}.698\,954$	1	0.575	20
$\bar{2}.18$	5.314 400 26	117	1.072 32	1999			$\bar{1}.698\,953$	0	0.595	20
$\bar{2}.19$	5.314 399 09	123	1.092 31	2000			$\bar{1}.698\,953$	1	0.615	20
$\bar{2}.20$	5.314 397 86	128	1.112 31	2000			$\bar{1}.698\,952$	1	0.635	20
$\bar{2}.21$	5.314 396 58	135	1.132 31	2000			$\bar{1}.698\,951$	1	0.655	20
$\bar{2}.22$	5.314 395 23	141	1.152 31	1999			$\bar{1}.698\,950$	1	0.675	20
$\bar{2}.23$	5.314 393 82	147	1.172 30	2000			$\bar{1}.698\,949$	1	0.695	20
$\bar{2}.24$	5.314 392 35	155	1.192 30	2000			$\bar{1}.698\,948$	1	0.715	20
$\bar{2}.25$	5.314 390 80	162	1.212 30	1999	$\bar{4}.207$	40	$\bar{1}.698\,947$	1	0.735	20
$\bar{2}.26$	5.314 389 18	169	1.232 29	2000	$\bar{4}.247$	40	$\bar{1}.698\,946$	1	0.755	20
$\bar{2}.27$	5.314 387 49	177	1.252 29	2000	$\bar{4}.287$	40	$\bar{1}.698\,945$	1	0.775	20
$\bar{2}.28$	5.314 385 72	186	1.272 29	1999	$\bar{4}.327$	40	$\bar{1}.698\,944$	2	0.795	20
$\bar{2}.29$	5.314 383 86	195	1.292 28	2000	$\bar{4}.367$	40	$\bar{1}.698\,942$	1	0.815	20
$\bar{2}.30$	5.314 381 91	203	1.312 28	1999	$\bar{4}.407$	40	$\bar{1}.698\,941$	1	0.835	20
$\bar{2}.31$	5.314 379 88	213	1.332 27	2000	$\bar{4}.447$	40	$\bar{1}.698\,940$	2	0.855	20
$\bar{2}.32$	5.314 377 75	224	1.352 27	2000	$\bar{4}.487$	40	$\bar{1}.698\,938$	1	0.875	20
$\bar{2}.33$	5.314 375 51	234	1.372 27	1999	$\bar{4}.527$	40	$\bar{1}.698\,937$	2	0.895	20
$\bar{2}.34$	5.314 373 17	244	1.392 26	2000	$\bar{4}.567$	40	$\bar{1}.698\,935$	1	0.915	20
$\bar{2}.35$	5.314 370 73	257	1.412 26	1999	$\bar{4}.607$	40	$\bar{1}.698\,934$	2	0.935	20
$\bar{2}.36$	5.314 368 16	268	1.432 25	2000	$\bar{4}.647$	40	$\bar{1}.698\,932$	2	0.955	20
$\bar{2}.37$	5.314 365 48	281	1.452 25	1999	$\bar{4}.687$	40	$\bar{1}.698\,930$	2	0.975	20
$\bar{2}.38$	5.314 362 67	295	1.472 24	1999	$\bar{4}.727$	40	$\bar{1}.698\,928$	2	0.995	20
$\bar{2}.39$	5.314 359 72	308	1.492 23	2000	$\bar{4}.767$	40	$\bar{1}.698\,926$	2	1.015	20
$\bar{2}.40$	5.314 356 64		1.512 23		$\bar{4}.807$		$\bar{1}.698\,924$		1.035	

TABLES for computing geodesics 3.

Arg	$\log\alpha$	$-\Delta$	$\log\beta$	Δ	$\log\gamma$	Δ	$\log\alpha'$	$-\Delta$	$\log\beta'$	Δ
$\bar{2}.40$	5.314 356 64	323	1.512 23	1999	$\bar{4}.807$	40	$\bar{1}.698\,924$	2	1.035	20
$\bar{2}.41$	5.314 353 41	338	1.532 22	1999	$\bar{4}.847$	40	$\bar{1}.698\,922$	2	1.055	20
$\bar{2}.42$	5.314 350 03	353	1.552 21	2000	$\bar{4}.887$	40	$\bar{1}.698\,920$	2	1.075	20
$\bar{2}.43$	5.314 346 50	371	1.572 21	1999	$\bar{4}.927$	40	$\bar{1}.698\,918$	3	1.095	20
$\bar{2}.44$	5.314 342 79	388	1.592 20	1999	$\bar{4}.967$	40	$\bar{1}.698\,915$	2	1.115	20
$\bar{2}.45$	5.314 338 91	406	1.612 19	1999	$\bar{3}.007$	40	$\bar{1}.698\,913$	3	1.135	20
$\bar{2}.46$	5.314 334 85	425	1.632 18	2000	$\bar{3}.047$	40	$\bar{1}.698\,910$	3	1.155	20
$\bar{2}.47$	5.314 330 60	446	1.652 18	1999	$\bar{3}.087$	40	$\bar{1}.698\,907$	3	1.175	20
$\bar{2}.48$	5.314 326 14	466	1.672 17	1999	$\bar{3}.127$	40	$\bar{1}.698\,904$	3	1.195	20
$\bar{2}.49$	5.314 321 48	489	1.692 16	1999	$\bar{3}.167$	40	$\bar{1}.698\,901$	3	1.215	20
$\bar{2}.50$	5.314 316 59	511	1.712 15	1999	$\bar{3}.207$	40	$\bar{1}.698\,898$	4	1.235	20
$\bar{2}.51$	5.314 311 48	535	1.732 14	1999	$\bar{3}.247$	40	$\bar{1}.698\,894$	3	1.255	20
$\bar{2}.52$	5.314 306 13	561	1.752 13	1999	$\bar{3}.287$	40	$\bar{1}.698\,891$	4	1.275	20
$\bar{2}.53$	5.314 300 52	587	1.772 12	1998	$\bar{3}.327$	40	$\bar{1}.698\,887$	4	1.295	20
$\bar{2}.54$	5.314 294 65	615	1.792 10	1999	$\bar{3}.367$	40	$\bar{1}.698\,883$	4	1.315	20
$\bar{2}.55$	5.314 288 50	644	1.812 09	1999	$\bar{3}.407$	40	$\bar{1}.698\,879$	4	1.335	20
$\bar{2}.56$	5.314 282 06	674	1.832 08	1999	$\bar{3}.447$	40	$\bar{1}.698\,875$	5	1.355	20
$\bar{2}.57$	5.314 275 32	705	1.852 07	1998	$\bar{3}.487$	40	$\bar{1}.698\,870$	5	1.375	20
$\bar{2}.58$	5.314 268 27	739	1.872 05	1999	$\bar{3}.527$	40	$\bar{1}.698\,865$	4	1.395	20
$\bar{2}.59$	5.314 260 88	774	1.892 04	1998	$\bar{3}.567$	40	$\bar{1}.698\,861$	6	1.415	20
$\bar{2}.60$	5.314 253 14	810	1.912 02	1998	$\bar{3}.607$	39	$\bar{1}.698\,855$	5	1.435	20
$\bar{2}.61$	5.314 245 04	848	1.932 00	1999	$\bar{3}.646$	40	$\bar{1}.698\,850$	6	1.455	20
$\bar{2}.62$	5.314 236 56	889	1.951 99	1998	$\bar{3}.686$	40	$\bar{1}.698\,844$	6	1.475	20
$\bar{2}.63$	5.314 227 67	930	1.971 97	1998	$\bar{3}.726$	40	$\bar{1}.698\,838$	6	1.495	20
$\bar{2}.64$	5.314 218 37	973	1.991 95	1998	$\bar{3}.766$	40	$\bar{1}.698\,832$	6	1.515	20
$\bar{2}.65$	5.314 208 64	1020	2.011 93	1998	$\bar{3}.806$	40	$\bar{1}.698\,826$	7	1.535	20
$\bar{2}.66$	5.314 198 44	1068	2.031 91	1998	$\bar{3}.846$	40	$\bar{1}.698\,819$	7	1.555	20
$\bar{2}.67$	5.314 187 76	1118	2.051 89	1998	$\bar{3}.886$	40	$\bar{1}.698\,812$	8	1.575	20
$\bar{2}.68$	5.314 176 58	1170	2.071 87	1997	$\bar{3}.926$	40	$\bar{1}.698\,804$	7	1.595	20
$\bar{2}.69$	5.314 164 88	1226	2.091 84	1998	$\bar{3}.966$	40	$\bar{1}.698\,797$	9	1.615	20
$\bar{2}.70$	5.314 152 62	1283	2.111 82	1997	$\bar{2}.006$	40	$\bar{1}.698\,788$	8	1.635	19
$\bar{2}.71$	5.314 139 79	1344	2.131 79	1998	$\bar{2}.046$	40	$\bar{1}.698\,780$	9	1.654	20
$\bar{2}.72$	5.314 126 35	1406	2.151 77	1997	$\bar{2}.086$	40	$\bar{1}.698\,771$	9	1.674	20
$\bar{2}.73$	5.314 112 29	1473	2.171 74	1997	$\bar{2}.126$	40	$\bar{1}.698\,762$	10	1.694	20
$\bar{2}.74$	5.314 097 56	1543	2.191 71	1997	$\bar{2}.166$	40	$\bar{1}.698\,752$	11	1.714	20
$\bar{2}.75$	5.314 082 13	1615	2.211 68	1997	$\bar{2}.206$	40	$\bar{1}.698\,741$	10	1.734	20
$\bar{2}.76$	5.314 065 98	1690	2.231 65	1996	$\bar{2}.246$	40	$\bar{1}.698\,731$	12	1.754	20
$\bar{2}.77$	5.314 049 08	1771	2.251 61	1997	$\bar{2}.286$	40	$\bar{1}.698\,719$	11	1.774	20
$\bar{2}.78$	5.314 031 37	1853	2.271 58	1996	$\bar{2}.326$	40	$\bar{1}.698\,708$	13	1.794	20
$\bar{2}.79$	5.314 012 84	1941	2.291 54	1996	$\bar{2}.366$	39	$\bar{1}.698\,695$	13	1.814	20
$\bar{2}.800$	5.313 993 43	1004	2.311 50	998	$\bar{2}.405$	20	$\bar{1}.698\,682$	6	1.834	10
$\bar{2}.805$	5.313 983 39	1028	2.321 48	998	$\bar{2}.425$	20	$\bar{1}.698\,676$	7	1.844	10
$\bar{2}.810$	5.313 973 11	1051	2.331 46	998	$\bar{2}.445$	20	$\bar{1}.698\,669$	7	1.854	10
$\bar{2}.815$	5.313 962 60	1076	2.341 44	998	$\bar{2}.465$	20	$\bar{1}.698\,662$	7	1.864	10
$\bar{2}.820$	5.313 951 84	1101	2.351 42	998	$\bar{2}.485$	20	$\bar{1}.698\,655$	8	1.874	10
$\bar{2}.825$	5.313 940 83	1127	2.361 40	997	$\bar{2}.505$	20	$\bar{1}.698\,647$	7	1.884	10
$\bar{2}.830$	5.313 929 56	1152	2.371 37	998	$\bar{2}.525$	20	$\bar{1}.698\,640$	8	1.894	10
$\bar{2}.835$	5.313 918 04	1180	2.381 35	998	$\bar{2}.545$	20	$\bar{1}.698\,632$	8	1.904	10
$\bar{2}.840$	5.313 906 24	1207	2.391 33	997	$\bar{2}.565$	20	$\bar{1}.698\,624$	8	1.914	10
$\bar{2}.845$	5.313 894 17	1234	2.401 30	998	$\bar{2}.585$	20	$\bar{1}.698\,616$	8	1.924	10
$\bar{2}.850$	5.313 881 83		2.411 28		$\bar{2}.605$		$\bar{1}.698\,608$		1.934	

TABLES for computing geodesics 4.

Arg	$\log\alpha$	$-\Delta$	$\log\beta$	Δ	$\log\gamma$	Δ	$\log\alpha'$	$-\Delta$	$\log\beta'$	Δ
$\bar{2}.850$	5.313 881 83	1264	2.411 279	9974	$\bar{2}.605$	20	$\bar{1}.698\,608$	8	1.934	10
$\bar{2}.855$	5.313 869 19	1293	2.421 253	9974	$\bar{2}.625$	20	$\bar{1}.698\,600$	9	1.944	10
$\bar{2}.860$	5.313 856 26	1323	2.431 227	9974	$\bar{2}.645$	20	$\bar{1}.698\,591$	9	1.954	10
$\bar{2}.865$	5.313 843 03	1353	2.441 201	9973	$\bar{2}.665$	20	$\bar{1}.698\,582$	9	1.964	10
$\bar{2}.870$	5.313 829 50	1385	2.451 174	9972	$\bar{2}.685$	20	$\bar{1}.698\,573$	9	1.974	10
$\bar{2}.875$	5.313 815 65	1417	2.461 146	9972	$\bar{2}.705$	20	$\bar{1}.698\,564$	10	1.984	10
$\bar{2}.880$	5.313 801 48	1450	2.471 118	9971	$\bar{2}.725$	20	$\bar{1}.698\,554$	9	1.994	10
$\bar{2}.885$	5.313 786 98	1484	2.481 089	9970	$\bar{2}.745$	20	$\bar{1}.698\,545$	10	2.004	10
$\bar{2}.890$	5.313 772 14	1518	2.491 059	9970	$\bar{2}.765$	20	$\bar{1}.698\,535$	10	2.014	9
$\bar{2}.895$	5.313 756 96	1553	2.501 029	9969	$\bar{2}.785$	19	$\bar{1}.698\,525$	11	2.023	10
$\bar{2}.900$	5.313 741 43	1590	2.510 998	9968	$\bar{2}.804$	20	$\bar{1}.698\,514$	10	2.033	10
$\bar{2}.905$	5.313 725 53	1626	2.520 966	9968	$\bar{2}.824$	20	$\bar{1}.698\,504$	11	2.043	10
$\bar{2}.910$	5.313 709 27	1664	2.530 934	9966	$\bar{2}.844$	20	$\bar{1}.698\,493$	11	2.053	10
$\bar{2}.915$	5.313 692 63	1702	2.540 900	9966	$\bar{2}.864$	20	$\bar{1}.698\,482$	11	2.063	10
$\bar{2}.920$	5.313 675 61	1742	2.550 866	9965	$\bar{2}.884$	20	$\bar{1}.698\,471$	12	2.073	10
$\bar{2}.925$	5.313 658 19	1783	2.560 831	9965	$\bar{2}.904$	20	$\bar{1}.698\,459$	12	2.083	10
$\bar{2}.930$	5.313 640 36	1824	2.570 796	9963	$\bar{2}.924$	20	$\bar{1}.698\,447$	12	2.093	10
$\bar{2}.935$	5.313 622 12	1866	2.580 759	9963	$\bar{2}.944$	20	$\bar{1}.698\,435$	12	2.103	10
$\bar{2}.940$	5.313 603 46	1909	2.590 722	9962	$\bar{2}.964$	20	$\bar{1}.698\,423$	13	2.113	10
$\bar{2}.945$	5.313 584 37	1953	2.600 684	9961	$\bar{2}.984$	20	$\bar{1}.698\,410$	13	2.123	10
$\bar{2}.950$	5.313 564 84	1999	2.610 645	9960	$\bar{1}.004$	20	$\bar{1}.698\,397$	13	2.133	10
$\bar{2}.955$	5.313 544 85	2045	2.620 605	9959	$\bar{1}.024$	20	$\bar{1}.698\,384$	14	2.143	10
$\bar{2}.960$	5.313 524 40	2093	2.630 564	9958	$\bar{1}.044$	20	$\bar{1}.698\,370$	14	2.153	10
$\bar{2}.965$	5.313 503 47	2141	2.640 522	9957	$\bar{1}.064$	19	$\bar{1}.698\,356$	14	2.163	10
$\bar{2}.970$	5.313 482 06	2191	2.650 479	9956	$\bar{1}.083$	20	$\bar{1}.698\,342$	15	2.173	10
$\bar{2}.975$	5.313 460 15	2241	2.660 435	9956	$\bar{1}.103$	20	$\bar{1}.698\,327$	15	2.183	10
$\bar{2}.980$	5.313 437 74	2293	2.670 391	9954	$\bar{1}.123$	20	$\bar{1}.698\,312$	15	2.193	10
$\bar{2}.985$	5.313 414 81	2347	2.680 345	9953	$\bar{1}.143$	20	$\bar{1}.698\,297$	16	2.203	9
$\bar{2}.990$	5.313 391 34	2400	2.690 298	9952	$\bar{1}.163$	20	$\bar{1}.698\,281$	15	2.212	10
$\bar{2}.995$	5.313 367 34	2457	2.700 250	9951	$\bar{1}.183$	20	$\bar{1}.698\,266$	17	2.222	10
$\bar{1}.000$	5.313 342 77	2513	2.710 201	9950	$\bar{1}.203$	20	$\bar{1}.698\,249$	17	2.232	10
$\bar{1}.005$	5.313 317 64	2571	2.720 151	9948	$\bar{1}.223$	20	$\bar{1}.698\,232$	17	2.242	10
$\bar{1}.010$	5.313 291 93	2631	2.730 099	9948	$\bar{1}.243$	20	$\bar{1}.698\,215$	17	2.252	10
$\bar{1}.015$	5.313 265 62	2691	2.740 047	9946	$\bar{1}.263$	19	$\bar{1}.698\,198$	18	2.262	10
$\bar{1}.020$	5.313 238 71	2754	2.749 993	9945	$\bar{1}.282$	20	$\bar{1}.698\,180$	18	2.272	10
$\bar{1}.025$	5.313 211 17	2818	2.759 938	9943	$\bar{1}.302$	20	$\bar{1}.698\,162$	19	2.282	10
$\bar{1}.030$	5.313 182 99	2883	2.769 881	9943	$\bar{1}.322$	20	$\bar{1}.698\,143$	19	2.292	10
$\bar{1}.035$	5.313 154 16	2949	2.779 824	9941	$\bar{1}.342$	20	$\bar{1}.698\,124$	20	2.302	10
$\bar{1}.040$	5.313 124 67	3018	2.789 765	9939	$\bar{1}.362$	20	$\bar{1}.698\,104$	20	2.312	10
$\bar{1}.045$	5.313 094 49	3087	2.799 704	9939	$\bar{1}.382$	20	$\bar{1}.698\,084$	20	2.322	10
$\bar{1}.050$	5.313 063 62	3159	2.809 643	9936	$\bar{1}.402$	20	$\bar{1}.698\,064$	21	2.332	10
$\bar{1}.055$	5.313 032 03	3232	2.819 579	9936	$\bar{1}.422$	20	$\bar{1}.698\,043$	22	2.342	9
$\bar{1}.060$	5.312 999 71	3306	2.829 515	9934	$\bar{1}.442$	19	$\bar{1}.698\,021$	22	2.351	10
$\bar{1}.065$	5.312 966 65	3383	2.839 449	9932	$\bar{1}.461$	20	$\bar{1}.697\,999$	22	2.361	10
$\bar{1}.070$	5.312 932 82	3460	2.849 381	9931	$\bar{1}.481$	20	$\bar{1}.697\,977$	23	2.371	10
$\bar{1}.075$	5.312 898 22	3541	2.859 312	9929	$\bar{1}.501$	20	$\bar{1}.697\,954$	24	2.381	10
$\bar{1}.080$	5.312 862 81	3623	2.869 241	9928	$\bar{1}.521$	20	$\bar{1}.697\,930$	24	2.391	10
$\bar{1}.085$	5.312 826 58	3706	2.879 169	9926	$\bar{1}.541$	20	$\bar{1}.697\,906$	25	2.401	10
$\bar{1}.090$	5.312 789 52	3791	2.889 095	9924	$\bar{1}.561$	20	$\bar{1}.697\,881$	25	2.411	10
$\bar{1}.095$	5.312 751 61	3879	2.899 019	9922	$\bar{1}.581$	19	$\bar{1}.697\,856$	26	2.421	10
$\bar{1}.100$	5.312 712 82		2.908 941		$\bar{1}.600$		$\bar{1}.697\,830$		2.431	

GEODESICS ON AN ELLIPSOID - BESSEL'S METHOD

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ABSTRACT

These notes provide a detailed derivation of the equations for computing the direct and inverse problems on the ellipsoid. These equations could be called *Bessel's method* and have a history dating back to F. W. Bessel's original paper on the topic titled: 'On the computation of geographical longitude and latitude from geodetic measurements', published in *Astronomische Nachrichten* (Astronomical Notes), Band 4 (Volume 4), Number 86, Speiten 241-254 (Columns 241-254), Altona 1826. The equations developed here are of a slightly different form than those presented by Bessel, but they lead directly to equations presented by Rainsford (1955) and Vincenty (1975) and the method of development closely follows that shown in *Geometric Geodesy* (Rapp, 1981). An understanding of the methods introduced in the following pages, in particular the evaluation of elliptic integrals by series expansion, will give the student an insight into other geodetic calculations.

INTRODUCTION

The direct and inverse problems on the ellipsoid are fundamental geodetic operations and can be likened to the equivalent operations of plane surveying; radiations (computing coordinates of points given bearings and distances radiating from a point of known coordinates) and joins (computing bearings and distances between points having known coordinates). In plane surveying, the coordinates are 2-Dimensional (2D) rectangular coordinates, usually designated East and North and the reference surface is a plane, either a local horizontal plane or a map projection plane.

In geodesy, the reference surface is an ellipsoid, the coordinates are latitudes and longitudes, directions are known as azimuths and distances are geodesic arc lengths.

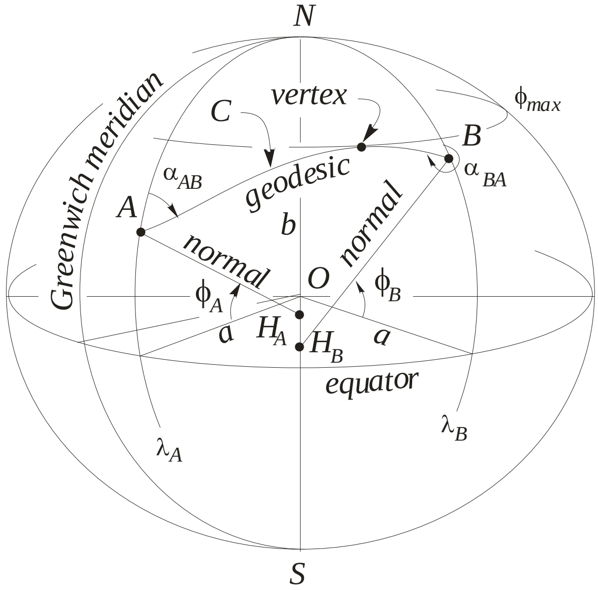


Fig. 1: Geodesic curve on an ellipsoid

The geodesic is a unique curve on the surface of an ellipsoid defining the shortest distance between two points. A geodesic will cut meridians of an ellipsoid at angles α, known as azimuths and measured clockwise from north 0° to 360°. Figure 1 shows a geodesic curve C between two points A (ϕ_A, λ_A) and B (ϕ_B, λ_B) on an ellipsoid. ϕ, λ are latitude and longitude respectively and an ellipsoid is taken to mean a surface of revolution created by rotating an ellipse about its minor axis, NS. The geodesic curve C of length s from A to B has a forward azimuth α_{AB} measured at A and a reverse azimuth α_{BA} measured at B.

The direct problem on an ellipsoid is: given latitude and longitude of A and azimuth α_{AB} and geodesic distance s, compute the latitude and longitude of B and the reverse azimuth α_{BA}.

The inverse problem is: given the latitudes and longitudes of A and B, compute the forward and reverse azimuths α_{AB}, α_{BA} and the geodesic distance s.

Formula for computing geodesic distances and longitude differences between points connected by geodesic curves are derived from solutions of elliptic integrals and in Bessel's method, these elliptic integrals are solutions of equations connecting differential elements on the ellipsoid with corresponding elements on an auxiliary sphere. These integrals do not have direct solutions but instead are solved by expanding them into trigonometric series and integrating term-by-term. Hence the equations developed here are series-type formula truncated at a certain number of terms that give millimetre precision for any length of line not exceeding 180° in longitude difference.

These formulae were first developed by Bessel (1826) who gave examples of their use using 10-place logarithms. A similar development is given in *Handbuch der Vermessungskunde* (Handbook of Geodesy) by Jordan/Eggert/Kneissl, 1958.

The British geodesist Hume Rainsford (1955) presented equations and computational methods for the direct and inverse problems that were applicable to machine computation of the mid 20th century. His formulae and iterative method for the inverse case were similar to Bessel's, although his equations contained different ellipsoid constants and geodesic curve parameters, but his equations for the direct case, different from Bessel's, were based on a direct technique given by G.T. McCaw (1932-33) which avoided iteration. For many years Rainsford's (and McCaw's) equations were the standard method of solving the direct and inverse problems on the ellipsoid when millimetre precision was required, even though they involved iteration and lengthy long-hand machine computation. In 1975, Thaddeus (Tom) Vincenty (1975-76), then working for the Geodetic Survey Squadron of the US Air Force, presented a set of compact nested equations that could be conveniently programmed on the then new electronic computers. His method and equations were based on Rainsford's inverse method combined with techniques developed by Professor Richard H. Rapp of the Ohio State University. Vincenty's equations for the direct and inverse problems on the ellipsoid have become a standard method of solution.

Vincenty's method (following on from Rainsford and Bessel) is not the only method of solving the direct and inverse problems on the ellipsoid. There are other techniques; some involving elegant solutions to integrals using recurrence relationships, e.g., Pittman (1986) and others using numerical integration techniques, e.g., Kivioja (1971) and Jank & Kivioja (1980).

In this paper, we present a development following Rapp (1981) and based on Bessel's method which yields Rainsford's equations for the inverse problem. We then show how Vincenty's equations are obtained and how they are used in practice. In addition, certain ellipsoid relationships are given, the mathematical definition of a geodesic is discussed and the characteristic equation of a geodesic derived. The characteristic equation of a geodesic is fundamental to all solutions of the direct and inverse problems on the ellipsoid.

SOME ELLIPSOID RELATIONSHIPS

The size and shape of an ellipsoid is defined by one of three pairs of parameters: (i) a, b where a and b are the semi-major and semi-minor axes lengths of an ellipsoid respectively, or (ii) a, f where f is the flattening of an ellipsoid, or (iii) a, e^2 where e^2 is the square of the first eccentricity of an ellipsoid. The ellipsoid parameters a, b, f, e^2 are related by the following equations

$$f = \frac{a-b}{a} = 1 - \frac{b}{a} \tag{1}$$

$$b = a(1-f) \tag{2}$$

$$e^2 = \frac{a^2-b^2}{a^2} = 1 - \frac{b^2}{a^2} = f(2-f) \tag{3}$$

$$1 - e^2 = \frac{b^2}{a^2} = 1 - f(2-f) = (1-f)^2 \tag{4}$$

The second eccentricity e' of an ellipsoid is also of use and

$$e'^2 = \frac{a^2-b^2}{b^2} = \frac{a^2}{b^2} - 1 = \frac{e^2}{1-e^2} = \frac{f(2-f)}{(1-f)^2} \tag{5}$$

$$e^2 = \frac{e'^2}{1+e'^2} \tag{6}$$

In Figure 1 the normals to the surface at A and B intersect the rotational axis of the ellipsoid (NS line) at H_A and H_B making angles ϕ_A, ϕ_B with the equatorial plane of the ellipsoid. These are the latitudes of A and B respectively. The longitudes λ_A, λ_B are the angles between the Greenwich meridian plane (a reference plane) and the meridian planes $ONAH_A$ and $ONBH_B$ containing the normals through A and B. ϕ and λ are curvilinear coordinates and meridians of longitude (curves of constant λ) and parallels of latitude (curves of constant ϕ) are parametric curves on the ellipsoidal surface.

For a general point P on the surface of the ellipsoid (see Fig. 2), planes containing the normal to the ellipsoid intersect the surface creating elliptical sections known as normal sections. Amongst the infinite number of possible normal sections at a point, each having a certain radius of curvature, two are of interest: (i) the meridian section, containing the axis of revolution of the ellipsoid and having the least radius of curvature, denoted by ρ, and (ii) the prime vertical section, perpendicular to the meridian plane and having the greatest radius of curvature, denoted by ν.

$$\rho = \frac{a\left(1-e^2\right)}{\left(1-e^2\sin^2\phi\right)^{\frac{3}{2}}} = \frac{a\left(1-e^2\right)}{W^3} \tag{7}$$

$$\nu = \frac{a}{\left(1-e^2\sin^2\phi\right)^{\frac{1}{2}}} = \frac{a}{W} \tag{8}$$

$$W^2 = 1-e^2\sin^2\phi \tag{9}$$

The centres of the radii of curvature of the prime vertical sections at A and B are at H_A and H_B, where H_A and H_B are the intersections of the normals at A and B and the rotational axis, and $\nu_A = PH_A$, $\nu_B = PH_B$. The centres of the radii of curvature of the meridian sections at A and B lie on the normals between P and H_A and P and H_B.

Alternative equations for the radii of curvature ρ and ν are given by

$$\rho = \frac{a^2}{b\left(1+e'^2\cos^2\phi\right)^{\frac{3}{2}}} = \frac{c}{V^3} \tag{10}$$

$$\nu = \frac{a^2}{b\left(1+e'^2\cos^2\phi\right)^{\frac{1}{2}}} = \frac{c}{V} \tag{11}$$

$$c = \frac{a^2}{b} = \frac{a}{1-f} \tag{12}$$

$$V^2 = 1+e'^2\cos^2\phi \tag{13}$$

and c is the polar radius of curvature of the ellipsoid.

The latitude functions W and V are related as follows

$$W^2 = \frac{V^2}{1+e'^2} \quad \text{and} \quad W = \frac{V}{\left(1+e'^2\right)^{\frac{1}{2}}} = \frac{b}{a}V \tag{14}$$

Points on the ellipsoidal surface have curvilinear coordinates ϕ,λ and Cartesian coordinates x,y,z where the x-z plane is the Greenwich meridian plane, the x-y plane is the equatorial plane and the y-z plane is a meridian plane 90° east of the Greenwich meridian plane. Cartesian and curvilinear coordinates are related by

$$\begin{aligned} x &= \nu\cos\phi\cos\lambda \\ y &= \nu\cos\phi\cos\lambda \\ z &= \nu\left(1-e^2\right)\sin\phi \end{aligned} \tag{15}$$

Note that $\nu\left(1-e^2\right)$ is the distance along the normal from a point on the surface to the point where the normal cuts the equatorial plane.

THE DIFFERENTIAL RECTANGLE ON THE ELLIPSOID

The derivation of equations relating to the geodesic requires an understanding of the connection between differentially small quantities on the surface of the ellipsoid. These relationships can be derived from the differential rectangle, with diagonal PQ in Figure 2 which shows P and Q on an ellipsoid, having semi-major axis *a,* flattening *f,* separated by differential changes in latitude $d\phi$ and longitude $d\lambda$. P and Q are connected by a curve of length ds making an angle α (the azimuth) with the meridian through *P.* The meridians λ and $\lambda + d\lambda$, and the parallels ϕ and $\phi + d\phi$ form a differential rectangle on the surface of the ellipsoid. The differential distances dp along the parallel ϕ and dm along the meridian λ are

$$dp = w\,d\lambda = \nu \cos\phi\, d\lambda \tag{16}$$

$$dm = \rho\, d\phi \tag{17}$$

where ρ and ν are radii of curvature in the meridian and prime vertical planes respectively and $w = \nu\cos\phi$ is the perpendicular distance from the rotational axis.

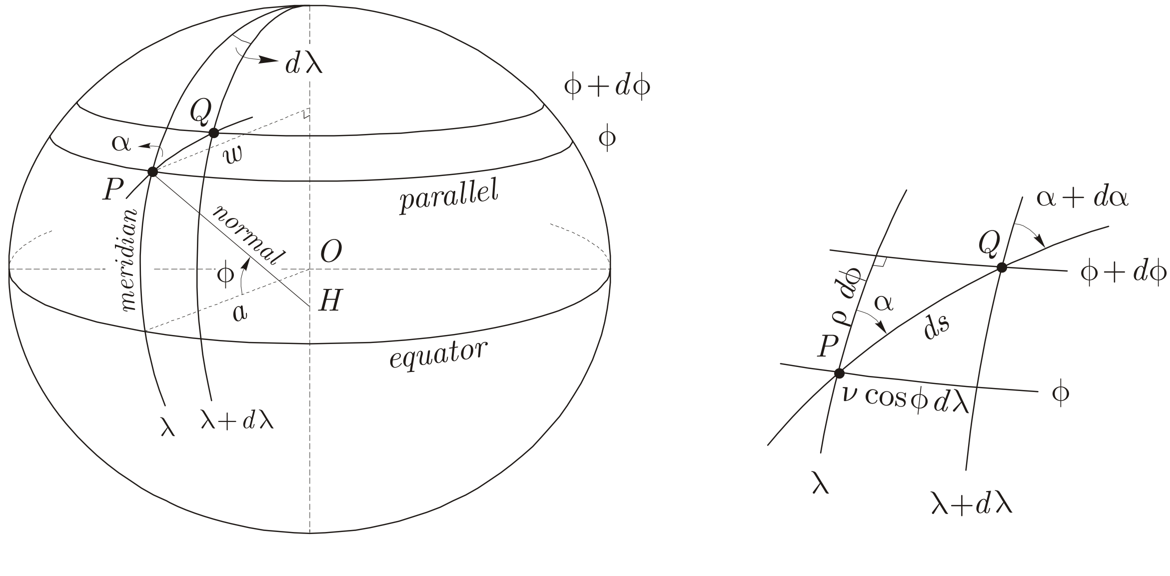


Figure 2: Differential rectangle on the ellipsoid

The differential distance ds is given by

$$ds = \sqrt{dp^2 + dm^2} = \sqrt{\left(\nu\cos\phi\, d\lambda\right)^2 + \left(\rho\, d\phi\right)^2} \tag{18}$$

and so

$$\frac{ds}{d\phi} = \sqrt{\nu^2 \cos^2 \phi \left(\frac{d\lambda}{d\phi}\right)^2 + \rho^2} \qquad \text{or} \qquad \frac{ds}{d\lambda} = \sqrt{\nu^2 \cos^2 \phi + \rho^2 \left(\frac{d\phi}{d\lambda}\right)^2}$$

while

$$\sin\alpha = \nu\cos\phi\,\frac{d\lambda}{ds} \qquad \text{and} \qquad \cos\alpha = \rho\,\frac{d\phi}{ds} \tag{19}$$

MATHEMATICAL DEFINITION OF A GEODESIC

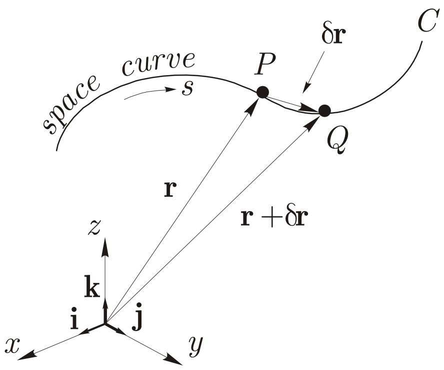


Figure 3: Space curve C

A geodesic can be defined mathematically by considering concepts associated with space curves and surfaces. A space curve may be defined as the locus of the terminal points P of a position vector $\mathbf{r}(t)$ defined by a single scalar parameter t,

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k} \tag{20}$$

$\mathbf{i}, \mathbf{j}, \mathbf{k}$ are fixed unit Cartesian vectors in the directions of the x,y,z coordinate axes. As the parameter t varies the terminal point P of the vector sweeps out the space curve C.

Let s be the arc-length of C measured from some convenient point on C, so that $\dfrac{ds}{dt} = \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 + \left(\dfrac{dz}{dt}\right)^2}$ or $s = \displaystyle\int \sqrt{\frac{d\mathbf{r}}{dt} \bullet \frac{d\mathbf{r}}{dt}}\, dt$. Hence s is a function of t and x,y,z are functions of s. Let Q, a small distance δs along the curve from P, have a position vector $\mathbf{r} + \delta\mathbf{r}$. Then $\delta\mathbf{r} = \overrightarrow{PQ}$ and $|\delta\mathbf{r}| \simeq |\delta s|$. Both when δs is positive or negative $\dfrac{\delta\mathbf{r}}{\delta s}$ approximates to a unit vector in the direction of s increasing and $\dfrac{d\mathbf{r}}{ds}$ is a tangent vector of unit length denoted by $\hat{\mathbf{t}}$; hence

$$\hat{\mathbf{t}} = \frac{d\mathbf{r}}{ds} = \frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} + \frac{dz}{ds}\mathbf{k} \tag{21}$$

Since $\hat{\mathbf{t}}$ is a unit vector then $\hat{\mathbf{t}} \bullet \hat{\mathbf{t}} = 1$ and differentiating with respect to s leads to $\hat{\mathbf{t}} \bullet \dfrac{d\hat{\mathbf{t}}}{ds} = 0$ from which we deduce that $\dfrac{d\hat{\mathbf{t}}}{ds}$ is orthogonal to $\hat{\mathbf{t}}$ and write

$$\frac{d\hat{\mathbf{t}}}{ds} = \kappa\hat{\mathbf{n}}, \qquad \kappa > 0 \tag{22}$$

$\dfrac{d\hat{\mathbf{t}}}{ds}$ is called the <u>curvature vector</u> $\mathbf{k}$, $\hat{\mathbf{n}}$ is a unit vector called the <u>principal normal</u> vector, κ the <u>curvature</u> and $\dfrac{1}{\kappa} = \rho$ is the <u>radius of curvature</u>. The circle through P, tangent to $\hat{\mathbf{t}}$ with this radius ρ is called the <u>osculating circle</u>. Also $\hat{\mathbf{n}} \bullet \dfrac{d\hat{\mathbf{t}}}{ds} = \kappa$; i.e., $\hat{\mathbf{n}}$ is the unit vector in the direction of $\mathbf{k}$. Let $\hat{\mathbf{b}}$ be a third unit vector defined by the vector cross product

$$\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}} \tag{23}$$

thus $\hat{\mathbf{t}}, \hat{\mathbf{b}}$ and $\hat{\mathbf{n}}$ form a right-handed triad. Differentiating equation (23) with respect to s gives

$$\frac{d\hat{\mathbf{b}}}{ds} = \frac{d}{ds}\left(\hat{\mathbf{t}} \times \hat{\mathbf{n}}\right) = \frac{d\hat{\mathbf{t}}}{ds} \times \hat{\mathbf{n}} + \hat{\mathbf{t}} \times \frac{d\hat{\mathbf{n}}}{ds} = \kappa\hat{\mathbf{n}} \times \hat{\mathbf{n}} + \hat{\mathbf{t}} \times \frac{d\hat{\mathbf{n}}}{ds} = \hat{\mathbf{t}} \times \frac{d\hat{\mathbf{n}}}{ds}$$

then

$$\hat{\mathbf{t}} \bullet \frac{d\hat{\mathbf{b}}}{ds} = \hat{\mathbf{t}} \bullet \left(\hat{\mathbf{t}} \times \frac{d\hat{\mathbf{n}}}{ds}\right) = \frac{d\hat{\mathbf{n}}}{ds} \bullet \left(\hat{\mathbf{t}} \times \hat{\mathbf{t}}\right) = 0$$

so that $\dfrac{d\hat{\mathbf{b}}}{ds}$ is orthogonal to $\hat{\mathbf{t}}$. But from $\hat{\mathbf{b}} \bullet \hat{\mathbf{b}} = 1$ it follows that $\hat{\mathbf{b}} \bullet \dfrac{d\hat{\mathbf{b}}}{ds} = 0$ so that $\dfrac{d\hat{\mathbf{b}}}{ds}$ is orthogonal to $\hat{\mathbf{b}}$ and so is in the plane containing $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$. Since $\dfrac{d\hat{\mathbf{b}}}{ds}$ is in the plane of $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$ and is orthogonal to $\hat{\mathbf{t}}$, it must be parallel to $\hat{\mathbf{n}}$. The direction of $\dfrac{d\hat{\mathbf{b}}}{ds}$ is opposite $\hat{\mathbf{n}}$ as it must be to ensure the cross product $\dfrac{d\hat{\mathbf{b}}}{ds} \times \hat{\mathbf{t}}$ is in the direction of $\hat{\mathbf{b}}$. Hence

$$\frac{d\hat{\mathbf{b}}}{ds} = -\tau\hat{\mathbf{n}}\,, \qquad \tau > 0 \tag{24}$$

We call $\hat{\mathbf{b}}$ the unit <u>binormal</u> vector, τ the <u>torsion</u>, and $\dfrac{1}{\tau}$ the <u>radius of torsion</u>. $\hat{\mathbf{t}}$, $\hat{\mathbf{n}}$ and $\hat{\mathbf{b}}$ form a right-handed set of orthogonal unit vectors along a space curve.

The plane containing $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$ is the <u>osculating plane</u>, the plane containing $\hat{\mathbf{n}}$ and $\hat{\mathbf{b}}$ is the <u>normal plane</u> and the plane containing $\hat{\mathbf{t}}$ and $\hat{\mathbf{b}}$ is the <u>rectifying plane</u>. Figure 4 shows these orthogonal unit vectors for a space curve.

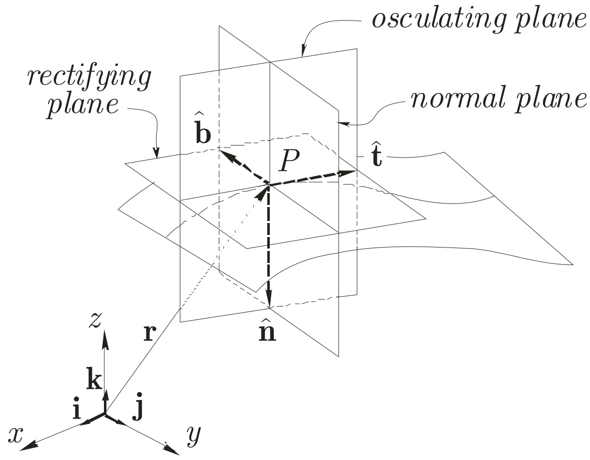


Figure 4: The tangent $\hat{\mathbf{t}}$, principal normal $\hat{\mathbf{n}}$ and binormal $\hat{\mathbf{b}}$ to a space curve

Also $\hat{\mathbf{n}} = \hat{\mathbf{b}} \times \hat{\mathbf{t}}$ and the derivative with respect to s is

$$\frac{d\hat{\mathbf{n}}}{ds} = \frac{d}{ds}\left(\hat{\mathbf{b}} \times \hat{\mathbf{t}}\right) = \frac{d\hat{\mathbf{b}}}{ds} \times \hat{\mathbf{t}} + \hat{\mathbf{b}} \times \frac{d\hat{\mathbf{t}}}{ds} = -\tau\hat{\mathbf{n}} \times \hat{\mathbf{t}} + \hat{\mathbf{b}} \times \kappa\hat{\mathbf{n}} = \tau\hat{\mathbf{b}} - \kappa\hat{\mathbf{t}} \tag{25}$$

Equations (22), (24) and (25) are known as the Frenet-Serret formulae.

$$\begin{aligned} \frac{d\hat{\mathbf{t}}}{ds} &= \kappa\hat{\mathbf{n}} \\ \frac{d\hat{\mathbf{b}}}{ds} &= -\tau\hat{\mathbf{n}} \\ \frac{d\hat{\mathbf{n}}}{ds} &= \tau\hat{\mathbf{b}} - \kappa\hat{\mathbf{t}} \end{aligned} \tag{26}$$

These formulae, derived independently by the French mathematicians Jean-Frédéric Frenet (1816–1900) and Joseph Alfred Serret (1819–1885) describe the dynamics of a point moving along a continuous and differentiable curve in three-dimensional space. Frenet derived these formulae in his doctoral thesis at the University of Toulouse; the latter part of which was published as 'Sur quelques propriétés des courbes à double courbure', (Some properties of curves with double curvature) in the *Journal de mathématiques pures et appliqués* (Journal of pure and applied mathematics), Vol. 17, pp.437-447, 1852. Frenet also explained their use in a paper titled 'Théorèmes sur les courbes gauches' (Theorems on awkward curves) published in 1853. Serret presented an independent derivation of the same formulae in 'Sur quelques formules relatives à la théorie des courbes à double courbure' (Some formulas relating to the theory of curves with double curvature) published in the *J. de Math.* Vol. 16, pp.241-254, 1851 (DSB 1971).

A geodesic may be defined in the following manner:

> A curve drawn on a surface so that its osculating plane at any point contains the normal to the surface at the point is a **geodesic**. It follows that the principal normal at any point on the curve is the normal to the surface and the geodesic is the shortest distance between two points on a surface.

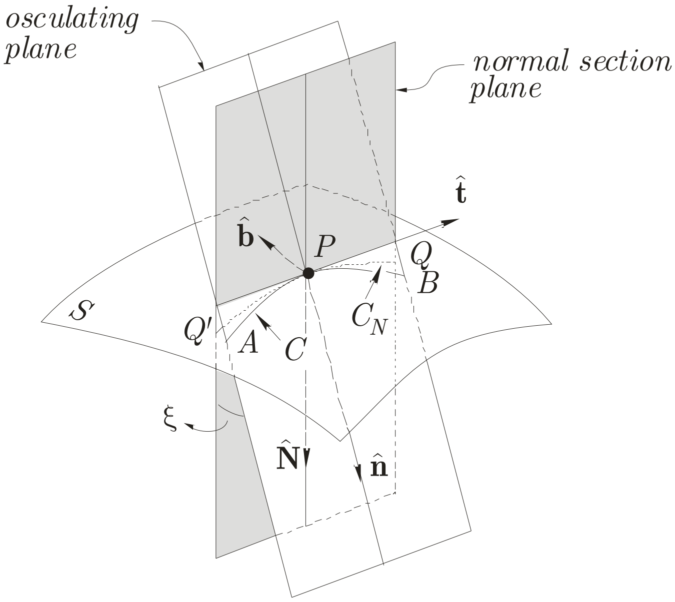


Figure 5: The osculating plane of a geodesic

To understand that the geodesic is the shortest path on a surface requires the use of *Meusnier's theorem,* a fundamental theorem on the nature of surfaces. Jean-Baptiste-Marie-Charles Meusnier de la Place (1754 - 1793) was a French mathematician who, in a paper titled *Mémoire sur la corbure des surfaces* (Memoir on the curvature of surfaces), read at the Paris Academy of Sciences in 1776 and published in 1785, derived his theorem on the curvature, at a point of a surface, of plane sections with a common tangent (DSB 1971). His theorem can be stated as:

> Between the radius ρ of the osculating circle of a plane slice C and the radius ρ_N of the osculating circle of a normal slice C_N, where both slices have the same tangent at P, there exists the relation
>
> $$\rho = \rho_N \cos\xi$$
>
> where ξ is the angle between the unit principal normals $\hat{\mathbf{n}}$ and $\hat{\mathbf{N}}$ to curves C and C_N at *P.*

In Figure 5, an infinitesimal arc PQ of a geodesic coincides with the section of the surface S by a plane containing $\hat{\mathbf{t}}$ and $\hat{\mathbf{N}}$ where $\hat{\mathbf{N}}$ is a unit vector normal to the surface at P.

This plane is a normal section plane through P and by Meusnier's theorem, the geodesic arc PQ is the arc of least curvature through P and Q; or the shortest distance on the surface between two adjacent points P and Q is along the geodesic through the points. In Figure 5, curve C (the arc APB) will have a smaller radius of curvature at P than curve C_N the normal section arc $Q'PQ$.

THE CHARACTERISTIC EQUATION OF A GEODESIC USING DIRECTION COSINES

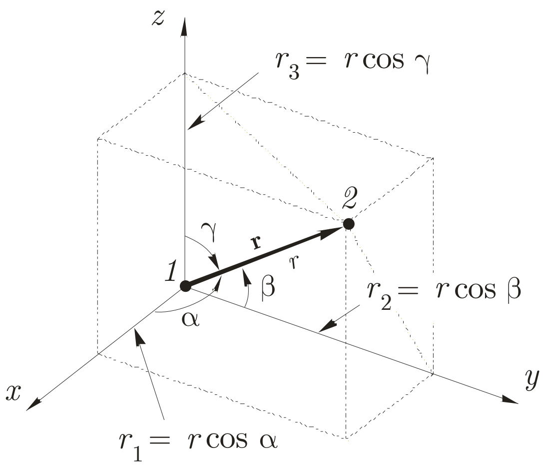


Figure 6: Direction cosines

The characteristic equation of a geodesic can be derived from relationships between the direction cosines of the principal normal to a curve and the normal to the surface. In Figure 6, $\mathbf{r} = r_1\,\mathbf{i} + r_2\,\mathbf{j} + r_3\,\mathbf{k}$ is a vector between two points in space having a magnitude $r = \sqrt{r_1^2 + r_2^2 + r_3^2}$. $\hat{\mathbf{r}} = \frac{\mathbf{r}}{r} = \frac{r_1}{r}\mathbf{i} + \frac{r_2}{r}\mathbf{j} + \frac{r_3}{r}\mathbf{k}$ is a unit vector and the scalar components $\frac{r_1}{r} = \cos\alpha$, $\frac{r_2}{r} = \cos\beta$ and $\frac{r_3}{r} = \cos\gamma$. $l = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$ are known as <u>direction cosines</u> and the unit vector can be expressed as $\hat{\mathbf{r}} = l\,\mathbf{i} + m\,\mathbf{j} + n\,\mathbf{k}$.

From equations (20) and (22) we may write the unit principal normal vector $\hat{\mathbf{n}}$ of a curve C as

$$\hat{\mathbf{n}} = \frac{1}{\kappa}\frac{d^2\mathbf{r}}{ds^2} = \frac{x''}{\kappa}\mathbf{i} + \frac{y''}{\kappa}\mathbf{j} + \frac{z''}{\kappa}\mathbf{k} = \rho\,x''\,\mathbf{i} + \rho\,y''\,\mathbf{j} + \rho\,z''\,\mathbf{k} \tag{27}$$

where $\rho = \dfrac{1}{\kappa}$. $x' = \dfrac{dx}{ds}$ and $x'' = \dfrac{d^2x}{ds^2}$ are first and second derivatives with respect to arc length respectively and similarly for y', z', y'', z''.

The unit normal $\hat{\mathbf{N}}$ to the ellipsoid surface is $\hat{\mathbf{N}} = \dfrac{N_1}{\nu}\mathbf{i} + \dfrac{N_2}{\nu}\mathbf{j} + \dfrac{N_3}{\nu}\mathbf{k}$ where N_1, N_2, N_3 are the Cartesian components of the normal vector $\overrightarrow{PH}$ and ν is the magnitude. $\dfrac{N_1}{\nu} = \cos\alpha$, $\dfrac{N_2}{\nu} = \cos\beta$ and $\dfrac{N_3}{\nu} = \cos\gamma$ are the direction cosines *l*, *m* and *n*. Note that the direction of the unit normal to the ellipsoid is towards the centre of curvature of normal sections passing through *P*.

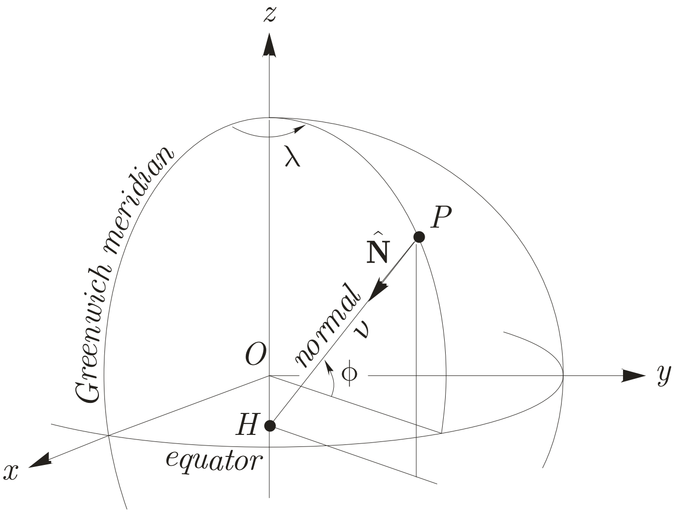


Figure 7: The unit normal $\hat{\mathbf{N}}$ to the ellipsoid

The unit normal $\hat{\mathbf{N}}$ to the ellipsoid surface is given by

$$\hat{\mathbf{N}} = \left(\frac{-x}{\nu}\right)\mathbf{i} + \left(\frac{-y}{\nu}\right)\mathbf{j} + \left(\frac{-\nu\sin\phi}{\nu}\right)\mathbf{k} \tag{28}$$

To ensure that the curve C is a geodesic, i.e., the unit principal normal $\hat{\mathbf{n}}$ to the curve must be coincident with the unit normal $\hat{\mathbf{N}}$ to the surface, the coefficients in equations (27) and (28) must be equal, thus

$$\frac{-x}{\nu} = \rho\,x''; \quad \frac{-y}{\nu} = \rho\,y''; \quad \frac{-\nu\sin\phi}{\nu} = \rho\,z''$$

This leads to

$$\frac{\rho\,x''}{x/\nu} = \frac{\rho\,y''}{y/\nu} = \frac{\rho\,z''}{\nu\sin\phi/\nu} \tag{29}$$

From the first two equations of (29) we have $\rho x'' \dfrac{\nu}{x} = \rho y'' \dfrac{\nu}{y}$ giving the second-order differential equation (provided $\rho\nu \neq 0$)

$$xy'' - yx'' = 0$$

which can be written as $\dfrac{d}{ds}\left(xy' - yx'\right) = 0$ and so a first integral is

$$xy' - yx' = C \tag{30}$$

where C is an arbitrary constant. Now, from equations (15), x and y are functions of ϕ and λ, and the chain rule gives

$$\begin{aligned} x' &= \frac{\partial x}{\partial \phi}\frac{d\phi}{ds} + \frac{\partial x}{\partial \lambda}\frac{d\lambda}{ds} \\ y' &= \frac{\partial y}{\partial \phi}\frac{d\phi}{ds} + \frac{\partial y}{\partial \lambda}\frac{d\lambda}{ds} \end{aligned} \tag{31}$$

Differentiating the first two equations of (15) with respect to ϕ, bearing in mind that ν is a function of ϕ gives

$$\begin{aligned} \frac{\partial x}{\partial \phi} &= -\nu \sin\phi \cos\lambda + \cos\phi \cos\lambda \frac{d\nu}{d\phi} \\ &= -\nu \sin\phi \cos\lambda + \cos\phi \cos\lambda \frac{ae^2 \sin\phi \cos\phi}{\left(1 - e^2 \sin^2\phi\right)^{\frac{3}{2}}} \end{aligned}$$

Using equation (8) and simplifying yields

$$\frac{\partial x}{\partial \phi} = -\rho \sin\phi \cos\lambda$$

Similarly

$$\frac{\partial y}{\partial \phi} = -\nu \sin\phi \sin\lambda + \cos\phi \sin\lambda \frac{d\nu}{d\phi} = -\rho \sin\phi \sin\lambda$$

Placing these results, together with the derivatives $\dfrac{\partial x}{\partial \lambda}$ and $\dfrac{\partial y}{\partial \lambda}$ into equations (31) gives

$$\begin{aligned} x' &= -\rho \sin\phi \cos\lambda \frac{d\phi}{ds} - \nu \cos\phi \sin\lambda \frac{d\lambda}{ds} \\ y' &= -\rho \sin\phi \sin\lambda \frac{d\phi}{ds} + \nu \cos\phi \cos\lambda \frac{d\lambda}{ds} \end{aligned}$$

These values of x' and y' together with x and y from equations (15) substituted into equation (30) gives

$$\nu^2 \cos^2\phi \frac{d\lambda}{ds} = C \tag{32}$$

which can be re-arranged to give an expression for the differential distance ds

$$ds = \frac{\nu^2 \cos^2 \phi}{C} d\lambda$$

ds is also given by equation (18) and equating the two and simplifying gives the differential equation of the geodesic (Thomas 1952)

$$C^2 \rho^2 d\phi^2 + \nu^2 \cos^2 \phi \left(C^2 - \nu^2 \cos^2 \phi\right) d\lambda^2 = 0 \tag{33}$$

From equation (19), $\sin\alpha = \nu\cos\phi \dfrac{d\lambda}{ds}$ and substituting into equation (32) gives the characteristic equation of the geodesic on the ellipsoid

$$\boxed{\nu \cos\phi \sin\alpha = C} \tag{34}$$

Equation (34) is also known as *Clairaut's equation* in honour of the French mathematical physicist Alexis-Claude Clairaut (1713-1765). In a paper in 1733 titled *Détermination géométrique de la perpendiculaire à la méridienne, tracée par M. Cassini, avec plusieurs methods d'en tirer la grandeur et la figure de la terre* (Geometric determination of the perpendicular to the meridian, traced by Mr. Cassini, … on the figure of the Earth.) Clairaut made an elegant study of the geodesics of quadrics of rotation. It included the property already pointed out by Johann Bernoulli: the osculating plane of the geodesic is normal to the surface (DSB 1971).

The characteristic equation of a geodesic shows that the geodesic on the ellipsoid has the intrinsic property that at any point, the product of the radius w of the parallel of latitude and the sine of the azimuth of the geodesic at that point is a constant. This means that as $w = \nu\cos\phi$ decreases in higher latitudes, in both the northern and southern hemispheres, $\sin\alpha$ increases until it reaches a maximum or minimum of ± 1, noting that the azimuth of a geodesic at a point will vary between 0° and 180° if the point is moving along a geodesic in an easterly direction or between 180° and 360° if the point is moving along a geodesic in a westerly direction. At the point when $\sin\alpha = \pm 1$, which is known as the *vertex*, w is a minimum and the latitude ϕ will be a maximum value ϕ_0, known as the geodetic latitude of the vertex. Thus the geodesic oscillates over the surface of the ellipsoid between two parallels of latitude having a maximum in the northern and southern hemispheres and crossing the equator at nodes; but as we will demonstrate later, due to the eccentricity of the ellipsoid the geodesic will not repeat after a complete cycle.

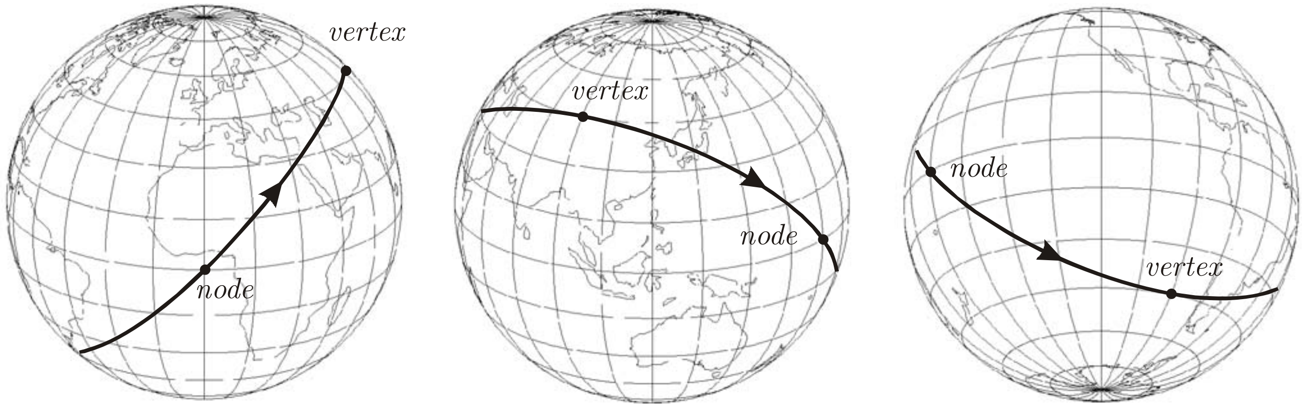


Figure 8a Figure 8b Figure 8c

Figure 8: A single cycle of a geodesic on the Earth

Figures 8a, 8b and 8c show a single cycle of a geodesic on the Earth. This particular geodesic reaches maximum latitudes of approximately $\pm 45^\circ$ and has an azimuth of approximately 45° as it crosses the equator at longitude 0°.

Figure 9 shows a schematic representation of the oscillation of a geodesic on an ellipsoid. P is a point on a geodesic that crosses the equator at A, heading in a north-easterly direction reaching a maximum northerly latitude $\phi_{\max}$ at the vertex P_0 (north), then descends in a south-easterly direction crossing the equator at B, reaching a maximum southerly latitude $\phi_{\min}$ at P_0 (south), then ascends in a north-easterly direction crossing the equator again at A'. This is one complete cycle of the geodesic, but $\lambda_{A'}$ does not equal λ_A due to the eccentricity of the ellipsoid, hence we say that the geodesic curve does not repeat after a complete cycle.

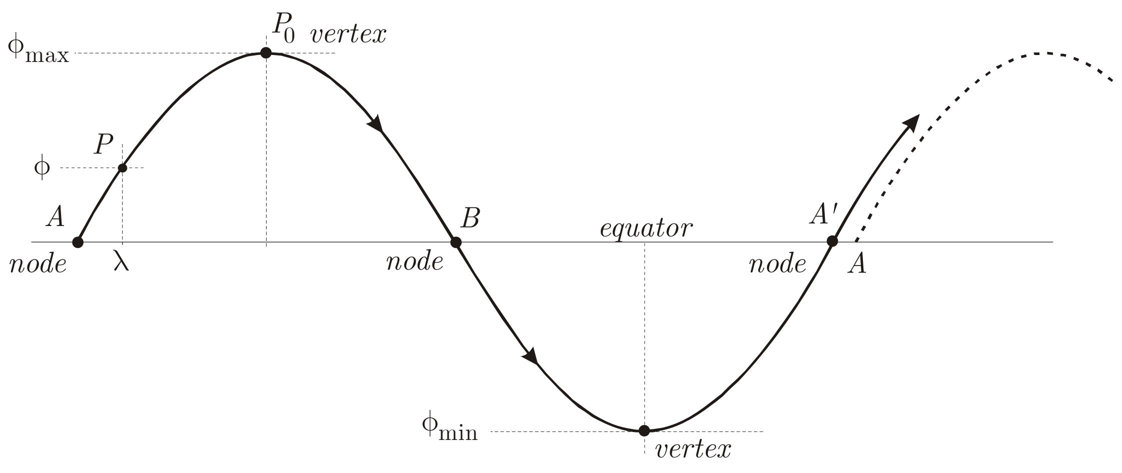


Figure 9: Schematic representation of the oscillation of a geodesic on an ellipsoid

RELATIONSHIPS BETWEEN PARAMETRIC LATITUDE ψ AND GEODETIC LATITUDE ϕ

The development of formulae is simplified if parametric latitude ψ is used rather than geodetic latitude ϕ. The connection between the two latitudes can be obtained from the following relationships.

Figure 10 shows a portion of a meridian NPE of an ellipsoid having semi-major axis $OE = a$ and semi-minor axis $ON = b$. P is a point on the ellipsoid and P' is a point on an auxiliary circle centred on O of radius a. P and P' have the same perpendicular distance w from the axis of revolution ON. The normal to the ellipsoid at P cuts the major axis at an angle ϕ (the geodetic latitude) and intersects the rotational axis at H. The distance $PH = \nu$. The angle $P'OE = \psi$ is the parametric latitude

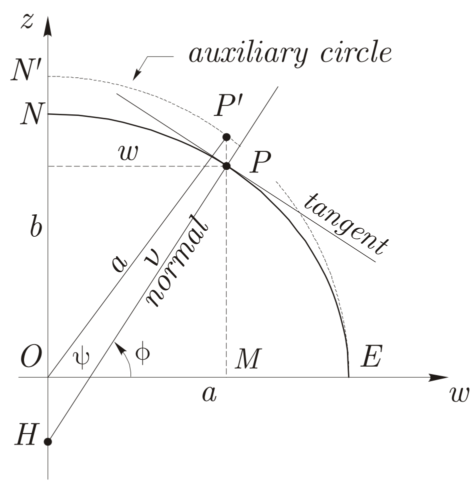


Figure 10: Meridian section of ellipsoid

The Cartesian equation of the ellipse and the auxiliary circle of Figure 10 are $\dfrac{w^2}{a^2} + \dfrac{z^2}{b^2} = 1$ and $w^2 + z^2 = a^2$ respectively. Now, since the w-coordinate of P and P' are the same then $a^2 - \dfrac{a^2}{b^2} z_P^2 = w_P^2 = w_{P'}^2 = a^2 - z_{P'}^2$ which leads to $z_P = \dfrac{b}{a} z_{P'}$. Using this relationship

$$\begin{aligned} w &= OM = a\cos\psi \\ z &= MP = b\sin\psi \end{aligned} \tag{35}$$

Note that writing equations (35) as $\dfrac{w}{a} = \cos\psi$ and $\dfrac{z}{b} = \sin\psi$ then squaring and adding gives $\dfrac{w^2}{a^2} + \dfrac{z^2}{b^2} = \cos^2\psi + \sin^2\psi = 1$ which is the Cartesian equation of an ellipse.

From Figure 10

$$w = \nu\cos\phi = a\cos\psi \tag{36}$$

and from the third of equations (15) $z = \nu\left(1 - e^2\right)\sin\phi$, hence using equations (35) we may write

$$
\begin{aligned}
w &= a\cos\psi = \nu\cos\phi \\
z &= b\sin\psi = \nu\left(1-e^2\right)\sin\phi
\end{aligned}
\tag{37}
$$

from which the following ratios are obtained

$$\frac{z}{w} = \frac{b}{a}\tan\psi = \left(1-e^2\right)\tan\phi$$

Since $e^2 = \dfrac{a^2-b^2}{a^2} = 1-\dfrac{b^2}{a^2}$ then $1-e^2 = \dfrac{b^2}{a^2}$ and we may define parametric latitude ψ by

$$\tan\psi = \frac{b}{a}\tan\phi = \left(1-e^2\right)^{\frac{1}{2}}\tan\phi = (1-f)\tan\phi \tag{38}$$

Alternatively, using equations (36) and (8) we may define the parametric latitude ψ by

$$\cos\psi = \frac{\cos\phi}{\left(1-e^2\sin^2\phi\right)^{\frac{1}{2}}} \tag{39}$$

or equivalently by

$$\sin\phi = \frac{\sin\psi}{\left(1-e^2\cos^2\psi\right)^{\frac{1}{2}}} \tag{40}$$

These three relationships are useful in the derivation of formulae for geodesic distance and longitude difference that follow.

THE LATITUDES ϕ_0 AND ψ_0 OF THE GEODESIC VERTEX

Now Clairaut's equation (34) is $\nu\cos\phi\sin\alpha = \text{constant} = C$, where $\nu = \dfrac{a}{\left(1-e^2\sin^2\phi\right)^{\frac{1}{2}}}$.

The term $\nu\cos\phi$ will be a minimum (and the latitude ϕ will be a maximum in the northern and southern hemispheres) when $|\sin\alpha|$ is a maximum of 1, and this occurs when $\alpha = 90^\circ$ or 270°. This point is known as the geodesic vertex.

Let $\nu_0\cos\phi_0$ be this smallest value, then

$$\nu_0\cos\phi_0 = C = \nu\cos\phi\sin\alpha \tag{41}$$

ϕ_0 is called the maximum geodetic latitude and the value of ψ corresponding to ϕ_0 is called the maximum parametric latitude and is denoted by ψ_0. Using this correspondence and equations (36) and (41) gives

$$a\cos\psi_0 = \nu\cos\phi\sin\alpha = a\cos\psi\sin\alpha \tag{42}$$

From this we may define the parametric latitude of the vertex ψ_0 as

$$\boxed{\cos\psi_0 = \cos\psi\sin\alpha} \tag{43}$$

and the azimuth α of the geodesic as

$$\cos\alpha = \frac{\sqrt{\cos^2\psi - \cos^2\psi_0}}{\cos\psi} \tag{44}$$

From equation (43) we see that if the azimuth α of a geodesic is known at a point P having parametric latitude ψ, the parametric latitude ψ_0 of the vertex P_0 can be computed. Conversely, given ψ and ψ_0 of points P and P_0 the azimuth of the geodesic between them may be computed from equation (44).

THE ELLIPSOID, THE AUXILIARY SPHERE AND THE DIFFERENTIAL EQUATIONS

The derivation of Bessel's formulae (or Rainsford's and Vincenty's equations) begins by developing relationships between the ellipsoid and a sphere. The sphere is an auxiliary surface and not an approximation of the ellipsoid; its radius therefore is immaterial and can be taken to be 1 (unit radius).

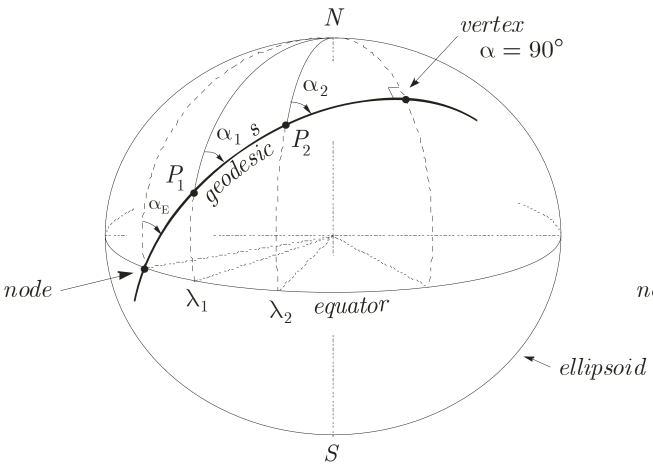


Figure 11a: The geodesic passing through P_1 and P_2 on the ellipsoid.

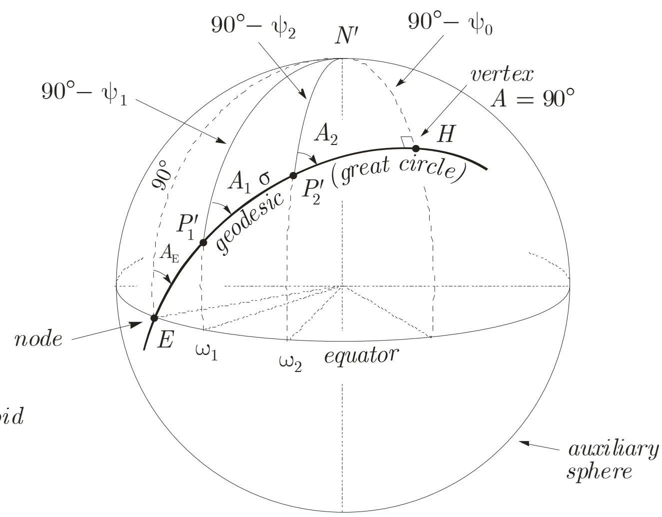


Figure 11b: The great circle passing through P_1' and P_2' on the auxiliary sphere.

Figure 11a shows a geodesic passing through P_1 and P_2 on an ellipsoid. The geodesic has azimuths α_E where it crosses the equator (a node), α_1 and α_2 at P_1 and P_2 respectively and reaches a maximum latitude at the vertex where its azimuth is $\alpha = 90^\circ$. The length of the geodesic between P_1 and P_2 is s and the longitudes of P_1 and P_2 are λ_1 and λ_2. Using equation (43) we may write

$$\cos\psi_1 \sin\alpha_1 = \cos\psi_2 \sin\alpha_2 = \cos\psi_0 \tag{45}$$

Figure 11b shows P_1' and P_2' on an auxiliary sphere (of unit radius) where latitudes on this sphere are defined to be equal to parametric latitudes on the ellipsoid. The geodesic, a great circle on a sphere, passing through P_1' and P_2' has azimuths A_E at the equator E, A_1 and A_2 at P_1' and P_2' respectively and $A = 90^\circ$ at the vertex H. The length of the great circle between P_1' and P_2' is σ and the longitudes of P_1' and P_2' are ω_1 and ω_2. Again, using equation (43), which holds for all geodesics (or great circles on auxiliary spheres) we may write

$$\cos\psi_1 \sin A_1 = \cos\psi_2 \sin A_2 = \cos\psi_0 \tag{46}$$

Now, since parametric latitudes are defined to be equal on the auxiliary sphere and the ellipsoid, equations (45) and (46) show that on these two surfaces $A = \alpha$, i.e., <u>azimuths of great circles on the auxiliary sphere are equal to azimuths of geodesics on the ellipsoid</u>.

Now, consider the differential rectangle on the ellipsoid and sphere shown in Figures 12a and 12b below

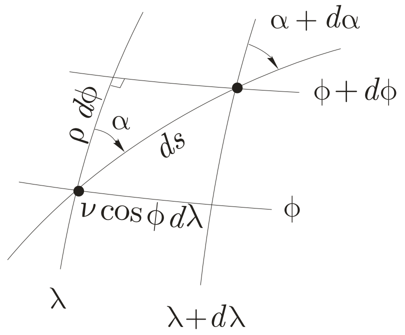


Figure 12a: Differential rectangle on ellipsoid

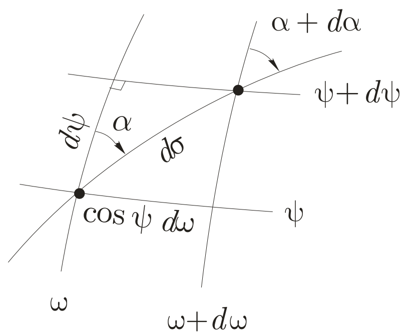


Figure 12b: Differential rectangle on sphere

We have for the ellipsoid [see Figure 2 and equations (19)]

$$\begin{aligned} ds\cos\alpha &= \rho\, d\phi \\ ds\sin\alpha &= \nu\cos\phi\; d\lambda \end{aligned} \tag{47}$$

and for the sphere

$$\begin{aligned} d\sigma\cos\alpha &= d\psi \\ d\sigma\sin\alpha &= \cos\psi \; d\omega \end{aligned} \tag{48}$$

Dividing equations (47) by equations (48) gives

$$\frac{ds\cos\alpha}{d\sigma\cos\alpha} = \frac{\rho\, d\phi}{d\psi}; \qquad \frac{ds\sin\alpha}{d\sigma\sin\alpha} = \frac{\nu\cos\phi \; d\lambda}{\cos\psi \; d\omega}$$

and noting from equation (36) that $\nu\cos\phi = a\cos\psi$, then cancelling terms gives

$$\frac{ds}{d\sigma} = \rho\frac{d\phi}{d\psi} = a\frac{d\lambda}{d\omega} \tag{49}$$

We may write these equations as two separate relationships

$$\frac{ds}{d\sigma} = \rho\frac{d\phi}{d\psi} \tag{50}$$

$$\frac{d\lambda}{d\omega} = \frac{1}{a}\frac{ds}{d\sigma} \tag{51}$$

and if we can obtain an expression for $\frac{d\phi}{d\psi}$ then we may develop two relatively simple differential equations; one involving distance $\frac{ds}{d\sigma}$ (s ellipsoid and σ sphere) and the other involving longitude $\frac{d\lambda}{d\omega}$ (λ ellipsoid and ω sphere). Integration yields equations that will enable us to compute geodesic lengths s on the ellipsoid given great circle distances σ on an auxiliary sphere, and equations to compute longitude differences $\Delta\lambda$ on the ellipsoid given longitude differences $\Delta\omega$ on the auxiliary sphere.

An expression for $\frac{d\phi}{d\psi}$ can be determined as follows.

From equation (38) we have

$$\tan\psi = \left(1-e^2\right)^{\frac{1}{2}}\tan\phi$$

and differentiating with respect to ψ gives

$$\frac{d}{d\psi}(\tan\psi) = \frac{d}{d\phi}\left\{\left(1-e^2\right)^{\frac{1}{2}}\tan\phi\right\}\frac{d\phi}{d\psi}$$

and
$$\sec^2\psi = \left(1-e^2\right)^{\frac{1}{2}}\sec^2\phi\,\frac{d\phi}{d\psi}$$

giving
$$\frac{d\phi}{d\psi}=\frac{1}{\left(1-e^2\right)^{\frac{1}{2}}}\frac{\cos^2\phi}{\cos^2\psi} \tag{52}$$

Substituting equation (52) into equation (50) gives

$$\frac{ds}{d\sigma}=\frac{\rho}{\left(1-e^2\right)^{\frac{1}{2}}}\frac{\cos^2\phi}{\cos^2\psi} \tag{53}$$

and substituting equation (53) into equation (51) gives

$$\frac{d\lambda}{d\omega}=\frac{\rho}{a\left(1-e^2\right)^{\frac{1}{2}}}\frac{\cos^2\phi}{\cos^2\psi} \tag{54}$$

Now from equation (36) we may write

$$\frac{\cos\phi}{\cos\psi}=\frac{a}{\nu}\quad\text{and}\quad\frac{\cos^2\phi}{\cos^2\psi}=\frac{a^2}{\nu^2}$$

and using the relationships given in equations (4), (10), (11) and (12) we may write

$$\frac{\cos^2\phi}{\cos^2\psi}=\frac{a^2}{\nu^2}=\frac{b^2V^2}{a^2};\quad\frac{\rho}{\left(1-e^2\right)^{\frac{1}{2}}}=\frac{c}{V^3}\frac{a}{b}=\frac{a^3}{b^2V^3};\quad\frac{\rho}{a\left(1-e^2\right)^{\frac{1}{2}}}=\frac{a^2}{b^2V^3} \tag{55}$$

Substituting these results into equations (53) and (54) gives

$$\frac{ds}{d\sigma}=\frac{a}{V} \tag{56}$$

and
$$\frac{d\lambda}{d\omega}=\frac{1}{V} \tag{57}$$

Now from equation (13) we may write $V^2=1+e'^2\cos^2\phi$ and also from equation (55) we may write $\cos^2\phi=\dfrac{b^2V^2}{a^2}\cos^2\psi$. Using these gives

$$V^2=1+e'^2\frac{b^2V^2}{a^2}\cos^2\psi$$

Now using equations (4) and (5) gives

$$\begin{aligned}V^2&=1+\frac{e^2}{1-e^2}\left(1-e^2\right)V^2\cos^2\psi\\&=1+e^2V^2\cos^2\psi\end{aligned}$$

and $V^2\left(1-e^2\cos^2\psi\right)=1$ from which we obtain

$$V=\frac{1}{\left(1-e^2\cos^2\psi\right)^{\frac{1}{2}}} \tag{58}$$

Substituting equation (58) into equations (56) and (57) gives

$$\frac{ds}{d\sigma} = a\left(1 - e^2 \cos^2 \psi\right)^{\frac{1}{2}} \tag{59}$$

and

$$\frac{d\lambda}{d\omega} = \left(1 - e^2 \cos^2 \psi\right)^{\frac{1}{2}} \tag{60}$$

Equations (59) and (60) are the two differential equations from which we obtain distance s and longitude difference $\omega - \lambda$.

FORMULA FOR COMPUTATION OF GEODESIC DISTANCE *s*

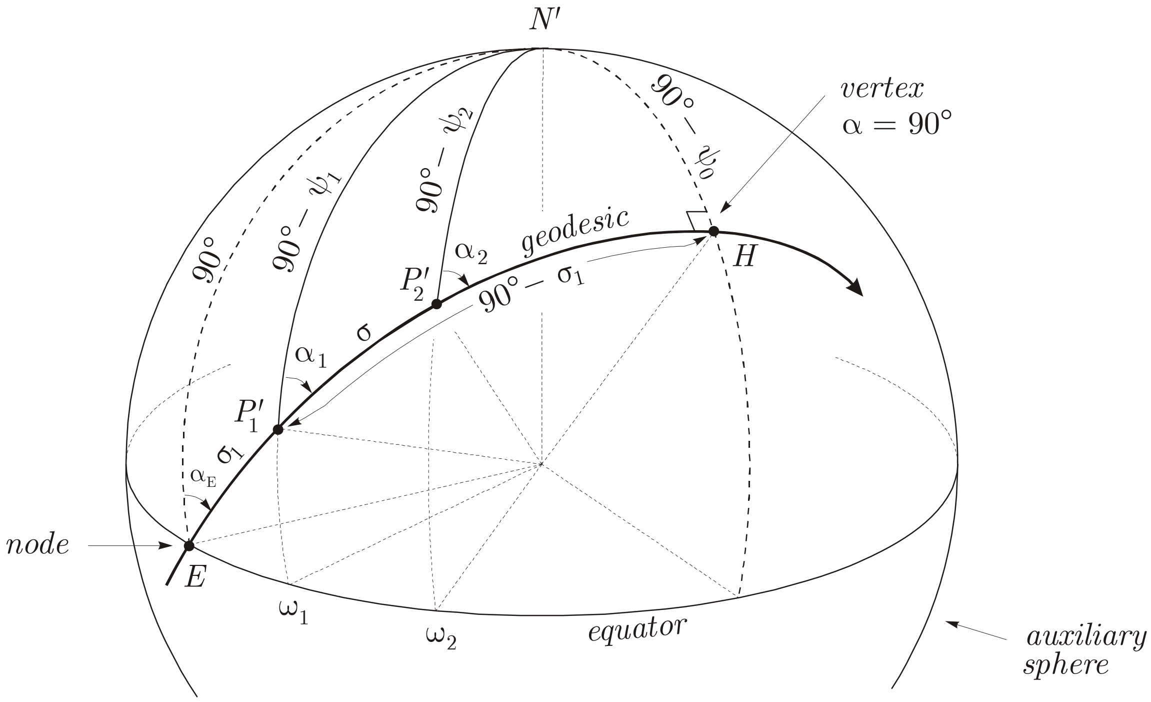


Figure 13: Geodesic on auxiliary sphere

Figure 13 shows P_1' and P_2' on an auxiliary sphere (of unit radius) where latitudes on this sphere are defined to be equal to parametric latitudes on the ellipsoid. The geodesic, a great circle on a sphere, passing through P_1' and P_2' has azimuths α_E at the equator *E*, α_1 at P_1', α_2 at P_2' and $\alpha = 90^\circ$ at the vertex *H*.

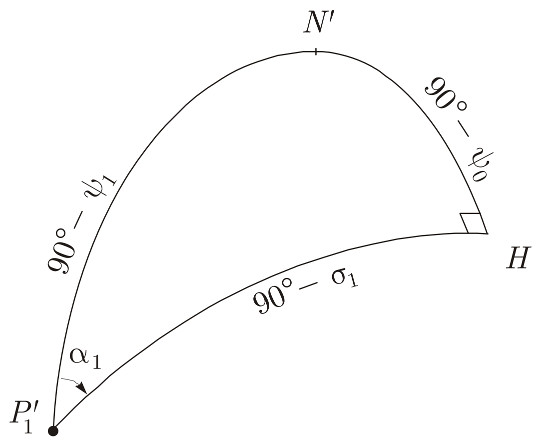
Note here that we have shown previously that for our auxiliary sphere, the azimuth of a great circle on the sphere is equal to the azimuth of the geodesic on the ellipsoid. The length of the great circle arc between P_1' and P_2' is σ and the longitudes of P_1' and P_2' are ω_1 and ω_2. Also note that σ_1 and σ_2 are angular distances along the great circle from the node E to P_1' and E to P_2' respectively and the angular distance from E to the vertex H is 90°. ψ_1, ψ_2 and ψ_0 are the parametric latitudes of P_1, P_2 and the vertex respectively, and they are also the latitudes of P_1', P_2' and the vertex H on the auxiliary sphere.

From the spherical triangle $P_1' N' H$ with the right-angle at H, using the sine rule (for spherical trigonometry)

$$\frac{\sin\alpha_1}{\sin(90^\circ - \psi_0)} = \frac{\sin(90^\circ)}{\sin(90^\circ - \psi_1)}$$

or $$\frac{\sin\alpha_1}{\cos\psi_0} = \frac{1}{\cos\psi_1}$$

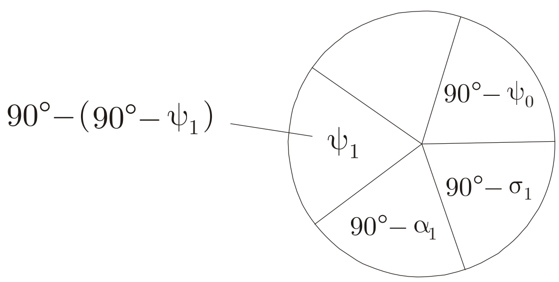
so $$\sin\alpha_1 \cos\psi_1 = \cos\psi_0 \qquad (61)$$



Note that equation (61) can also be obtained from equation (43) and at the equator where $\psi = 90^\circ$ and $\cos\psi = 1$ we have

$$\boxed{\sin\alpha_E = \cos\psi_0} \qquad (62)$$

Using Napier's Rules for circular parts in the right-angled spherical triangle $P_1' N' H$



$$\sin(\text{mid-part}) = \text{product of } \tan(\text{adjacent-parts})$$

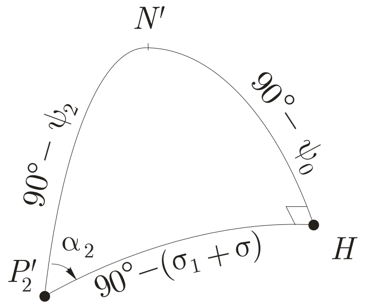
$$\sin(90^\circ - \alpha_1) = \tan\psi_1 \tan(90^\circ - \sigma_1)$$

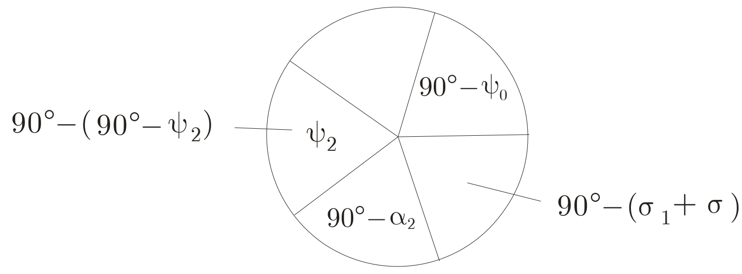
$$\cos\alpha_1 = \tan\psi_1 \cot\sigma_1 = \frac{\tan\psi_1}{\tan\sigma_1}$$

and

$$\boxed{\tan\sigma_1 = \frac{\tan\psi_1}{\cos\alpha_1}} \qquad (63)$$

Using Napier's Rules for circular parts in the right-angled spherical triangle $P_2' N' H$





$$\sin(\text{mid-part}) = \text{product of } \cos(\text{opposite-parts})$$

$$\sin\psi_2 = \cos\big(90^\circ - (\sigma_1 + \sigma)\big)\cos\left(90^\circ - \psi_0\right)$$

$$\sin\psi_2 = \sin(\sigma_1 + \sigma)\sin\psi_0 \tag{64}$$

Note: The subscript 2 can be dropped and we can just refer to a general point P' and the distance from P_1' to P' is σ, hence

$$\boxed{\sin\psi = \sin(\sigma_1 + \sigma)\sin\psi_0} \tag{65}$$

Referring to equations (59) and (60), we need to develop an expression for $\cos^2\psi$. This can be achieved in the following manner.

Squaring both sides of equation (65) and using the trigonometric identity $\sin^2\psi + \cos^2\psi = 1$ we have

$$\sin^2\psi = 1 - \cos^2\psi = \sin^2(\sigma_1 + \sigma)\sin^2\psi_0$$

so that

$$\cos^2\psi = 1 - \sin^2(\sigma_1 + \sigma)\sin^2\psi_0 \tag{66}$$

Let

$$\boxed{x = \sigma_1 + \sigma} \tag{67}$$

and equation (66) becomes

$$\cos^2\psi = 1 - \sin^2 x \sin^2\psi_0 \tag{68}$$

We may now write equation (59) with $dx = d\sigma$ since σ_1 is constant, as

$$\begin{aligned} ds &= a\left(1 - e^2\cos^2\psi\right)^{\frac{1}{2}} d\sigma \\ &= a\left(1 - e^2\left[1 - \sin^2 x\sin^2\psi_0\right]\right)^{\frac{1}{2}} dx \\ &= a\left(1 - e^2 + e^2\sin^2 x\sin^2\psi_0\right)^{\frac{1}{2}} dx \end{aligned}$$

Now using equations (4), (5) and (6)

$$\begin{aligned} ds &= a\left(\frac{1}{1+e'^2}+\frac{e'^2}{1+e'^2}\sin^2 x\sin^2\psi_0\right)^{\frac{1}{2}}dx \\ &= \frac{a}{\left(1+e'^2\right)^{\frac{1}{2}}}\left(1+e'^2\sin^2 x\sin^2\psi_0\right)^{\frac{1}{2}}dx \\ &= b\left(1+e'^2\sin^2 x\sin^2\psi_0\right)^{\frac{1}{2}}dx \end{aligned}$$

Now, since e'^2 is a constant for the ellipsoid and ψ_0 is a constant for a particular geodesic we may write

$$\boxed{u^2 = e'^2\sin^2\psi_0 = e'^2\cos^2\alpha_E} \tag{69}$$

where α_E is the azimuth of the geodesic at the node or equator crossing, and

$$ds = b\left(1+u^2\sin^2 x\right)^{\frac{1}{2}}dx \tag{70}$$

The length of the geodesic arc s between P_1 and P_2 is found by integration as

$$s = b\int_{x=\sigma_1}^{x=\sigma_1+\sigma}\left(1+u^2\sin^2 x\right)^{\frac{1}{2}}dx \tag{71}$$

where the integration terminals are $x=\sigma_1$ and $x=\sigma_1+\sigma$ remembering that at P_1', $\sigma=0$ and $x=\sigma_1$, and at P_2', $x=\sigma_1+\sigma$.

Equation (71) is an elliptic integral and does not have a simple closed-form solution. However, the integrand $\left(1+u^2\sin^2 x\right)^{\frac{1}{2}}$ can be expanded in a series and then evaluated by term-by-term integration.

The integrand in equation (71) can be expanded by use of the binomial series

$$(1+x)^\beta = \sum_{n=0}^{\infty} B_n^\beta x^n \tag{72}$$

An infinite series where n is a positive integer, β is any real number and the binomial coefficients B_n^β are given by

$$B_n^\beta = \frac{\beta(\beta-1)(\beta-2)(\beta-3)\cdots(\beta-n+1)}{n!} \tag{73}$$

The binomial series (72) is convergent when $-1<x<1$. In equation (73) $n!$ denotes n-factorial and $n! = n(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1$. zero-factorial is defined as $0! = 1$ and the binomial coefficient $B_0^\beta = 1$.

In the case where β is a positive integer, say k, the binomial series (72) can be expressed as the finite sum

$$(1+x)^k = \sum_{n=0}^{k} B_n^k \, x^n \tag{74}$$

where the binomial coefficients B_n^k in series (74) are given by

$$B_n^k = \frac{k!}{n!(k-n)!} \tag{75}$$

The binomial coefficients $B_n^{\frac{1}{2}}$ for the series (72) are given by equation (73) with the following results for $n = 0, 1, 2$ and 3

$$
\begin{aligned}
&n=0 \qquad B_0^{\frac{1}{2}} = 1\\
&n=1 \qquad B_1^{\frac{1}{2}} = \frac{1}{2}\\
&n=2 \qquad B_2^{\frac{1}{2}} = \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!} = -\frac{1}{8}\\
&n=3 \qquad B_3^{\frac{1}{2}} = \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!} = \frac{1}{16}
\end{aligned}
$$

Inspecting the results above, we can see that the binomial coefficients $B_n^{\frac{1}{2}}$ form a sequence

$$1,\ \frac{1}{2},\ -\frac{1\cdot 1}{2\cdot 4},\ \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6},\ -\frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8},\ \frac{1\cdot 1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 10},\ -\frac{1\cdot 1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12},\cdots$$

Using these results

$$
\begin{aligned}
\left(1+u^2\sin^2 x\right)^{\frac{1}{2}} = 1 &+ \frac{1}{2}u^2\sin^2 x - \frac{1\cdot 1}{2\cdot 4}u^4\sin^4 x + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}u^6\sin^6 x\\
&- \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}u^8\sin^8 x + \frac{1\cdot 1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 10}u^{10}\sin^{10} x + \cdots
\end{aligned} \tag{76}
$$

To simplify this expression, and make the eventual integration easier, the powers of $\sin x$ can be expressed in terms of multiple angles using the standard form

$$
\begin{aligned}
\sin^{2n} x = \frac{1}{2^{2n}}\binom{2n}{n} + \frac{(-1)^n}{2^{2n-1}}\Bigg\{&\cos 2nx - \binom{2n}{1}\cos(2n-2)x + \binom{2n}{2}\cos(2n-4)x\\
&- \binom{2n}{3}\cos(2n-6)x + \cdots (-1)^n\binom{2n}{n-1}\cos 2x\Bigg\}
\end{aligned} \tag{77}
$$

Using equation (77) and the binomial coefficients $B_n^{2n} = \binom{2n}{n}$ computed using equation (75) gives

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$$

$$\sin^4 x = \frac{3}{8} + \frac{1}{8}\cos 4x - \frac{1}{2}\cos 2x$$

$$\sin^6 x = \frac{5}{16} - \frac{1}{32}\cos 6x + \frac{3}{16}\cos 4x - \frac{15}{32}\cos 2x$$

$$\sin^8 x = \frac{35}{128} + \frac{1}{128}\cos 8x - \frac{1}{16}\cos 6x + \frac{7}{32}\cos 4x - \frac{7}{16}\cos 2x$$

$$\sin^{10} x = \frac{63}{256} - \frac{1}{512}\cos 10x + \frac{5}{256}\cos 8x - \frac{45}{512}\cos 6x + \frac{15}{64}\cos 4x - \frac{105}{256}\cos 2x \tag{78}$$

Substituting equations (78) into equation (76) and arranging according to $\cos 2x$, $\cos 4x$, etc, we obtain (Rapp 1981, p. 7-8)

$$\left(1 + u^2 \sin^2 x\right)^{\frac{1}{2}} = A + B\cos 2x + C\cos 4x + D\cos 6x + E\cos 8x + F\cos 10x + \cdots \tag{79}$$

where the coefficients A, B, C, etc., are

$$\begin{aligned}
A &= 1 + \frac{1}{4}u^2 - \frac{3}{64}u^4 + \frac{5}{256}u^6 - \frac{175}{16384}u^8 + \frac{441}{65536}u^{10} - \cdots \\
B &= -\frac{1}{4}u^2 + \frac{1}{16}u^4 - \frac{15}{512}u^6 + \frac{35}{2048}u^8 - \frac{735}{65536}u^{10} + \cdots \\
C &= -\frac{1}{64}u^4 + \frac{3}{256}u^6 - \frac{35}{4096}u^8 + \frac{105}{16384}u^{10} - \cdots \\
D &= -\frac{1}{512}u^6 + \frac{5}{2048}u^8 - \frac{35}{131072}u^{10} + \cdots \\
E &= -\frac{5}{16384}u^8 + \frac{35}{65536}u^{10} - \cdots \\
F &= -\frac{7}{131072}u^{10} + \cdots
\end{aligned} \tag{80}$$

Substituting equation (79) into equation (71) gives

$$s = b\int_{\sigma_1}^{\sigma_1+\sigma} \left\{A + B\cos 2x + C\cos 4x + D\cos 6x + E\cos 8x + F\cos 10x + \cdots\right\} dx \tag{81}$$

or

$$\begin{aligned}
\frac{s}{b} &= A\int_{\sigma_1}^{\sigma_1+\sigma} dx + B\int_{\sigma_1}^{\sigma_1+\sigma} \cos 2x\, dx + C\int_{\sigma_1}^{\sigma_1+\sigma} \cos 4x\, dx + D\int_{\sigma_1}^{\sigma_1+\sigma} \cos 6x\, dx \\
&\quad + E\int_{\sigma_1}^{\sigma_1+\sigma} \cos 8x\, dx + F\int_{\sigma_1}^{\sigma_1+\sigma} \cos 10x\, dx \cdots
\end{aligned} \tag{82}$$

The evaluation of the integral

$$\int_{\sigma_1}^{\sigma_1+\sigma} \cos nx\, dx = \frac{1}{n}\left[\sin nx\right]_{\sigma_1}^{\sigma_1+\sigma} = \frac{1}{n}\left\{\sin n\left(\sigma_1+\sigma\right) - \sin n\sigma_1\right\} \tag{83}$$

combined with the trigonometric identity

$$\sin nX - \sin nY = 2\cos\left[\frac{n}{2}(X+Y)\right]\sin\left[\frac{n}{2}(X-Y)\right]$$

where $X = \sigma_1 + \sigma$ and $Y = \sigma_1$ so that $X + Y = 2\sigma_1 + \sigma$ and $X - Y = \sigma$ gives

$$\int_{\sigma_1}^{\sigma_1+\sigma} \cos nx\, dx = \frac{2}{n}\cos n\sigma_m \sin\frac{n}{2}\sigma \tag{84}$$

Noting that

$$\sin n\left(\sigma_1+\sigma\right) - \sin n\sigma_1 = 2\cos\frac{n}{2}\left(2\sigma_1+\sigma\right)\sin\frac{n}{2}\sigma$$

and with $\sigma = \sigma_2 - \sigma_1$, then $2\sigma_1 + \sigma = 2\sigma_1 + \left(\sigma_2 - \sigma_1\right) = \sigma_1 + \sigma_2$

and putting
$$\sigma_m = \frac{\sigma_1+\sigma_2}{2} \tag{85}$$

then

$$\boxed{2\sigma_m = 2\sigma_1 + \sigma} \tag{86}$$

and

$$\sin n\left(\sigma_1+\sigma\right) - \sin n\sigma_1 = 2\cos n\sigma_m \sin\frac{n}{2}\sigma \tag{87}$$

Using this result, equation (82) becomes

$$\begin{aligned}\frac{s}{b} &= A\sigma + B\left(\cos 2\sigma_m \sin\sigma\right) + C\left(\tfrac{1}{2}\cos 4\sigma_m \sin 2\sigma\right) + D\left(\tfrac{1}{3}\cos 6\sigma_m \sin 3\sigma\right)\\ &\quad + E\left(\tfrac{1}{4}\cos 8\sigma_m \sin 4\sigma\right) + F\left(\tfrac{1}{5}\cos 10\sigma_m \sin 5\sigma\right) + \cdots\end{aligned}$$

or re-arranged as (Rapp 1981, equation 39, p. 9)

$$\begin{aligned}s &= b\left\{A\sigma + B\cos 2\sigma_m \sin\sigma + \frac{C}{2}\cos 4\sigma_m \sin 2\sigma + \frac{D}{3}\cos 6\sigma_m \sin 3\sigma\right.\\ &\quad \left. + \frac{E}{4}\cos 8\sigma_m \sin 4\sigma + \frac{F}{5}\cos 10\sigma_m \sin 5\sigma + \cdots\right\}\end{aligned} \tag{88}$$

Equation (88) may be modified by adopting another set of constants; defined as

$$B_0 = A;\quad B_2 = B;\quad B_4 = \frac{C}{2};\quad B_6 = \frac{D}{3};\quad B_8 = \frac{E}{4};\quad B_{10} = \frac{F}{5} \tag{89}$$

to give

$$
\boxed{\begin{aligned} s &= b\{B_0\sigma + B_2\cos 2\sigma_m \sin\sigma + B_4\cos 4\sigma_m \sin 2\sigma + B_6\cos 6\sigma_m \sin 3\sigma \\ &\qquad + B_8\cos 8\sigma_m \sin 4\sigma + B_{10}\cos 10\sigma_m \sin 5\sigma + \cdots \\ &\qquad + B_{2n}\cos 2n\sigma_m \sin n\sigma + \cdots\} \end{aligned}} \tag{90}
$$

where the coefficients B_0, B_2, B_4, … are

$$
\begin{aligned}
B_0 &= 1 + \frac{1}{4}u^2 - \frac{3}{64}u^4 + \frac{5}{256}u^6 - \frac{175}{16384}u^8 + \frac{441}{65536}u^{10} - \cdots \\
B_2 &= -\frac{1}{4}u^2 + \frac{1}{16}u^4 - \frac{15}{512}u^6 + \frac{35}{2048}u^8 - \frac{735}{65536}u^{10} + \cdots \\
B_4 &= -\frac{1}{128}u^4 + \frac{3}{512}u^6 - \frac{35}{8192}u^8 + \frac{105}{32768}u^{10} - \cdots \\
B_6 &= -\frac{1}{1536}u^6 + \frac{5}{6144}u^8 - \frac{35}{393216}u^{10} + \cdots \\
B_8 &= -\frac{5}{65536}u^8 + \frac{35}{262144}u^{10} - \cdots \\
B_{10} &= -\frac{7}{655360}u^{10} + \cdots
\end{aligned}
$$

Since each of these convergent series is alternating, an upper bound of the error committed in truncating the series is the first term omitted – keeping terms up to u^8 only commits an error of order u^{10} – and equation (90) can be approximated by

$$
\boxed{\begin{aligned} s &= b\{B_0\sigma + B_2\cos 2\sigma_m \sin\sigma + B_4\cos 4\sigma_m \sin 2\sigma + B_6\cos 6\sigma_m \sin 3\sigma \\ &\qquad + B_8\cos 8\sigma_m \sin 4\sigma\} \end{aligned}} \tag{91}
$$

where

$$
\begin{aligned}
B_0 &= 1 + \frac{1}{4}u^2 - \frac{3}{64}u^4 + \frac{5}{256}u^6 - \frac{175}{16384}u^8 \\
B_2 &= -\frac{1}{4}u^2 + \frac{1}{16}u^4 - \frac{15}{512}u^6 + \frac{35}{2048}u^8 \\
B_4 &= -\frac{1}{128}u^4 + \frac{3}{512}u^6 - \frac{35}{8192}u^8 \\
B_6 &= -\frac{1}{1536}u^6 + \frac{5}{6144}u^8 \\
B_8 &= -\frac{5}{65536}u^8
\end{aligned} \tag{92}
$$

The approximation (91) and the coefficients given by equations (92) are the same as Rainsford (1955, equations 18 and 19, p.15) and also Rapp (1981, equations 40 and 41, p. 9).

Equation (91) can be used in two ways which will be discussed in detail later. Briefly, however, the first way is in the direct problem – where s, u^2 and σ_1 are known – to solve iteratively for σ (and hence σ_m from $2\sigma_m = 2\sigma_1 + \sigma$; and $x = \sigma_1 + \sigma$) by using Newton-Raphson iteration for the real roots of the equation $f(\sigma) = 0$ given in the form of an iterative equation

$$\sigma_{(n+1)} = \sigma_{(n)} - \frac{f\left(\sigma_{(n)}\right)}{f'\left(\sigma_{(n)}\right)} \tag{93}$$

where n denotes the n^{th} iteration and $f(\sigma)$ can be obtained from equation (91) as

$$\begin{aligned} f(\sigma) = B_0\sigma &+ B_2 \cos 2\sigma_m \sin\sigma + B_4 \cos 4\sigma_m \sin 2\sigma + B_6 \cos 6\sigma_m \sin 3\sigma \\ &+ B_8 \cos 8\sigma_m \sin 4\sigma - \frac{s}{b} \end{aligned} \tag{94}$$

and the derivative $f'(\sigma) = \dfrac{d}{d\sigma}\{f(\sigma)\}$ is given by

$$f'\left(\sigma\right) = \left(1 + u^2 \sin^2 x\right)^{\frac{1}{2}} \tag{95}$$

[Note here that $f\left(\sigma\right)$ is the result of integrating the function $\left(1 + u^2 \sin^2 x\right)^{\frac{1}{2}}$ with respect to *dx;* so then the derivative $f'\left(\sigma\right)$ must be the original function.]

An initial value, $\sigma_{(1)}$ (σ for $n = 1$) can be computed from $\sigma_{(1)} = \dfrac{s}{B_0 b}$ and the functions $f\left(\sigma_{(1)}\right)$ and $f'\left(\sigma_{(1)}\right)$ evaluated from equations (94) and (95) using $\sigma_{(1)}$. $\sigma_{(2)}$ (σ for $n = 2$) can now be computed from equation (93) and this process repeated to obtain values $\sigma_{(3)}, \sigma_{(4)}, \ldots$. This iterative process can be concluded when the difference between $\sigma_{(n+1)}$ and $\sigma_{(n)}$ reaches an acceptably small value.

The second application of equation (91) is in the inverse problem where s is computed once σ has been determined by spherical trigonometry.

FORMULA FOR COMPUTATION OF LONGITUDE DIFFERENCE BETWEEN TWO POINTS ON A GEODESIC

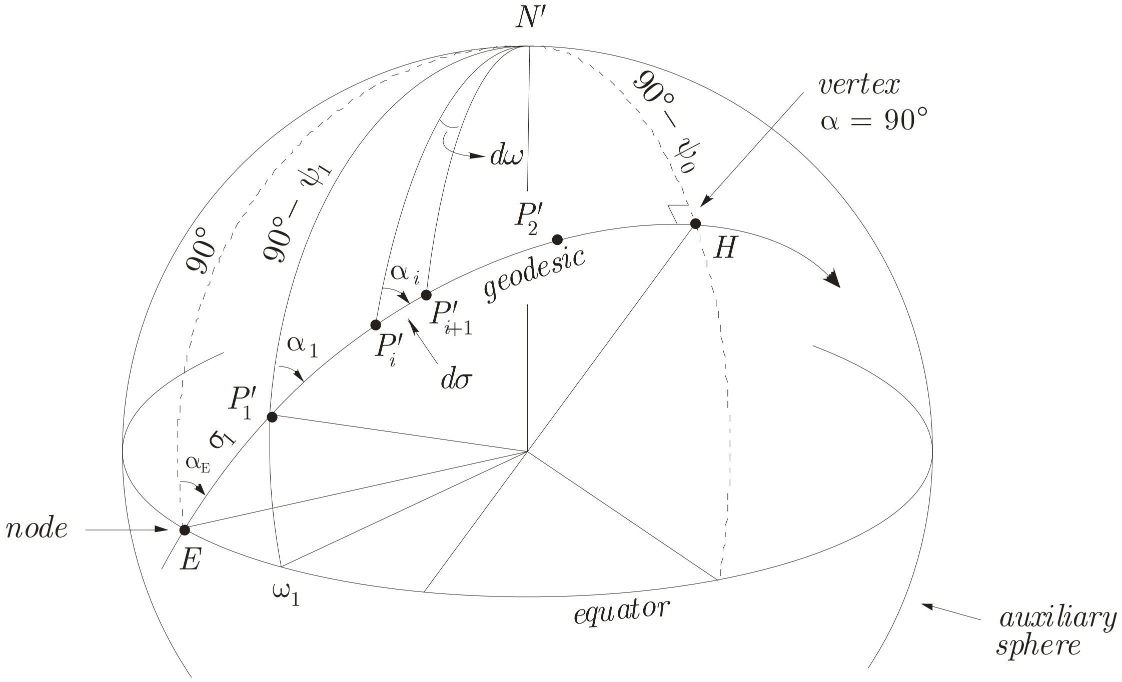


Figure 14: Geodesic on auxiliary sphere

Figure 14 shows P_1' and P_2' on an auxiliary sphere (of unit radius) where latitudes on this sphere are defined to be equal to parametric latitudes on the ellipsoid. P_i' and P_{i+1}' are arbitrary points on the geodesic (a great circle) between P_1' and P_2' separated by the angular distance $d\sigma$.

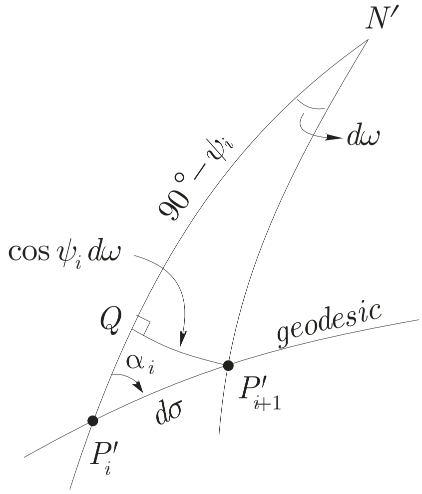


Figure 15

Figure 15 shows the differential spherical triangle $P_i' N' P_{i+1}'$ broken into two right-angled spherical triangles $P_i' Q P_{i+1}'$ and $Q N' P_{i+1}'$. The great circle arc $Q P_{i+1}'$ is defined as $\cos\psi_1 \, d\omega$, which is the differential arc length of the parallel of parametric latitude ψ_1. Approximating the spherical triangle $P_i' Q P_{i+1}'$ with a plane right-angled triangle gives $\cos\psi_i \, d\omega = d\sigma \sin\alpha_i$ and

$$d\omega = \frac{\sin\alpha_i}{\cos\psi_i} d\sigma \tag{96}$$

From equation (43)

$$\sin\alpha_i = \frac{\cos\psi_0}{\cos\psi_i} \tag{97}$$

and substituting equation (97) into (96) gives the relationship (dropping the subscript i)

$$d\omega = \frac{\cos\psi_0}{\cos^2\psi} d\sigma \tag{98}$$

Substituting equation (98) into equation (60) and re-arranging gives

$$d\lambda = \cos\psi_0 \frac{\left(1 - e^2\cos^2\psi\right)^{\frac{1}{2}}}{\cos^2\psi} d\sigma \tag{99}$$

Subtracting equation (98) from equation (99) gives an expression for the difference between differentials of two measures of longitude; $d\omega$ on the auxiliary sphere and $d\lambda$ on the ellipsoid

$$d\lambda - d\omega = \cos\psi_0 \left[\frac{\left(1 - e^2\cos^2\psi\right)^{\frac{1}{2}}}{\cos^2\psi} - \frac{1}{\cos^2\psi}\right] d\sigma \tag{100}$$

Equation (100) can be simplified by expanding $\left(1 - e^2\cos^2\psi\right)^{\frac{1}{2}}$ using the binomial series (72)

$$\left(1 - e^2\cos^2\psi\right)^{\frac{1}{2}} = \sum_{n=0}^{\infty} B_n^{\frac{1}{2}} \left(-e^2\cos^2\psi\right)^n$$

and from the previous development, the binomial coefficients $B_n^{\frac{1}{2}}$ form a sequence

$$1,\ \frac{1}{2},\ -\frac{1\cdot1}{2\cdot4},\ \frac{1\cdot1\cdot3}{2\cdot4\cdot6},\ -\frac{1\cdot1\cdot3\cdot5}{2\cdot4\cdot6\cdot8},\ \frac{1\cdot1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8\cdot10},\ -\frac{1\cdot1\cdot3\cdot5\cdot7\cdot9}{2\cdot4\cdot6\cdot8\cdot10\cdot12},\ldots$$

Using these results

$$\left(1-e^2\cos^2\psi\right)^{\frac{1}{2}} = 1-\frac{1}{2}e^2\cos^2\psi-\frac{1\cdot 1}{2\cdot 4}e^4\cos^4\psi-\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}e^6\cos^6\psi$$
$$-\frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}e^8\cos^8\psi-\frac{1\cdot 1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 10}e^{10}\cos^{10}\psi+\cdots \tag{101}$$

so that

$$\frac{\left(1-e^2\cos^2\psi\right)^{\frac{1}{2}}}{\cos^2\psi} = \frac{1}{\cos^2\psi}-\frac{1}{2}e^2-\frac{1}{8}e^4\cos^2\psi-\frac{1}{16}e^6\cos^4\psi$$
$$-\frac{5}{128}e^8\cos^6\psi-\frac{7}{256}e^{10}\cos^8\psi+\cdots \tag{102}$$

Now, subtracting $\dfrac{1}{\cos^2\psi}$ from both sides of equation (102) gives a new equation whose left-hand-side is the term inside the brackets [ ] in equation (100), and using this result we may write equation (100) as

$$d\lambda-d\omega = \cos\psi_0\left\{-\frac{1}{2}e^2-\frac{1}{8}e^4\cos^2\psi-\frac{1}{16}e^6\cos^4\psi\right.$$
$$\left.-\frac{5}{128}e^8\cos^6\psi-\frac{7}{256}e^{10}\cos^8\psi+\cdots\right\}d\sigma \tag{103}$$

which can be re-arranged as

$$d\omega-d\lambda- = \frac{e^2}{2}\cos\psi_0\left\{1+\frac{1}{4}e^2\cos^2\psi+\frac{1}{8}e^4\cos^4\psi\right.$$
$$\left.+\frac{5}{64}e^6\cos^6\psi+\frac{7}{128}e^8\cos^8\psi+\cdots\right\}d\sigma \tag{104}$$

From equations (65) and (67) we have $\sin\psi=\sin\left(\sigma_1+\sigma\right)\sin\psi_0$ and $x=\sigma_1+\sigma$ respectively, which gives $\sin\psi=\sin x\sin\psi_0$ and $\sin^2\psi=\sin^2 x\sin^2\psi_0=1-\cos^2\psi$. This result can be re-arranged as

$$\cos^2\psi = 1-\sin^2\psi_0\sin^2 x$$

Now $\cos^4\psi=\left(1-\sin^2\psi_0\sin^2 x\right)^2$, $\cos^6\psi=\left(1-\sin^2\psi_0\sin^2 x\right)^3$, $\cos^8\psi=\left(1-\sin^2\psi_0\sin^2 x\right)^4$, etc., and using the binomial series (74) we may write

$$\cos^4\psi = 1-2\sin^2\psi_0\sin^2 x+\sin^4\psi_0\sin^4 x$$

$$\cos^6\psi = 1-3\sin^2\psi_0\sin^2 x+3\sin^4\psi_0\sin^4 x-\sin^6\psi_0\sin^6 x$$

$$\cos^8\psi = 1-4\sin^2\psi_0\sin^2 x+6\sin^4\psi_0\sin^4 x-4\sin^6\psi_0\sin^6 x+\sin^8\psi_0\sin^8 x$$

Substituting these relationships into equation (104) and noting that $dx = d\sigma$ gives

$$\begin{aligned} d\omega - d\lambda - = \frac{e^2}{2}\cos\psi_0 \Big\{ 1 &+ \frac{1}{4}e^2\left(1-\sin^2\psi_0\sin^2 x\right) \\ &+ \frac{1}{8}e^4\left(1-2\sin^2\psi_0\sin^2 x+\sin^4\psi_0\sin^4 x\right) \\ &+ \frac{5}{64}e^6\left(1-3\sin^2\psi_0\sin^2 x+3\sin^4\psi_0\sin^4 x-\sin^6\psi_0\sin^6 x\right) \\ &+ \frac{7}{128}e^8\Big(1-4\sin^2\psi_0\sin^2 x+6\sin^4\psi_0\sin^4 x \\ &\qquad -4\sin^6\psi_0\sin^6 x+\sin^8\psi_0\sin^8 x\Big) \\ &+\cdots\Big\}\,dx \end{aligned} \tag{105}$$

Now, expressions for $\sin^2 x$, $\sin^4 x$, … have been developed previously and are given in equations (78). These even powers of $\sin x$ may be substituted into equation (105) to give

$$\begin{aligned} d\omega - d\lambda - = \frac{e^2}{2}\cos\psi_0 \Big\{ 1 &+ \frac{1}{4}e^2\left(1-\sin^2\psi_0\left[\frac{1}{2}-\frac{1}{2}\cos 2x\right]\right) \\ &+ \frac{1}{8}e^4\Big(1-2\sin^2\psi_0\left[\frac{1}{2}-\frac{1}{2}\cos 2x\right] \\ &\qquad +\sin^4\psi_0\left[\frac{3}{8}+\frac{1}{8}\cos 4x-\frac{1}{2}\cos 2x\right]\Big) \\ &+ \frac{5}{64}e^6\Big(1-3\sin^2\psi_0\left[\frac{1}{2}-\frac{1}{2}\cos 2x\right] \\ &\qquad +3\sin^4\psi_0\left[\frac{3}{8}+\frac{1}{8}\cos 4x-\frac{1}{2}\cos 2x\right] \\ &\qquad -\sin^6\psi_0\left[\frac{5}{16}-\frac{1}{32}\cos 6x+\frac{3}{16}\cos 4x-\frac{15}{32}\cos 2x\right]\Big) \\ &+ \frac{7}{128}e^8\Big(1-4\sin^2\psi_0\left[\frac{1}{2}-\frac{1}{2}\cos 2x\right] \\ &\qquad +6\sin^4\psi_0\left[\frac{3}{8}+\frac{1}{8}\cos 4x-\frac{1}{2}\cos 2x\right] \\ &\qquad -4\sin^6\psi_0\left[\frac{5}{16}-\frac{1}{32}\cos 6x+\frac{3}{16}\cos 4x-\frac{15}{32}\cos 2x\right] \\ &\qquad +\sin^8\psi_0\Big[\frac{35}{128}+\frac{1}{128}\cos 8x-\frac{1}{16}\cos 6x \\ &\qquad\qquad +\frac{7}{32}\cos 4x-\frac{7}{16}\cos 2x\Big] \\ &+\cdots\Big\}\,dx \end{aligned} \tag{106}$$

Expanding the components of equation (106) associated with the even powers of e we have

$$\frac{1}{4}e^2\left(1-\frac{1}{2}\sin^2\psi_0+\frac{1}{2}\sin^2\psi_0\cos 2x\right) \tag{107}$$

$$\begin{aligned}&\frac{1}{8}e^4\Big(1-\sin^2\psi_0+\sin^2\psi_0\cos 2x\\&\qquad+\frac{3}{8}\sin^4\psi_0+\frac{1}{8}\sin^4\psi_0\cos 4x-\frac{1}{2}\sin^4\psi_0\cos 2x\Big)\end{aligned} \tag{108}$$

$$\begin{aligned}&\frac{5}{64}e^6\Big(1-\sin^2\psi_0+\sin^2\psi_0\cos 2x\\&\qquad+\frac{9}{8}\sin^4\psi_0+\frac{3}{8}\sin^4\psi_0\cos 4x-\frac{3}{2}\sin^4\psi_0\cos 2x\\&\qquad-\frac{5}{16}\sin^6\psi_0+\frac{1}{32}\sin^6\psi_0\cos 6x-\frac{3}{16}\sin^6\psi_0\cos 4x+\frac{15}{32}\sin^6\psi_0\cos 2x\Big)\end{aligned} \tag{109}$$

$$\begin{aligned}&\frac{7}{128}e^8\Big(1-\sin^2\psi_0+\sin^2\psi_0\cos 2x\\&\qquad+\frac{9}{4}\sin^4\psi_0+\frac{3}{4}\sin^4\psi_0\cos 4x-3\sin^4\psi_0\cos 2x\\&\qquad-\frac{5}{4}\sin^6\psi_0+\frac{1}{8}\sin^6\psi_0\cos 6x-\frac{3}{4}\sin^6\psi_0\cos 4x\\&\qquad+\frac{15}{8}\sin^6\psi_0\cos 2x\\&\qquad+\frac{35}{128}\sin^8\psi_0+\frac{1}{128}\sin^8\psi_0\cos 8x-\frac{1}{16}\sin^8\psi_0\cos 6x\\&\qquad+\frac{7}{132}\sin^8\psi_0\cos 4x-\frac{7}{16}\sin^8\psi_0\cos 2x\Big)\end{aligned} \tag{110}$$

Gathering together the constant terms and the coefficients of $\cos 2x$, $\cos 4x$, $\cos 6x$, etc. in equations (107) to (110), we can write equation (106) as

$$d\omega-d\lambda=\frac{e^2}{2}\cos\psi_0\left\{C_0+C_2\cos 2x+C_4\cos 4x+C_6\cos 6x+C_8\cos 8x+\cdots\right\}dx \tag{111}$$

where the coefficients C_0, C_2, C_4, etc. are

$$
\begin{aligned}
C_0 = 1 &+ \frac{1}{4}e^2 + \frac{1}{8}e^4 + \frac{5}{64}e^6 + \frac{7}{128}e^8 + \cdots \\
&-\left(\frac{1}{8}e^2 + \frac{1}{8}e^4 + \frac{15}{128}e^6 + \frac{7}{64}e^8 + \cdots\right)\sin^2\psi_0 \\
&+\left(\frac{3}{64}e^4 + \frac{45}{512}e^6 + \frac{63}{512}e^8 + \cdots\right)\sin^4\psi_0 \\
&-\left(\frac{25}{1024}e^6 + \frac{35}{512}e^8 + \cdots\right)\sin^6\psi_0 \\
&+\left(\frac{245}{16384}e^8 + \cdots\right)\sin^8\psi_0 \\
&-\cdots
\end{aligned} \tag{112}
$$

$$
\begin{aligned}
C_2 = &\left(\frac{1}{8}e^2 + \frac{1}{8}e^4 + \frac{15}{128}e^6 + \frac{7}{64}e^8 + \cdots\right)\sin^2\psi_0 \\
&-\left(\frac{1}{16}e^4 + \frac{15}{128}e^6 + \frac{21}{128}e^8 + \cdots\right)\sin^4\psi_0 \\
&+\left(\frac{75}{2048}e^6 + \frac{105}{1024}e^8 + \cdots\right)\sin^6\psi_0 \\
&+\left(\frac{49}{2048}e^8 + \cdots\right)\sin^8\psi_0 \\
&-\cdots
\end{aligned} \tag{113}
$$

$$
\begin{aligned}
C_4 = &\left(\frac{1}{64}e^4 + \frac{15}{512}e^6 + \frac{21}{512}e^8 + \cdots\right)\sin^4\psi_0 \\
&-\left(\frac{15}{1024}e^6 + \frac{21}{512}e^8 + \cdots\right)\sin^6\psi_0 \\
&+\left(\frac{49}{1096}e^8 + \cdots\right)\sin^8\psi_0 \\
&-\cdots
\end{aligned} \tag{114}
$$

$$
\begin{aligned}
C_6 = &\left(\frac{5}{2048}e^6 + \frac{7}{1024}e^8 + \cdots\right)\sin^6\psi_0 \\
&-\left(\frac{7}{2048}e^8 + \cdots\right)\sin^8\psi_0 \\
&+\cdots
\end{aligned} \tag{115}
$$

$$
C_8 = \left(\frac{7}{16384}e^8 + \cdots\right)\sin^8\psi_0 - \cdots \tag{116}
$$

The longitude differences (spherical ω minus geodetic λ) are given by the integral

$$
\Delta\omega - \Delta\lambda = \frac{e^2}{2}\cos\psi_0 \int_{x=\sigma_1}^{x=\sigma_1+\sigma} \left\{C_0 + C_2\cos 2x + C_4\cos 4x + C_6\cos 6x + C_8\cos 8x + \cdots\right\} dx \tag{117}
$$

where $\Delta\omega = \omega_2 - \omega_1$ is the difference in longitudes of P_1' and P_2' on the auxiliary sphere and $\Delta\lambda = \lambda_2 - \lambda_1$ is the difference in longitudes of P_1 and P_2 on the ellipsoid.

Equation (117) has a similar form to equation (81) and the solution of the integral in equation (117) can be achieved by the same method used to solve the integral in equation (81). Hence, similarly to equation (88) and also Rapp (1981 equation (55), p. 13)

$$\Delta\omega - \Delta\lambda = \frac{e^2}{2}\cos\psi_0 \left\{ C_0\sigma + C_2\cos 2\sigma_m \sin\sigma + \frac{C_4}{2}\cos 4\sigma_m \sin 2\sigma + \frac{C_6}{3}\cos 6\sigma_m \sin 3\sigma + \frac{C_8}{4}\cos 8\sigma_m \sin 4\sigma + \cdots \right\} \tag{118}$$

Rainsford (1955, p. 14, equations 10 and 11) has the differences in longitudes $\Delta\omega - \Delta\lambda$ as a function of the flattening f and the azimuth of the geodesic at the equator α_E; noting that from either equations (61) or (69) we may obtain the relationships

$$\sin\alpha_E = \cos\psi_0 \tag{119}$$

$$1 - \sin^2\alpha_E = \sin^2\psi_0 \tag{120}$$

Also, since $e^2 = f(2-f) = 2f - f^2$, even powers of the eccentricity e can be expressed as functions of the flattening f

$$\begin{aligned} e^2 &= 2f - f^2 \\ e^4 &= 4f^2 - 4f^3 + f^4 \\ e^6 &= 8f^3 - 12f^4 + 6f^5 - f^6 \\ e^8 &= 16f^4 - 32f^5 + 24f^6 - 8f^7 + f^8 \end{aligned} \tag{121}$$

Re-arranging equation (118) and using equation (119) gives

$$\Delta\omega - \Delta\lambda = \sin\alpha_E \left\{ \frac{e^2}{2}C_0\sigma + \frac{e^2}{2}C_2\cos 2\sigma_m \sin\sigma + \frac{e^2}{4}C_4\cos 4\sigma_m \sin 2\sigma + \frac{e^2}{6}C_6\cos 6\sigma_m \sin 3\sigma + \frac{e^2}{8}C_8\cos 8\sigma_m \sin 4\sigma + \cdots \right\} \tag{122}$$

Now, with equations (112) and (120) the coefficient $\frac{e^2}{2}C_0$ can be written as

$$\begin{aligned} \frac{e^2}{2}C_0 = {} & \frac{e^2}{2} + \frac{1}{8}e^4 + \frac{1}{16}e^6 + \frac{5}{128}e^8 + \cdots \\ & - \left(\frac{1}{16}e^4 + \frac{1}{16}e^6 + \frac{15}{256}e^8 \cdots\right)\left(1 - \sin^2\alpha_E\right) \\ & + \left(\frac{3}{128}e^6 + \frac{45}{1024}e^8 + \cdots\right)\left(1 - \sin^2\alpha_E\right)^2 \\ & - \left(\frac{25}{2048}e^8 + \cdots\right)\left(1 - \sin^2\alpha_E\right)^3 \\ & + \cdots \end{aligned} \tag{123}$$

noting here that terms greater than e^8 have been ignored.

Using equations (121) in equation (123) with the trigonometric identity $\cos^2\alpha_E + \sin^2\alpha_E = 1$ gives

$$\begin{aligned}\frac{e^2}{2}C_0 = f - \frac{7}{8}f^5 + \cdots \\ -\left(\frac{1}{4}f^2 + \frac{1}{4}f^3 + \frac{1}{4}f^4 - \frac{3}{2}f^5 + \cdots\right)\cos^2\alpha_E \\ +\left(\frac{3}{16}f^3 + \frac{27}{64}f^4 - \frac{81}{64}f^5 + \cdots\right)\cos^4\alpha_E \\ -\left(\frac{25}{128}f^4 - \frac{25}{64}f^5 + \cdots\right)\cos^6\alpha_E \\ +\cdots\end{aligned} \tag{124}$$

Now for any geodetic ellipsoid $e^8 \simeq 2.01\text{e-}009$ and $f^4 \simeq 1.26\text{e-}010$, and since terms greater than e^8 have been ignored in the development of equation (123) then no additional errors will be induced by ignoring terms greater than f^4 in equation (124). Hence we define

$$\begin{aligned}\frac{e^2}{2}C_0 \equiv f\left\{1 - \frac{1}{4}f\left(1 + f + f^2\right)\cos^2\alpha_E\right. \\ +\frac{3}{16}f^2\left(1 + \frac{9}{4}f\right)\cos^4\alpha_E \\ \left.-\frac{25}{128}f^3\cos^6\alpha_E\right\}\end{aligned} \tag{125}$$

Using similar reasoning we also define

$$\frac{e^2}{2}C_2 \equiv f\left\{\frac{1}{4}f\left(1 + f + f^2\right)\cos^2\alpha_E - \frac{1}{4}f^2\left(1 + \frac{9}{4}f\right)\cos^4\alpha_E + \frac{75}{256}f^3\cos^6\alpha_E\right\} \tag{126}$$

$$\frac{e^2}{4}C_4 \equiv f\left\{\frac{1}{32}f^2\left(1 + \frac{9}{4}f\right)\cos^4\alpha_E - \frac{15}{256}f^3\cos^6\alpha_E\right\} \tag{127}$$

$$\frac{e^2}{6}C_6 \equiv f\left\{\frac{5}{768}f^3\cos^6\alpha_E\right\} \tag{128}$$

Using equations (125) to (128) enables equation (122) to be approximated by

$$\boxed{\Delta\omega - \Delta\lambda = f\sin\alpha_E\left\{A_0\sigma + A_2\cos 2\sigma_m\sin\sigma + A_4\cos 4\sigma_m\sin 2\sigma + A_6\cos 6\sigma_m\sin 3\sigma\right\} \quad (129)}$$

where $\Delta\omega = \omega_2 - \omega_1$ is the difference in longitudes of P_1' and P_2' on the auxiliary sphere and $\Delta\lambda = \lambda_2 - \lambda_1$ is the difference in longitudes of P_1 and P_2 on the ellipsoid, and the coefficients are

$$
\begin{aligned}
A_0 &= 1-\frac{1}{4}f\left(1+f+f^2\right)\cos^2\alpha_E+\frac{3}{16}f^2\left(1+\frac{9}{4}f\right)\cos^4\alpha_E-\frac{25}{128}f^3\cos^6\alpha_E \\
A_2 &= \frac{1}{4}f\left(1+f+f^2\right)\cos^2\alpha_E-\frac{1}{4}f^2\left(1+\frac{9}{4}f\right)\cos^4\alpha_E+\frac{75}{256}f^3\cos^6\alpha_E \\
A_4 &= \frac{1}{32}f^2\left(1+\frac{9}{4}f\right)\cos^4\alpha_E-\frac{15}{256}f^3\cos^6\alpha_E \\
A_6 &= \frac{5}{768}f^3\cos^6\alpha_E
\end{aligned}
\tag{130}
$$

The approximation (129) and the coefficients (130) are the same as Rainsford (1955, equations 10 and 11, p. 14) and also Rapp (1981, equation 56, p. 13).

Equation (129) can be used in two ways which will be discussed in detail later. Briefly, however, the first way is in the direct problem – after σ (and σ_m from $2\sigma_m = 2\sigma_1 + \sigma$) has been solved iteratively – to compute the difference $\Delta\omega - \Delta\lambda$. And in the inverse problem to compute the longitude difference iteratively.

VINCENTY'S MODIFICATIONS OF RAINSFORD'S EQUATIONS

In 1975, T. Vincenty (1975) produced other forms of equations (91) and (129) more suited to computer evaluation and requiring a minimum of trigonometric function evaluations. These equations may be obtained in the following manner.

Vincenty's modification of Rainsford's equation for distance

The starting point here is equation (91) [Rainsford's equation for distance] that can be re-arranged as

$$
\begin{aligned}
\sigma &= \frac{s}{bB_0}-\frac{B_2}{B_0}\cos 2\sigma_m\sin\sigma-\frac{B_4}{B_0}\cos 4\sigma_m\sin 2\sigma-\frac{B_6}{B_0}\cos 6\sigma_m\sin 3\sigma \\
&\quad -\frac{B_8}{B_0}\cos 8\sigma_m\sin 4\sigma
\end{aligned}
\tag{131}
$$

or

$$
\sigma = \frac{s}{bB_0}+\Delta\sigma \tag{132}
$$

where

$$\Delta\sigma = -\frac{B_2}{B_0}\cos 2\sigma_m \sin\sigma - \frac{B_4}{B_0}\cos 4\sigma_m \sin 2\sigma - \frac{B_6}{B_0}\cos 6\sigma_m \sin 3\sigma$$
$$-\frac{B_8}{B_0}\cos 8\sigma_m \sin 4\sigma \tag{133}$$

Now, from equations (92) $B_0 = 1 + \frac{1}{4}u^2 - \frac{3}{64}u^4 + \frac{5}{256}u^6 - \frac{175}{16384}u^8 = 1 + x$ and $\frac{1}{B_0} = (1+x)^{-1}$. Using a special case of the binomial series [equation (72) with $\beta = -1$ and with $|x| < 1$]

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots$$

allows us to write

$$\begin{aligned}\frac{1}{B_0} &= 1 - \left(\frac{1}{4}u^2 - \frac{3}{64}u^4 + \cdots\right) + \left(\frac{1}{4}u^2 - \frac{3}{64}u^4 + \cdots\right)^2 - \left(\frac{1}{4}u^2 - \frac{3}{64}u^4 + \cdots\right)^3 \\ &\quad + \left(\frac{1}{4}u^2 - \frac{3}{64}u^4 + \cdots\right)^4 - \cdots \\ &= 1 - \frac{1}{4}u^2 + \frac{7}{64}u^4 - \frac{15}{256}u^6 + \frac{579}{16384}u^8 - \cdots\end{aligned}$$

and using this result gives

$$\begin{aligned}\frac{B_2}{B_0} &= \left(-\frac{1}{4}u^2 + \frac{1}{16}u^4 - \frac{15}{512}u^6 + \frac{35}{2048}u^8 - \cdots\right)\left(1 - \frac{1}{4}u^2 + \frac{7}{64}u^4 - \frac{15}{256}u^6 + \frac{579}{16384}u^8 - \cdots\right) \\ &= -\frac{1}{4}u^2 + \frac{1}{8}u^4 - \frac{37}{512}u^6 + \frac{47}{1024}u^8 - \cdots\end{aligned}$$

Similarly, the other ratios are obtained and

$$\begin{aligned}\frac{B_2}{B_0} &= -\frac{1}{4}u^2 + \frac{1}{8}u^4 - \frac{37}{512}u^6 + \frac{47}{1024}u^8 - \cdots \\ \frac{B_4}{B_0} &= -\frac{1}{128}u^4 + \frac{1}{128}u^6 - \frac{27}{4096}u^8 + \cdots \\ \frac{B_6}{B_0} &= -\frac{1}{1536}u^6 + \frac{1}{1024}u^8 - \cdots \\ \frac{B_8}{B_0} &= -\frac{5}{65536}u^8 + \cdots\end{aligned} \tag{134}$$

For a geodesic on the GRS80 ellipsoid, having $\alpha_E = 0^\circ$ (which makes u^2 a maximum) and with $\sigma = 22.5^\circ$, $\sigma_m = 22.5^\circ$ (which makes $\cos 8\sigma_m \sin 4\sigma = 1$) the maximum value of the last term in equations (131) and (133) is $\frac{B_8}{B_0}\cos 8\sigma_m \sin 4\sigma = 1.5739827\text{e-}013$ radians.

This is equivalent to an arc length of 0.000001 m on a sphere of radius 6378137 m. This term will be ignored and $\Delta\sigma$ is defined as

$$\Delta\sigma \equiv -\frac{B_2}{B_0}\cos 2\sigma_m \sin\sigma - \frac{B_4}{B_0}\cos 4\sigma_m \sin 2\sigma - \frac{B_6}{B_0}\cos 6\sigma_m \sin 3\sigma \tag{135}$$

Now, using the trigonometric identities

$$\begin{aligned} \sin 2A &= 2\sin A\cos A & \cos 2A &= 2\cos^2 A - 1 \\ \sin 3A &= 3\sin A - 4\sin^3 A & \cos 3A &= 4\cos^3 A - 3\cos A \end{aligned}$$

then

$$\begin{aligned} \cos 4A &= 2\cos^2 2A - 1 \\ \cos 6A &= 4\cos^3 2A - 3\cos 2A \end{aligned}$$

and using these identities in equation (135) gives

$$\begin{aligned} \Delta\sigma = &-\frac{B_2}{B_0}\cos 2\sigma_m \sin\sigma - \frac{B_4}{B_0}\left(2\cos^2 2\sigma_m - 1\right)(2\sin\sigma\cos\sigma) \\ &-\frac{B_6}{B_0}\left(4\cos^3 2\sigma_m - 3\cos 2\sigma_m\right)\left(3\sin\sigma - 4\sin^3\sigma\right) \end{aligned}$$

which may be written as

$$\begin{aligned} \Delta\sigma = \sin\sigma \Bigg\{ &-\frac{B_2}{B_0}\cos 2\sigma_m - 2\frac{B_4}{B_0}\cos\sigma\left(2\cos^2 2\sigma_m - 1\right) \\ &-\frac{B_6}{B_0}\cos 2\sigma_m\left(3 - 4\sin^2\sigma\right)\left(4\cos^2 2\sigma_m - 3\right)\Bigg\} \end{aligned} \tag{136}$$

Now

$$\begin{aligned} \left(\frac{-B_2}{B_0}\right)^2 &= \frac{1}{16}u^4 - \frac{1}{16}u^6 + \frac{53}{1024}u^8 - \cdots \\ \left(\frac{-B_2}{B_0}\right)^3 &= \qquad\qquad \frac{1}{64}u^6 - \frac{3}{128}u^8 + \cdots \end{aligned} \tag{137}$$

Comparing equations (137) with equations (134) we have

$$\begin{aligned} -2\left(\frac{B_4}{B_0}\right) &= \frac{1}{64}u^4 - \frac{1}{64}u^6 + \frac{54}{4096}u^8 \\ \frac{1}{4}\left(\frac{-B_2}{B_0}\right)^2 &= \frac{1}{64}u^4 - \frac{1}{64}u^6 + \frac{53}{4096}u^8 \end{aligned}$$

and these two equations differ by $\frac{1}{4096}u^8$ which would be equivalent to a maximum error of 5.0367e-013 radians or 0.000003 m on a sphere of radius 6378137 m. Ignoring this small difference, we define

$$-2\left(\frac{B_4}{B_0}\right) \equiv \frac{1}{4}\left(\frac{-B_2}{B_0}\right)^2 \tag{138}$$

Again, comparing equations (137) with equations (134) we have

$$\begin{aligned} -\left(\frac{B_6}{B_0}\right) &= \frac{1}{1536}u^6 + \frac{1}{1024}u^8 \\ \frac{1}{24}\left(\frac{-B_2}{B_0}\right)^2 &= \frac{1}{1536}u^6 + \frac{3}{3072}u^8 \end{aligned}$$

and noting that $\frac{1}{1024}u^8 = \frac{3}{3072}u^8$ we may say

$$-\left(\frac{B_6}{B_0}\right) = \frac{1}{24}\left(\frac{-B_2}{B_0}\right)^3 \tag{139}$$

Using equations (138) and (139) we may write equation (136) as

$$\begin{aligned} \Delta\sigma = \sin\sigma \Bigg\{ & \left(\frac{-B_2}{B_0}\right)\cos 2\sigma_m + \frac{1}{4}\left(\frac{-B_2}{B_0}\right)^2 \cos\sigma\left(2\cos^2 2\sigma_m - 1\right) \\ & + \frac{1}{24}\left(\frac{-B_2}{B_0}\right)^3 \cos 2\sigma_m\left(3 - 4\sin^2\sigma\right)\left(4\cos^2 2\sigma_m - 3\right)\Bigg\} \end{aligned}$$

We may now express the great circle arc length σ as

$$\boxed{\sigma = \frac{s}{bA'} + \Delta\sigma} \tag{140}$$

where

$$\boxed{\begin{aligned} \Delta\sigma = B'\sin\sigma\Bigg\{ & \cos 2\sigma_m + \frac{1}{4}B'\Big[\cos\sigma\left(2\cos^2 2\sigma_m - 1\right) \\ & - \frac{1}{6}B'\cos 2\sigma_m\left(-3 + 4\sin^2\sigma\right)\left(-3 + 4\cos^2 2\sigma_m\right)\Big]\Bigg\} \end{aligned}} \tag{141}$$

and

$$
\begin{aligned}
A' = B_0 &= 1+\frac{1}{4}u^2-\frac{3}{64}u^4+\frac{5}{256}u^6-\frac{175}{16384}u^8 \\
&= 1+\frac{4096}{16384}u^2-\frac{768}{16384}u^4+\frac{320}{16384}u^6-\frac{175}{16384}u^8 \\
&= 1+\frac{u^2}{16384}\left(4096+u^2\left(-768+u^2\left(320-175u^2\right)\right)\right)
\end{aligned} \tag{142}
$$

$$
\begin{aligned}
B' = \frac{-B_2}{B_0} &= \frac{1}{4}u^2-\frac{1}{8}u^4+\frac{37}{512}u^6-\frac{47}{1024}u^8 \\
&= \frac{256}{1024}u^2-\frac{128}{1024}u^4+\frac{74}{1024}u^6-\frac{47}{1024}u^8 \\
&= \frac{u^2}{1024}\left(256+u^2\left(-128+u^2\left(74-47u^2\right)\right)\right)
\end{aligned} \tag{143}
$$

Equations (140) to (143) are the same as those given by Vincenty (1975, equations 7, 6, 3 and 4, p. 89). Vincenty notes in his paper that these equations were derived from Rainsford's inverse formula and that most significant terms in u^8 were retained, but he gave no outline of his method.

Vincenty's modification of Rainsford's equation for longitude difference

The starting point here is equation (129) [Rainsford's equation for longitude differences] with coefficients A_0, A_2, A_4 and A_6. Referring to this equation, Rainsford (1955, p. 14) states:

"The A coefficients are given as functions of f since they converge more rapidly than when given as functions of e^2. The maximum value of any term in f^4 (i.e. f^3 in the $A's$) is less than $0''.00001$ even for a line half round the world. Thus the A_6 term may be omitted altogether and the following simplified forms used even for precise results:"

Rainsford's simplified formula is

$$
\Delta\omega - \Delta\lambda = f\sin\alpha_E\left\{A_0'\sigma + A_2'\cos 2\sigma_m \sin\sigma + A_4'\cos 4\sigma_m \sin 2\sigma\right\} \tag{144}
$$

where $\Delta\omega = \omega_2 - \omega_1$ is the difference in longitudes of P_1' and P_2' on the auxiliary sphere and $\Delta\lambda = \lambda_2 - \lambda_1$ is the difference in longitudes of P_1 and P_2 on the ellipsoid, and the coefficients are

$$
\begin{aligned}
A_0' &= 1 - \frac{1}{4} f(1+f)\cos^2\alpha_E - \frac{3}{16} f^2 \cos^4\alpha_E \\
A_2' &= \frac{1}{4} f(1+f)\cos^2\alpha_E - \frac{1}{4} f^2 \cos^4\alpha_E \\
A_4' &= \frac{1}{32} f^2 \cos^4\alpha_E
\end{aligned}
\tag{145}
$$

Equation (144) can be written as

$$
\Delta\omega - \Delta\lambda = A_0' f \sin\alpha_E \left\{ \sigma + \frac{A_2'}{A_0'} \cos 2\sigma_m \sin\sigma + \frac{A_4'}{A_0'} \cos 4\sigma_m \sin 2\sigma \right\} \tag{146}
$$

Using the trigonometric double angle formulas $\sin 2A = 2\sin A\cos A$, $\cos 2A = 2\cos^2 A - 1$ we can write

$$
\begin{aligned}
\sin 2\sigma &= 2\sin\sigma\cos\sigma \\
\cos 4\sigma_m &= 2\cos^2 2\sigma_m - 1
\end{aligned}
$$

and equation (146) becomes

$$
\begin{aligned}
\Delta\omega - \Delta\lambda &= A_0' f \sin\alpha_E \left\{ \sigma + \frac{A_2'}{A_0'} \cos 2\sigma_m \sin\sigma + \frac{A_4'}{A_0'} \left(2\cos^2 2\sigma_m - 1\right)\left(2\sin\sigma\cos\sigma\right) \right\} \\
&= A_0' f \sin\alpha_E \left\{ \sigma + \sin\sigma \left[\frac{A_2'}{A_0'} \cos 2\sigma_m + 2\frac{A_4'}{A_0'} \cos\sigma \left(2\cos^2 2\sigma_m - 1\right) \right] \right\}
\end{aligned}
\tag{147}
$$

Now the coefficient A_0' may be re-arranged as follows

$$
\begin{aligned}
A_0' &= 1 - \frac{1}{4} f(1+f)\cos^2\alpha_E + \frac{3}{16} f^2 \cos^4\alpha_E \\
&= 1 - \left(\frac{4}{16} f(1+f)\cos^2\alpha_E - \frac{3}{16} f^2 \cos^4\alpha_E \right) \\
&= 1 - \frac{f}{16} \cos^2\alpha_E \left(4(1+f) - 3f\cos^2\alpha_E \right) \\
&= 1 - \frac{f}{16} \cos^2\alpha_E \left(4 + f\left(4 - 3\cos^2\alpha_E\right) \right)
\end{aligned}
$$

or

$$
A_0' = 1 - C
$$

where

$$
C = \frac{f}{16} \cos^2\alpha_E \left(4 + f\left(4 - 3\cos^2\alpha_E\right) \right)
$$

Now using these relationships and a special result of the binomial series [equation (72) with $x = -C$ and $\beta = -1$] we may write

$$\frac{1}{A_0'} = \frac{1}{1-C} = (1-C)^{-1} = 1 + C + C^2 + C^3 + \cdots$$

and

$$\frac{A_2'}{A_0'} = \frac{1}{4} f \cos^2 \alpha_E + \frac{1}{4} f^2 \cos^2 \alpha_E - \frac{3}{16} f^2 \cos^4 \alpha_E + \frac{1}{8} f^3 \cos^4 \alpha_E + \cdots$$

Ignoring terms greater than f^3 (greater than f^2 in $\frac{A_2'}{A_0'}$) we have

$$\begin{aligned} \frac{A_2'}{A_0'} &\equiv \frac{1}{4} f \cos^2 \alpha_E + \frac{1}{4} f^2 \cos^2 \alpha_E - \frac{3}{16} f^2 \cos^4 \alpha_E \\ &= \frac{f}{16} \cos^2 \alpha_E \left(4 + f\left(4 - 3\cos^2 \alpha_E\right)\right) \\ &= C \end{aligned}$$

Also

$$\frac{A_4'}{A_0'} = \frac{1}{32} f^2 \cos^4 \alpha_E + \frac{1}{128} f^3 \cos^6 \alpha_E + \cdots$$

and ignoring terms greater than f^3 (greater than f^2 in $\frac{A_4'}{A_0'}$) we have

$$\frac{A_4'}{A_0'} \equiv \frac{1}{32} f^2 \cos^4 \alpha_E \quad \text{and} \quad 2\frac{A_4'}{A_0'} = \frac{1}{16} f^2 \cos^4 \alpha_E$$

Now

$$C^2 = \frac{1}{16} f^2 \cos^4 \alpha_E + \frac{1}{8} f^3 \cos^4 \alpha_E - \frac{3}{32} f^3 \cos^6 \alpha_E + \cdots$$

and ignoring terms greater than f^3 (greater than f^2 in C^2) we have

$$C^2 \equiv \frac{1}{16} f^2 \cos^4 \alpha_E = 2\frac{A_4'}{A_0'}$$

Using these results we may write equation (147) as

$$\boxed{\Delta\lambda = \Delta\omega - (1-C) f \sin \alpha_E \left\{\sigma + C \sin\sigma \left[\cos 2\sigma_m + C \cos\sigma \left(-1 + 2\cos^2 2\sigma_m\right)\right]\right\} \qquad (148)}$$

where $\Delta\omega = \omega_2 - \omega_1$ is the difference in longitudes of P_1' and P_2' on the auxiliary sphere and $\Delta\lambda = \lambda_2 - \lambda_1$ is the difference in longitudes of P_1 and P_2 on the ellipsoid, and

$$C = \frac{f}{16} \cos^2 \alpha_E \left(4 + f\left(4 - 3\cos^2 \alpha_E\right)\right) \tag{149}$$

Equations (148) and (149) are essentially the same as Vincenty (1975, equations 11 and 10, p.89) – Vincenty uses L and λ where we have used $\Delta\lambda$ and $\Delta\omega$ respectively – although he gives no outline of his method of deriving his equations from Rainsford's.

SOLVING THE DIRECT AND INVERSE PROBLEMS ON THE ELLIPSOID USING VINCENTY'S EQUATIONS

Vincenty (1975) set out methods of solving the direct and inverse problems on the ellipsoid. His methods were different from those proposed by Rainsford (1955) even though his equations (140) to (143) for spherical arc length σ and (148) and (149) for longitude λ were simplifications of Rainsford's equations. His approach was to develop solutions more applicable to computer programming rather than the mechanical methods used by Rainsford. Vincenty's method relies upon the auxiliary sphere and there are several equations using spherical trigonometry. Since distances are often small when compared with the Earth's circumference, resulting spherical triangles can have very small sides and angles. In such cases, usual spherical trigonometry formula, e.g., sine rule and cosine rule, may not furnish accurate results and other, less common formula, are used. Vincenty's equations and his methods are now widely used in geodetic computations.

In the solutions of the direct and inverse problems set out in subsequent sections, the following notation and relationships are used.

- a, f semi-major axis length and flattening of ellipsoid.
- b semi-minor axis length of the ellipsoid, $b = a(1-f)$
- e^2 eccentricity of ellipsoid squared, $e^2 = f(2-f)$
- e'^2 2nd-eccentricity of ellipsoid squared, $e'^2 = \dfrac{e^2}{1-e^2}$
- ϕ, λ latitude and longitude on ellipsoid: ϕ measured 0° to $\pm 90^\circ$ (north latitudes positive and south latitudes negative) and λ measured 0° to $\pm 180^\circ$ (east longitudes positive and west longitudes negative).
- s length of the geodesic on the ellipsoid.
- α_1, α_2 azimuths of the geodesic, clockwise from north 0° to 360°; α_2 in the direction P_1P_2 produced.

α_{12} azimuth of geodesic P_1P_2; $\alpha_{12} = \alpha_1$

α_{21} reverse azimuth; azimuth of geodesic P_2P_1; $\alpha_{21} = \alpha_2 \pm 180^\circ$

α_E azimuth of geodesic at the equator, $\sin\alpha_E = \cos\psi_0$

u^2 $= e'^2 \sin^2\psi_0$

ψ parametric latitude, $\tan\psi = (1-f)\tan\phi$

ψ_0 parametric latitude of geodesic vertex, $\cos\psi_0 = \cos\psi\sin\alpha = \sin\alpha_E$

ψ, ω latitude and longitude on auxiliary sphere: ψ measured 0° to $\pm 90^\circ$ (north latitudes positive and south latitudes negative) and ω measured 0° to $\pm 180^\circ$ (east longitudes positive and west longitudes negative).

$\Delta\lambda, \Delta\omega$ longitude differences; $\Delta\lambda = \lambda_2 - \lambda_1$ (ellipsoid) and $\Delta\omega = \omega_2 - \omega_1$ (spherical)

σ angular distance (great circle arc) $P_1'P_2'$ on the auxiliary sphere.

σ_1 angular distance from equator to P_1' on the auxiliary sphere, $\tan\sigma_1 = \dfrac{\tan\psi_1}{\cos\alpha_1}$

σ_m angular distance from equator to mid-point of great circle arc $P_1'P_2'$ on the auxiliary sphere, $2\sigma_m = 2\sigma_1 + \sigma$

THE DIRECT PROBLEM ON THE ELLIPSOID USING VINCENTY'S EQUATIONS

Using Vincenty's equations the direct problem on the ellipsoid

> [given latitude and longitude of P_1 on the ellipsoid and azimuth α_{12} and geodesic distance s to P_2 on the ellipsoid, compute the latitude and longitude of P_2 and the reverse azimuth α_{21}]

may be solved by the following sequence.

With the ellipsoid constants a, f, $b = a(1-f)$, $e^2 = f(2-f)$ and $e'^2 = \dfrac{e^2}{1-e^2}$ and given $\phi_1, \lambda_1, \alpha_1 = \alpha_{12}$ and s

1. Compute parametric latitude ψ_1 of P_1 from

$$\tan\psi_1 = (1-f)\tan\phi_1$$

2. Compute the parametric latitude of the geodesic vertex ψ_0 from

$$\cos\psi_0 = \cos\psi_1 \sin\alpha_1$$

3. Compute the geodesic constant u^2 from

$$u^2 = e'^2 \sin^2\psi_0$$

4. Compute angular distance σ_1 on the auxiliary sphere from the equator to P_1' from

$$\tan\sigma_1 = \frac{\tan\psi_1}{\cos\alpha_1}$$

5. Compute the azimuth of the geodesic at the equator α_E from

$$\sin\alpha_E = \cos\psi_0 = \cos\psi_1 \sin\alpha_1$$

6. Compute Vincenty's constants A' and B' from

$$\begin{aligned} A' &= 1 + \frac{u^2}{16384}\left(4096 + u^2\left(-768 + u^2\left(320 - 175u^2\right)\right)\right) \\ B' &= \frac{u^2}{1024}\left(256 + u^2\left(-128 + u^2\left(74 - 47u^2\right)\right)\right) \end{aligned}$$

7. Compute angular distance σ on the auxiliary sphere from P_1' to P_2' by iteration using the following sequence of equations until there is negligible change in σ

$$\begin{aligned} 2\sigma_m &= 2\sigma_1 + \sigma \\ \Delta\sigma &= B'\sin\sigma\left\{\cos 2\sigma_m + \frac{1}{4}B'\left[\cos\sigma\left(2\cos^2 2\sigma_m - 1\right)\right.\right. \\ &\quad \left.\left. - \frac{1}{6}B'\cos 2\sigma_m\left(-3 + 4\sin^2\sigma\right)\left(-3 + 4\cos^2 2\sigma_m\right)\right]\right\} \\ \sigma &= \frac{s}{bA'} + \Delta\sigma \end{aligned}$$

The first approximation for σ in this iterative solution can be taken as $\sigma \simeq \dfrac{s}{bA'}$

8. After computing the spherical arc length σ the latitude of P_2 can be computed using spherical trigonometry and the relationship $\tan\phi_2 = \dfrac{\tan\psi_2}{(1-f)}$

$$\tan\phi_2 = \frac{\sin\psi_1\cos\sigma + \cos\psi_1\sin\sigma\cos\alpha_1}{(1-f)\sqrt{\sin^2\alpha_E + \left(\sin\psi_1\sin\sigma - \cos\psi_1\cos\sigma\cos\alpha_1\right)^2}}$$

9. Compute the longitude difference $\Delta\omega$ on the auxiliary sphere from

$$\tan\Delta\omega = \frac{\sin\sigma\sin\alpha_1}{\cos\psi_1\cos\sigma - \sin\psi_1\sin\sigma\cos\alpha_1}$$

10. Compute Vincenty's constant C from

$$C = \frac{f}{16}\cos^2\alpha_E\left(4 + f\left(4 - 3\cos^2\alpha_E\right)\right)$$

11. Compute the longitude difference $\Delta\lambda$ on the ellipsoid from

$$\Delta\lambda = \Delta\omega - (1 - C)\,f\sin\alpha_E\left\{\sigma + C\sin\sigma\left[\cos 2\sigma_m + C\cos\sigma\left(-1 + 2\cos^2 2\sigma_m\right)\right]\right\}$$

12. Compute azimuth α_2 from

$$\tan\alpha_2 = \frac{\sin\alpha_E}{\cos\psi_1\cos\sigma\cos\alpha_1 - \sin\psi_1\sin\sigma}$$

13. Compute reverse azimuth α_{21}

$$\alpha_{21} = \alpha_2 \pm 180^\circ$$

Shown below is the output of a MATLAB function *Vincenty_Direct.m* that solves the direct problem on the ellipsoid.

The ellipsoid is the GRS80 ellipsoid and ϕ,λ for P_1 are -45° and 132° respectively with $\alpha_{12} = 1^\circ\,43'\,25.876544''$ and $s = 3880\,275.684\,153$ m. ϕ,λ computed for P_2 are -10° and 133° respectively with the reverse azimuth $\alpha_{21} = 181^\circ\,14'\,22.613213''$

```
>> Vincenty_Direct

//////////////////////////////////////////////////
// DIRECT CASE on ellipsoid: Vincenty's method //
//////////////////////////////////////////////////

ellipsoid parameters
a    =  6378137.000000000
f    = 1/298.257222101000
b    =  6356752.314140356100
e2   =  6.694380022901e-003
ep2  =  6.739496775479e-003

Latitude & Longitude of P1
latP1 = -45  0  0.000000 (D M S)
lonP1 = 132  0  0.000000 (D M S)

Azimuth & Distance P1-P2
az12 =    1 43 25.876544 (D M S)
s    =    3880275.684153

Parametric Latitude of P1
psiP1 = -44 54 13.636256 (D M S)

Parametric Latitude of vertex P0
psiP0 =  88 46 44.750547 (D M S)

Geodesic constant u2 (u-squared)
u2 =  6.736437077728e-003
```

```
angular distance on auxiliary sphere from equator to P1'
sigma1 = -7.839452835875e-001 radians

Vincenty's constants A and B
A =  1.001681988050e+000
B =  1.678458818215e-003

angular distance sigma on auxiliary sphere from P1' to P2'
sigma =  6.099458753810e-001 radians
iterations =  5

Latitude of P2
latP2 = -10  0  0.000000 (D M S)

Vincenty's constant C
C =  8.385253517062e-004

Longitude difference P1-P2
dlon =   1  0  0.000000 (D M S)

Longitude of P2
lon2 = 133  0  0.000000 (D M S)

Reverse azimuth
alpha21 = 181 14 22.613213 (D M S)

>>
```

THE INVERSE PROBLEM ON THE ELLIPSOID USING VINCENTY'S EQUATIONS

Using Vincenty's equations the inverse problem on the ellipsoid

> [given latitudes and longitudes of P_1 and P_2 on the ellipsoid compute the forward and reverse azimuths α_{12} and α_{21} and the geodesic distance s]

may be solved by the following sequence.

With the ellipsoid constants a, f, $b = a(1-f)$, $e^2 = f(2-f)$ and $e'^2 = \dfrac{e^2}{1-e^2}$ and given ϕ_1, λ_1 and ϕ_2, λ_2

1. Compute parametric latitudes ψ_1 and ψ_2 of P_1 and P_2 from

$$\tan\psi = (1-f)\tan\phi$$

2. Compute the longitude difference $\Delta\lambda$ on the ellipsoid

$$\Delta\lambda = \lambda_2 - \lambda_1$$

3. Compute the longitude difference $\Delta\omega$ on the auxiliary sphere between P_1' to P_2' by iteration using the following sequence of equations until there is negligible change in $\Delta\omega$. Note that σ should be computed using the atan2 function after evaluating $\sin\sigma = \sqrt{\sin^2\sigma}$ and $\cos\sigma$. This will give $-180^\circ < \sigma \le 180^\circ$.

$$\sin^2\sigma = \left(\cos\psi_2 \sin\Delta\omega\right)^2 + \left(\cos\psi_1 \sin\psi_2 - \sin\psi_1 \cos\psi_2 \cos\Delta\omega\right)^2$$

$$\cos\sigma = \sin\psi_1 \sin\psi_2 + \cos\psi_1 \cos\psi_2 \cos\Delta\omega$$

$$\tan\sigma = \frac{\sin\sigma}{\cos\sigma}$$

$$\sin\alpha_E = \frac{\cos\psi_1 \cos\psi_2 \sin\Delta\omega}{\sin\sigma}$$

$$\cos 2\sigma_m = \cos\sigma - \frac{2\sin\psi_1 \sin\psi_2}{\cos^2\alpha_E}$$

$$C = \frac{f}{16}\cos^2\alpha_E\left(4 + f\left(4 - 3\cos^2\alpha_E\right)\right)$$

$$\Delta\omega = \Delta\lambda + (1 - C) f \sin\alpha_E \left\{\sigma + C\sin\sigma\left[\cos 2\sigma_m + C\cos\sigma\left(-1 + 2\cos^2 2\sigma_m\right)\right]\right\}$$

The first approximation for $\Delta\omega$ in this iterative solution can be taken as $\Delta\omega \simeq \Delta\lambda$

4. Compute the parametric latitude of the geodesic vertex ψ_0 from

$$\cos\psi_0 = \sin\alpha_E$$

5. Compute the geodesic constant u^2 from

$$u^2 = e'^2 \sin^2\psi_0$$

6. Compute Vincenty's constants A' and B' from

$$A' = 1 + \frac{u^2}{16384}\left(4096 + u^2\left(-768 + u^2\left(320 - 175u^2\right)\right)\right)$$

$$B' = \frac{u^2}{1024}\left(256 + u^2\left(-128 + u^2\left(74 - 47u^2\right)\right)\right)$$

7. Compute geodesic distance s from

$$\begin{aligned}\Delta\sigma &= B'\sin\sigma\left\{\cos 2\sigma_m + \frac{1}{4}B'\left[\cos\sigma\left(2\cos^2 2\sigma_m - 1\right)\right.\right.\\ &\quad \left.\left. - \frac{1}{6}B'\cos 2\sigma_m\left(-3 + 4\sin^2\sigma\right)\left(-3 + 4\cos^2 2\sigma_m\right)\right]\right\}\\ s &= bA(\sigma - \Delta\sigma)\end{aligned}$$

8. Compute the forward azimuth $\alpha_{12} = \alpha_1$ from

$$\tan\alpha_1 = \frac{\cos\psi_2 \sin\Delta\omega}{\cos\psi_1 \sin\psi_2 - \sin\psi_1 \cos\psi_2 \cos\Delta\omega}$$

9. Compute azimuth α_2 from

$$\tan\alpha_2 = \frac{\cos\psi_1 \sin\Delta\omega}{-\sin\psi_1 \cos\psi_2 + \cos\psi_1 \sin\psi_2 \cos\Delta\omega}$$

10. Compute reverse azimuth α_{21}

$$\alpha_{21} = \alpha_2 \pm 180^\circ$$

Shown below is the output of a MATLAB function *Vincenty_Inverse.m* that solves the inverse problem on the ellipsoid.

The ellipsoid is the GRS80 ellipsoid. ϕ,λ for P_1 are -10° and 110° respectively and ϕ,λ for P_2 are -45° and 155° respectively. <u>Computed</u> azimuths are $\alpha_{12} = 140^\circ\, 30'\, 03.017703''$ and $\alpha_{21} = 297^\circ\, 48'\, 47.310738''$, and geodesic distance $s = 5783\,228.548429$ m.

```
>> Vincenty_Inverse

////////////////////////////////////////////////////
// INVERSE CASE on ellipsoid: Vincenty's method //
////////////////////////////////////////////////////

ellipsoid parameters
a    =  6378137.000000000
f    = 1/298.257222101000
b    =  6356752.314140356100
e2   =  6.694380022901e-003
ep2  =  6.739496775479e-003

Latitude & Longitude of P1
latP1 = -10  0  0.000000 (D M S)
lonP1 = 110  0  0.000000 (D M S)

Latitude & Longitude of P2
latP2 = -45  0  0.000000 (D M S)
lonP2 = 155  0  0.000000 (D M S)

Parametric Latitudes of P1 and P2
psiP1 =  -9 58  1.723159 (D M S)
psiP2 = -44 54 13.636256 (D M S)

Longitude difference on ellipsoid P1-P2
dlon =  45  0  0.000000 (D M S)

Longitude difference on auxiliary sphere P1'-P2'
domega =  9.090186019005e-001 radians
iterations =  5

Parametric Latitude of vertex P0
psiP0 =  51 12 36.239192 (D M S)

Geodesic constant u2 (u-squared)
u2 =  4.094508823114e-003

Vincenty's constants A and B
A =  1.001022842684e+000
B =  1.021536528199e-003
```

```
Azimuth & Distance P1-P2
az12 =  140 30  3.017703 (D M S)
s    =    5783228.548429

Reverse azimuth
alpha21 = 297 48 47.310738 (D M S)

>>
```

EXCEL WORKBOOK vincenty.xls FROM GEOSCIENCE AUSTRALIA

Geoscience Australia has made available an Excel workbook vincenty.xls containing four spreadsheets labelled Ellipsoids, Direct Solution, Inverse Solution and Test Data. The Direct Solution and Inverse Solution spreadsheets are implementations of Vincenty's equations. The Excel workbook vincenty.xls can be downloaded via the Internet at the Geoscience Australia website (http://www.ga.gov.au/) following the links to Geodetic Calculations then Calculate Bearing Distance from Latitude Longitude. At this web page the spreadsheet vincenty.xls is available for use or downloading. Alternatively, the Intergovernmental Committee on Surveying and Mapping (ICSM) has produced an on-line publication *Geocentric Datum of Australia Technical Manual* Version 2.2 (GDA Technical Manual, ICSM 2002) with a link to vincenty.xls.

The operation of vincenty.xls is relatively simple, but since the spreadsheets use the Excel *solver* for the iterative solutions of certain equations then the Iteration box must be checked on the Calculation sheet. The Calculation sheet is found under Tools/Options on the Excel toolbar. Also, on the Calculation sheet make sure the Maximum change box has a value of 0.000000000001.

The Direct Solution and Inverse Solution spreadsheets have statements that the spreadsheets have been tested in the Australian region but not exhaustively tested worldwide.

To test vincenty.xls, direct and inverse solutions between points on a geographic rectangle $ABCD$ covering Australia were computed using vincenty.xls and MATLAB functions *Vincenty_Direct.m* and *Vincenty_Inverse.m.* Figure 16 shows the geographic rectangle $ABCD$ whose sides are the meridians of longitude 110° and 155° and parallels of latitude -10° and -45°. Several lines were chosen on and across this rectangle.

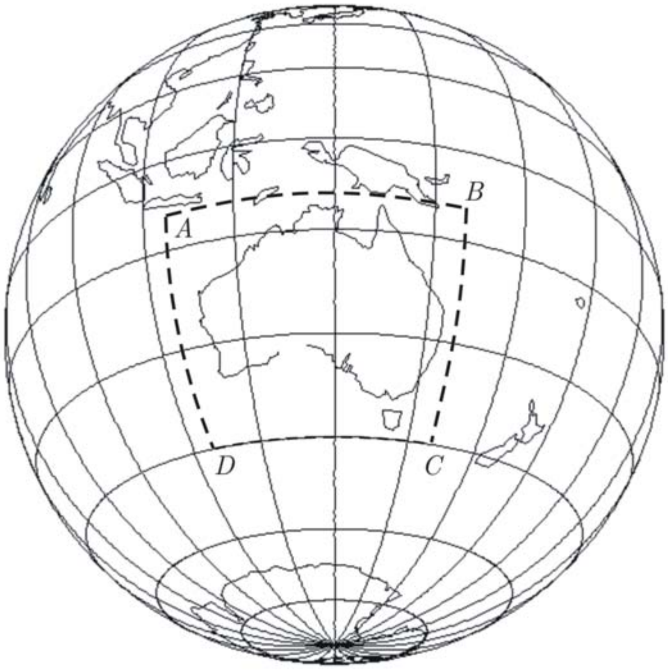


Figure 16: Geographic rectangle covering Australia

P_1	P_2	azimuth α	distance s
$\phi = -10^\circ$ $\lambda = 110^\circ$	$\phi = -10^\circ$ $\lambda = 155^\circ$	$\alpha_{12} = 94^\circ\ 06'\ 55.752182''$ $\alpha_{21} = 265^\circ\ 53'\ 04.247818''$	$s = 4\,929\,703.675\,416$ m
$\phi = -10^\circ$ $\lambda = 110^\circ$	$\phi = -45^\circ$ $\lambda = 155^\circ$	$\alpha_{12} = 140^\circ\ 30'\ 03.017\,703''$ $\alpha_{21} = 297^\circ\ 48'\ 47.310\,738''$	$s = 5\,783\,228.548\,429$ m
$\phi = -10^\circ$ $\lambda = 110^\circ$	$\phi = -45^\circ$ $\lambda = 110^\circ$	$\alpha_{12} = 180^\circ\ 00'\ 00.000\,000''$ $\alpha_{21} = 0^\circ\ 00'\ 00.000\,000''$	$s = 3\,879\,089.544\,659$ m
$\phi = -10^\circ$ $\lambda = 155^\circ$	$\phi = -45^\circ$ $\lambda = 110^\circ$	$\alpha_{12} = 219^\circ\ 29'\ 56.982\,297''$ $\alpha_{21} = 62^\circ\ 11'\ 12.689\,262''$	$s = 5\,783\,228.548\,429$ m
$\phi = -45^\circ$ $\lambda = 132^\circ$	$\phi = -10^\circ$ $\lambda = 133^\circ$	$\alpha_{12} = 1^\circ\ 43'\ 25.876\,544''$ $\alpha_{21} = 181^\circ\ 14'\ 22.613\,213''$	$s = 3\,880\,275.684\,153$ m
$\phi = -35^\circ$ $\lambda = 110^\circ$	$\phi = -36^\circ$ $\lambda = 155^\circ$	$\alpha_{12} = 105^\circ\ 00'\ 10.107\,712''$ $\alpha_{21} = 257^\circ\ 56'\ 53.869\,209''$	$s = 4\,047\,421.887\,193$ m

Table 1: Geodesic curves between P_1 and P_2 on the GRS80 ellipsoid

Table 1 shows a number of long geodesics that are either bounding meridians of the rectangle or geodesics crossing the rectangle. All of these results have been computed using the MATLAB function *Vincenty_Inverse.m* and verified by using the MATLAB function *Vincenty_Direct.m.* Each of the lines were then computed using the Inverse Solution spreadsheet of the Excel workbook **vincenty.xls**; all azimuths were identical and the differences between distances were 0.000002 m on one line and 0.000001 m on two other lines. Each of the lines were then verified by using the Direct Solution spreadsheet (all computed latitudes and longitudes we in exact agreement). It could be concluded that the Excel workbook **vincenty.xls** gives results accurate to at least the 5th decimal of distance and the 6th decimal of seconds of azimuth for any geodesic in Australia.

Vincenty (1975) verifies his equations by comparing his results with Rainsford's over five test lines (Rainsford 1955). On one of these lines – line (a) $\phi_1 = 55^\circ\, 45'$, $\lambda_1 = 0^\circ\, 00'$, $\alpha_{12} = 96^\circ\, 36'\, 08.79960''$, $s = 14\,110\,526.170$ m on Bessel's ellipsoid $a = 6377397.155$ m $1/f = 299.1528128$ – Vincenty finds his direct solution gives $\phi_2 = -33^\circ\, 26'\, 00.000012''$, $\lambda_2 = 108^\circ\, 13'\, 00.000\,007''$ and $\alpha_{21} = 137^\circ\, 52'\, 22.014\,528''$. We can confirm that the MATLAB function *Vincenty_Direct.m* also gives these results, <u>but it is interesting to note that the Direct Solution spreadsheet of the Excel workbook **vincenty.xls** does not give these results</u>. This is due to the Excel *solver* – used to determine a value by iteration – returning an incorrect value. Whilst the error in the Excel *solver* result is small, it is, nonetheless, significant and users should be aware of the likelihood or erroneous results over very long geodesics using **vincenty.xls**.

MATLAB FUNCTIONS

Shown below are two MATLAB functions *Vincenty_Direct.m* and *Vincenty_Inverse.m* that have been written to test Vincenty's equations and his direct and inverse methods of solution. Both functions call another function *DMS.m* that is also shown.

MATLAB function *Vincenty_ Direct.m*

```
function Vincenty_Direct
%  Vincenty_Direct computes the "direct case" on the ellipsoid using
%  Vinventy's method.
%  Given the size and shape of the ellipsoid and the latitude and
%  longitude of P1 and the azimuth and geodesic distance of P1 to P2,
%  this function computes the latitude and longitude of P2 and the
%  reverse azimuth P2 to P1.

%============================================================================
% Function:  Vincenty_Direct
%
% Useage:    Vincenty_Direct;
%
% Author:
%  Rod Deakin,
%  Department of Mathematical and Geospatial Sciences,
%  RMIT University,
%  GPO Box 2476V, MELBOURNE VIC 3001
%  AUSTRALIA
%  email: rod.deakin@rmit.edu.au
%
% Date:
%  Version 1.0   2 March 2008
%
% Functions Required:
%      [D,M,S] = DMS(DecDeg)
%
% Remarks:
%  This function computes the DIRECT CASE on the ellipsoid.  Given the size
%  and shape of an ellipsoid (defined by parameters a and f, semi-major
%  axis and flattening respectively) and the latitude and longitude of P1
%  and the azimuth (az12) P1 to P2 and the geodesic distance (s) P1 to P2,
%  the function computes the latitude and longitude of P2 and the reverse
%  azimuth (az21) P2 to P1.  Latitudes and longitudes of the geodesic
%  vertices P0 and P0' are also output as well as distances and longitude
%  difference from P1 and P2 to the relevant vertices.
%
% References:
%  [1] Deakin, R.E, and Hunter, M.N., 2007. 'Geodesics on an Ellipsoid -
%         Bessels' Method', School of Mathematical and Geospatial Sciences,
%         RMIT University, January 2007.
%  [2] Vincenty, T., 1975. 'Direct and Inverse solutions of geodesics on
%         the ellipsoid with application of nested equations', Survey
%         Review, Vol. 23, No. 176, pp.88-93, April 1975.
%
% Variables:
%  a            - semi-major axis of ellipsoid
%  A            - Vincenty's constant for computation of sigma
%  alpha1       - azimuth P1-P2 (radians)
%  az12         - azimuth P1-P2 (degrees)
%  az21         - azimuth P2-P1 (degrees)
%  b            - semi-minor axis of ellipsoid
%  A            - Vincenty's constant for computation of sigma
%  cos_alpha1   - cosine of azimuth of geodesic P1-P2 at P1
%  dlambda      - longitude difference P1 to P2 (radians)
%  domega       - longitude difference P1' to P2' (radians)
%  d2r          - degree to radian conversion factor
%  e2           - eccentricity of ellipsoid squared
%  ep2          - 2nd eccentricity squared
%  f            - flattening of ellipsoid
%  flat         - denominator of flattening, f = 1/flat
%  lambda1      - longitude of P1 (radians)
%  lambda2      - longitude of P2 (radians)
%  lat1         - latitude of P1 (degrees)
```

```
%  lat2         - latitude of P2 (degrees)
%  lon1         - longitude of P1 (degrees)
%  lon2         - longitude of P2 (degrees)
%  phi1         - latitude of P1 (radians)
%  phi2         - latitude of P2 (radians)
%  pion2        - pi/2
%  psi0         - parametric latitude of P0 (radians)
%  psi1         - parametric latitude of P1 (radians)
%  psi2         - parametric latitude of P2 (radians)
%  s            - geodesic distance P1 to P2
%  sigma1       - angular distance (radians) on auxiliary sphere from
%                 equator to P1'
%  sin_alpha1   - sine of azimuth of geodesic P1-P2 at P1
%  twopi        - 2*pi
%  u2           - geodesic constant u-squared
%
%
%============================================================================

% Define some constants
d2r   = 180/pi;
twopi = 2*pi;
pion2 = pi/2;

% Set defining ellipsoid parameters
a    = 6378137;          % GRS80
flat = 298.257222101;
% a    = 6377397.155;        % Bessel (see Ref [2], p.91)
% flat = 299.1528128;

% Compute derived ellipsoid constants
f   = 1/flat;
b   = a*(1-f);
e2  = f*(2-f);
ep2 = e2/(1-e2);

%---------------------------------------
% latitude and longitude of P1 (degrees)
%---------------------------------------
 lat1 =  -45;
 lon1 =  132;

% lat and lon of P1 (radians)
phi1    = lat1/d2r;
lambda1 = lon1/d2r;

%------------------------------------
% azimuth of geodesic P1-P2 (degrees)
%------------------------------------
az12 = 1 + 43/60 + 25.876544/3600;
%
% azimuth of geodesic P1-P2 (radians)
alpha1 = az12/d2r;

% sine and cosine of azimuth P1-P2
sin_alpha1 = sin(alpha1);
cos_alpha1 = cos(alpha1);

%------------------
% geodesic distance
%------------------
s = 3880275.684153;

% [1] Compute parametric latitude psi1 of P1
psi1 = atan((1-f)*tan(phi1));

% [2] Compute parametric latitude of vertex
psi0 = acos(cos(psi1)*sin_alpha1);
```

```
% [3] Compute geodesic constant u2 (u-squared)
u2 = ep2*(sin(psi0)^2);

% [4] Compute angular distance sigma1 on the auxiliary sphere from equator
%     to P1'
sigma1 = atan2(tan(psi1),cos_alpha1);

% [5] Compute the sine of the azimuth of the geodesic at the equator
sin_alphaE = cos(psi0);

% [6] Compute Vincenty's constants A and B
A = 1 + u2/16384*(4096 + u2*(-768 + u2*(320-175*u2)));
B = u2/1024*(256 + u2*(-128 + u2*(74-47*u2)));

% [7] Compute sigma by iteration
sigma = s/(b*A);
iter = 1;
while 1
    two_sigma_m = 2*sigma1 + sigma;
    s1 = sin(sigma);
    s2 = s1*s1;
    c1 = cos(sigma);
    c1_2m = cos(two_sigma_m);
    c2_2m = c1_2m*c1_2m;
    t1 = 2*c2_2m-1;
    t2 = -3+4*s2;
    t3 = -3+4*c2_2m;
    delta_sigma = B*s1*(c1_2m+B/4*(c1*t1-B/6*c1_2m*t2*t3));
    sigma_new = s/(b*A)+delta_sigma;
    if abs(sigma_new-sigma)<1e-12
        break;
    end;
    sigma = sigma_new;
    iter = iter + 1;
end;
s1 = sin(sigma);
c1 = cos(sigma);

% [8] Compute latitude of P2
y = sin(psi1)*c1+cos(psi1)*s1*cos_alpha1;
x = (1-f)*sqrt(sin_alphaE^2+(sin(psi1)*s1-cos(psi1)*c1*cos_alpha1)^2);
phi2 = atan2(y,x);
lat2 = phi2*d2r;

% [9] Compute longitude difference domega on the auxiliary sphere
y = s1*sin_alpha1;
x = cos(psi1)*c1-sin(psi1)*s1*cos_alpha1;
domega = atan2(y,x);

% [10] Compute Vincenty's constant C
x = 1-sin_alphaE^2;
C = f/16*x*(4+f*(4-3*x));

% [11] Compute longitude difference on ellipsoid
two_sigma_m = 2*sigma1 + sigma;
c1_2m = cos(two_sigma_m);
c2_2m = c1_2m*c1_2m;
dlambda = domega-(1-C)*f*sin_alphaE*(sigma+C*s1*(c1_2m+C*c1*(-1+2*c2_2m)));
dlon = dlambda*d2r;
lon2 = lon1+dlon;

% [12] Compute azimuth alpha2
y = sin_alphaE;
x = cos(psi1)*c1*cos_alpha1-sin(psi1)*s1;
alpha2 = atan2(y,x);

% [13] Compute reverse azimuth az21
```

```
az21 = alpha2*d2r + 180;
if az21 > 360
    az21 = az21-360;
end;



%-------------------------------------------------
% Print computed quantities, latitudes and azimuth
%-------------------------------------------------


fprintf('\n///////////////////////////////////////////////////');
fprintf('\n// DIRECT CASE on ellipsoid: Vincenty''s method //');
fprintf('\n///////////////////////////////////////////////////');
fprintf('\n\nellipsoid parameters');
fprintf('\na    = %18.9f',a);
fprintf('\nf    = 1/%16.12f',flat);
fprintf('\nb    = %21.12f',b);
fprintf('\ne2   = %20.12e',e2);
fprintf('\nep2  = %20.12e',ep2);

fprintf('\n\nLatitude & Longitude of P1');
[D,M,S] = DMS(lat1);
if D==0 && lat1<0
    fprintf('\nlatP1 =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nlatP1 = %3d %2d %9.6f (D M S)',D,M,S);
end;
[D,M,S] = DMS(lon1);
if D==0 && lon1<0
    fprintf('\nlonP1 =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nlonP1 = %3d %2d %9.6f (D M S)',D,M,S);
end;
fprintf('\n\nAzimuth & Distance P1-P2');
[D,M,S] = DMS(az12);
fprintf('\naz12 = %4d %2d %9.6f (D M S)',D,M,S);
fprintf('\ns    = %17.6f',s);

fprintf('\n\nParametric Latitude of P1');
[D,M,S] = DMS(psi1*d2r);
if D==0 && psi1<0
    fprintf('\npsiP1 =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\npsiP1 = %3d %2d %9.6f (D M S)',D,M,S);
end;

fprintf('\n\nParametric Latitude of vertex P0');
[D,M,S] = DMS(psi0*d2r);
if D==0 && psi0<0
    fprintf('\npsiP0 =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\npsiP0 = %3d %2d %9.6f (D M S)',D,M,S);
end;

fprintf('\n\nGeodesic constant u2 (u-squared)');
fprintf('\nu2 = %20.12e',u2);

fprintf('\n\nangular distance on auxiliary sphere from equator to P1''');
fprintf('\nsigma1 = %20.12e radians',sigma1);

fprintf('\n\nVincenty''s constants A and B');
fprintf('\nA = %20.12e',A);
fprintf('\nB = %20.12e',B);

fprintf('\n\nangular distance sigma on auxiliary sphere from P1'' to P2''');
fprintf('\nsigma = %20.12e radians',sigma);
```

```
fprintf('\niterations = %2d',iter);

fprintf('\n\nLatitude of P2');
[D,M,S] = DMS(lat2);
if D==0 && lat2<0
    fprintf('\nlatP2 =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nlatP2 = %3d %2d %9.6f (D M S)',D,M,S);
end;

fprintf('\n\nVincenty''s constant C');
fprintf('\nC = %20.12e',C);

fprintf('\n\nLongitude difference P1-P2');
[D,M,S] = DMS(dlon);
if D==0 && dlon<0
    fprintf('\ndlon =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\ndlon = %3d %2d %9.6f (D M S)',D,M,S);
end;

fprintf('\n\nLongitude of P2');
[D,M,S] = DMS(lon2);
if D==0 && lon2<0
    fprintf('\nlon2 =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nlon2 = %3d %2d %9.6f (D M S)',D,M,S);
end;

fprintf('\n\nReverse azimuth');
[D,M,S] = DMS(az21);
fprintf('\nalpha21 = %3d %2d %9.6f (D M S)',D,M,S);

fprintf('\n\n');
```

MATLAB function *Vincenty_ Inverse.m*

```
function Vincenty_Inverse
%  Vincenty_Inverse computes the "inverse case" on the ellipsoid using
%  Vinventy's method.
%  Given the size and shape of the ellipsoid and the latitudes and
%  longitudes of P1 and P2 this function computes the geodesic distance
%  P1 to P2 and the forward and reverse azimuths

%============================================================================
% Function:  Vincenty_Inverse
%
% Useage:    Vincenty_Inverse;
%
% Author:
%  Rod Deakin,
%  Department of Mathematical and Geospatial Sciences,
%  RMIT University,
%  GPO Box 2476V, MELBOURNE VIC 3001
%  AUSTRALIA
%  email: rod.deakin@rmit.edu.au
%
% Date:
%  Version 1.0   7 March 2008
%
% Functions Required:
%      [D,M,S] = DMS(DecDeg)
%
% Remarks:
%  This function computes the INVERSE CASE on the ellipsoid.  Given the size
```

```
%  and shape of an ellipsoid (defined by parameters a and f, semi-major
%  axis and flattening respectively) and the latitudes and longitudes of P1
%  this function computes the forward azimuth (az12) P1 to P2, the reverse
%  azimuth (az21) P2 to P1 and the geodesic distance (s) P1 to P2.
%
% References:
%  [1] Deakin, R.E, and Hunter, M.N., 2007. 'Geodesics on an Ellipsoid -
%         Bessels' Method', School of Mathematical and Geospatial Sciences,
%         RMIT University, January 2007.
%  [2] Vincenty, T., 1975. 'Direct and Inverse solutions of geodesics on
%         the ellipsoid with application of nested equations', Survey
%         Review, Vol. 23, No. 176, pp.88-93, April 1975.
%
% Variables:
%  A            - Vincenty's constant for computation of sigma
%  a            - semi-major axis of ellipsoid
%  alpha1       - azimuth at P1 for the line P1-P2 (radians)
%  alpha2       - azimuth at P2 for the line P1-P2 extended (radians)
%  az12         - azimuth P1-P2 (degrees)
%  az21         - azimuth P2-P1 (degrees)
%  B            - Vincenty's constant for computation of sigma
%  b            - semi-minor axis of ellipsoid
%  C            - Vincenty's constant for computation of longitude
%                 difference
%  cdo          - cos(domega)
%  cos_sigma    - cos(sigma)
%  delta_sigma  - small change in sigma
%  dlambda      - longitude difference P1 to P2 (radians)
%  domega       - longitude difference P1' to P2' (radians)
%  d2r          - degree to radian conversion factor
%  e2           - eccentricity of ellipsoid squared
%  ep2          - 2nd eccentricity squared
%  f            - flattening of ellipsoid
%  flat         - denominator of flattening, f = 1/flat
%  lambda1      - longitude of P1 (radians)
%  lambda2      - longitude of P2 (radians)
%  lat1         - latitude of P1 (degrees)
%  lat2         - latitude of P2 (degrees)
%  lon1         - longitude of P1 (degrees)
%  lon2         - longitude of P2 (degrees)
%  phi1         - latitude of P1 (radians)
%  phi2         - latitude of P2 (radians)
%  pion2        - pi/2
%  psi0         - parametric latitude of P0 (radians)
%  psi1         - parametric latitude of P1 (radians)
%  psi2         - parametric latitude of P2 (radians)
%  s            - geodesic distance P1 to P2
%  sdo          - sin(domega)
%  sigma        - angular distance (radians) on auxiliary sphere from P1'
%                 to P2'
%  sin_alphaE   - sine of azimuth of geodesic P1-P2 at equator
%  sin_sigma    - sin(sigma)
%  twopi        - 2*pi
%  u2           - geodesic constant u-squared
%
%
%============================================================================

% Define some constants
d2r   = 180/pi;
twopi = 2*pi;
pion2 = pi/2;

% Set defining ellipsoid parameters
 a    = 6378137;           % GRS80
 flat = 298.257222101;
% a    = 6377397.155;        % Bessel (see Ref [2], p.91)
% flat = 299.1528128;
```

```
% Compute derived ellipsoid constants
f   = 1/flat;
b   = a*(1-f);
e2  = f*(2-f);
ep2 = e2/(1-e2);

%----------------------------------------
% latitude and longitude of P1 (degrees)
%----------------------------------------
lat1 =  -10;
lon1 =  110;

% lat and lon of P1 (radians)
phi1    = lat1/d2r;
lambda1 = lon1/d2r;

%----------------------------------------
% latitude and longitude of P2 (degrees)
%----------------------------------------
lat2 =  -45;
lon2 =  155;

% lat and lon of P2 (radians)
phi2    = lat2/d2r;
lambda2 = lon2/d2r;

% [1] Compute parametric latitudes psi1 and psi2 of P1 and P2
psi1 = atan((1-f)*tan(phi1));
psi2 = atan((1-f)*tan(phi2));

s1 = sin(psi1);
s2 = sin(psi2);
c1 = cos(psi1);
c2 = cos(psi2);

% [2] Compute longitude difference dlambda on the ellipsoid
dlambda = lambda2-lambda1; % (radians)
dlon = lon2-lon1;          % (degrees)

% [3] Compute longitude difference domega on the auxiliary sphere by
%     iteration
domega = dlambda;
iter = 1;
while 1
    sdo = sin(domega);
    cdo = cos(domega);
    x = c2*sdo;
    y = c1*s2 - s1*c2*cdo;
    sin_sigma = sqrt(x*x + y*y);
    cos_sigma = s1*s2 + c1*c2*cdo;
    sigma = atan2(sin_sigma,cos_sigma);
    sin_alphaE = c1*c2*sdo/sin_sigma;
    % Compute c1_2m = cos(2*sigma_m)
    x = 1-(sin_alphaE*sin_alphaE);
    c1_2m = cos_sigma - (2*s1*s2/x);
    % Compute Vincenty's constant C
    C = f/16*x*(4+f*(4-3*x));
    % Compute domega
    c2_2m = c1_2m*c1_2m;
    domega_new = dlambda+(1-C)*f*sin_alphaE*(sigma+C*sin_sigma*(c1_2m+C*cos_sigma*(-
1+2*c2_2m)));
    if abs(domega-domega_new)<1e-12
        break;
    end;
    domega = domega_new;
    iter = iter + 1;
end;
```

```
% [4] Compute parametric latitude of vertex
psi0 = acos(sin_alphaE);

% [5] Compute geodesic constant u2 (u-squared)
u2 = ep2*(sin(psi0)^2);

% [6] Compute Vincenty's constants A and B
A = 1 + u2/16384*(4096 + u2*(-768 + u2*(320-175*u2)));
B = u2/1024*(256 + u2*(-128 + u2*(74-47*u2)));

% [7] Compute geodesic distance s
t1 = 2*c2_2m-1;
t2 = -3+4*sin_sigma*sin_sigma;
t3 = -3+4*c2_2m;
delta_sigma = B*sin_sigma*(c1_2m+B/4*(cos_sigma*t1-B/6*c1_2m*t2*t3));
s = b*A*(sigma-delta_sigma);

% [8] Compute forward azimuth alpha1
y = c2*sdo;
x = c1*s2 - s1*c2*cdo;
alpha1 = atan2(y,x);
if alpha1<0
    alpha1 = alpha1+twopi;
end;
az12 = alpha1*d2r;

% [9] Compute azimuth alpha2
y = c1*sdo;
x = -s1*c2 + c1*s2*cdo;
alpha2 = atan2(y,x);

% [10] Compute reverse azimuth az21
az21 = alpha2*d2r + 180;
if az21 > 360
    az21 = az21-360;
end;

%-------------------------------------------------
% Print computed quantities, latitudes and azimuth
%-------------------------------------------------

fprintf('\n/////////////////////////////////////////////////////');
fprintf('\n// INVERSE CASE on ellipsoid: Vincenty''s method //');
fprintf('\n/////////////////////////////////////////////////////');
fprintf('\n\nellipsoid parameters');
fprintf('\na    = %18.9f',a);
fprintf('\nf    = 1/%16.12f',flat);
fprintf('\nb    = %21.12f',b);
fprintf('\ne2   = %20.12e',e2);
fprintf('\nep2  = %20.12e',ep2);

fprintf('\n\nLatitude & Longitude of P1');
[D,M,S] = DMS(lat1);
if D==0 && lat1<0
    fprintf('\nlatP1 =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nlatP1 = %3d %2d %9.6f (D M S)',D,M,S);
end;
[D,M,S] = DMS(lon1);
if D==0 && lon1<0
    fprintf('\nlonP1 =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nlonP1 = %3d %2d %9.6f (D M S)',D,M,S);
end;
fprintf('\n\nLatitude & Longitude of P2');
[D,M,S] = DMS(lat2);
if D==0 && lat2<0
```

```
    fprintf('\nlatP2 =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nlatP2 = %3d %2d %9.6f (D M S)',D,M,S);
end;
[D,M,S] = DMS(lon2);
if D==0 && lon2<0
    fprintf('\nlonP2 =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nlonP2 = %3d %2d %9.6f (D M S)',D,M,S);
end;

fprintf('\n\nParametric Latitudes of P1 and P2');
[D,M,S] = DMS(psi1*d2r);
if D==0 && psi1<0
    fprintf('\npsiP1 =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\npsiP1 = %3d %2d %9.6f (D M S)',D,M,S);
end;
[D,M,S] = DMS(psi2*d2r);
if D==0 && psi2<0
    fprintf('\npsiP2 =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\npsiP2 = %3d %2d %9.6f (D M S)',D,M,S);
end;

fprintf('\n\nLongitude difference on ellipsoid P1-P2');
[D,M,S] = DMS(dlon);
if D==0 && dlon<0
    fprintf('\ndlon =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\ndlon = %3d %2d %9.6f (D M S)',D,M,S);
end;

fprintf('\n\nLongitude difference on auxiliary sphere P1''-P2''');
fprintf('\ndomega = %20.12e radians',sigma);
fprintf('\niterations = %2d',iter);

fprintf('\n\nParametric Latitude of vertex P0');
[D,M,S] = DMS(psi0*d2r);
if D==0 && psi0<0
    fprintf('\npsiP0 =  -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\npsiP0 = %3d %2d %9.6f (D M S)',D,M,S);
end;

fprintf('\n\nGeodesic constant u2 (u-squared)');
fprintf('\nu2 = %20.12e',u2);

fprintf('\n\nVincenty''s constants A and B');
fprintf('\nA = %20.12e',A);
fprintf('\nB = %20.12e',B);

fprintf('\n\nAzimuth & Distance P1-P2');
[D,M,S] = DMS(az12);
fprintf('\naz12 = %4d %2d %9.6f (D M S)',D,M,S);
fprintf('\ns    = %17.6f',s);

fprintf('\n\nReverse azimuth');
[D,M,S] = DMS(az21);
fprintf('\nalpha21 = %3d %2d %9.6f (D M S)',D,M,S);

fprintf('\n\n');
```

MATLAB function $DMS.m$

```
function [D,M,S] = DMS(DecDeg)
% [D,M,S] = DMS(DecDeg)  This function takes an angle in decimal degrees and returns
%   Degrees, Minutes and Seconds

val = abs(DecDeg);
D = fix(val);
M = fix((val-D)*60);
S = (val-D-M/60)*3600;

if abs(S-60) < 5.0e-10
    M = M + 1;
    S = 0.0;
end
if M == 60
    D = D + 1;
    M = 0.0;
end
if D >=360
   D = D - 360;
end

if(DecDeg<=0)
    D = -D;
end
return
```

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GEODESICS ON AN ELLIPSOID – PITTMAN'S METHOD

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ABSTRACT

The direct and inverse problems of the geodesic on an ellipsoid are fundamental geodetic operations. This paper presents a detailed derivation of a set of recurrence relationships that can be used to obtain solutions to the direct and inverse problems with sub-millimetre accuracies for any length of line anywhere on an ellipsoid. These recurrence relationships were first described by Pittman (1986), but since then, little or nothing about them has appeared in the geodetic literature. This is unusual for such an elegant technique and it is hoped that this paper can redress this situation. Pittman's method has much to recommend it.

BIOGRAPHIES OF PRESENTERS

Rod Deakin and Max Hunter are lecturers in the School of Mathematical and Geospatial Sciences, RMIT University; Rod is a surveyor and Max is a mathematician, and both have extensive experience teaching undergraduate students.

INTRODUCTION

Twenty-one years ago (March 1986), Michael E. Pittman, an assistant professor of mathematical physics with the Department of Physics, University of New Orleans, Louisiana USA, published a paper titled 'Precision Direct and Inverse Solutions of the Geodesic' in *Surveying and Mapping* (the journal of the American Congress on Surveying & Mapping, now called *Surveying and Land Information Systems*). It was probably an unusual event – a physicist writing a technical article on geodetic computation – but even more unusual was Pittman's method; or as he put it in his paper, "The following method is rather different." And it certainly is.

Usual approaches could be roughly divided into two groups: (i) numerical integration schemes and (ii) series expansion of elliptic integrals. The first group could be further divided into integration schemes based on simple differential relationships of the ellipsoid (e.g., Kivioja 1971, Jank & Kivioja 1980, Thomas & Featherstone 2005), or numerical integration of elliptic integrals that are usually functions of elements of the ellipsoid and an auxiliary sphere (e.g., Saito 1970, 1979 and Sjöberg 2006). The second group includes the original method of F. W. Bessel (1826) that used an auxiliary sphere and various modifications to his method (e.g., Rainsford 1955, Vincenty 1975, 1976 and Bowring 1983, 1984).

Pittman developed simple recurrence relationships for the evaluation of elliptic integrals that yield distance and longitude difference between a point on a geodesic and the geodesic vertex. These equations can then be used to solve the direct and inverse problems. Pittman's technique is not limited by distance, does not involve any auxiliary surfaces, does not use arbitrarily truncated series and its accuracy is limited only by capacity of the computer used.

Pittman's paper was eight pages long and five of those contained a FORTRAN computer program. In the remaining three pages he presented a very concise development of two recurrence relationships and how they can be used to solve the direct and inverse problems of the geodesic on an ellipsoid (more about this later). His paper, a masterpiece of brevity, contained a single reference and an acknowledgement to Clifford J. Mugnier – then a lecturer in the Department of Civil Engineering, University of New Orleans – for numerous discussions. Unlike other published methods which have been discussed and developed in detail over the years, Pittman's method seems to have received no further treatment to our knowledge in the academic literature, excepting brief mentions in bibliographies and reference lists. Our purpose, in this paper, is to explain Pittman's elegant method as well as provide some useful information about the properties of the geodesic on an ellipsoid.

The Direct and Inverse problems of the geodesic on an ellipsoid

In geodesy, the *geodesic* is a unique curve on the surface of an ellipsoid defining the shortest distance between two points. A geodesic will cut meridians of an ellipsoid at angles α, known as *azimuths* and measured clockwise from north 0º to 360°. Figure 1 shows a geodesic curve C between two points A (ϕ_A, λ_A) and B (ϕ_B, λ_B) on an ellipsoid. ϕ, λ are geodetic latitude and longitude respectively and an ellipsoid is taken to mean a surface of revolution created by rotating an ellipse about its minor axis, *NS*.

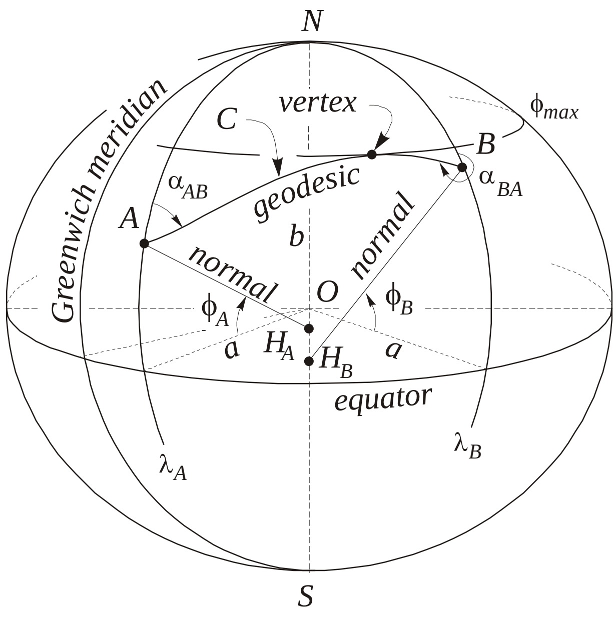

Fig. 1: Geodesic curve on an ellipsoid

The geodesic curve C of length s from A to B has a *forward azimuth* α_{AB} measured at A and a *reverse azimuth* α_{BA} measured at B and $\alpha_{AB} \neq \alpha_{BA}$. The *direct* problem on an ellipsoid is: given latitude and longitude of A and azimuth α_{AB} and geodesic distance *s*, compute the latitude and longitude of *B* and the reverse azimuth α_{BA}. The *inverse* problem is: given the latitudes and longitudes of *A* and *B*, compute the forward and reverse azimuths α_{AB}, α_{BA} and the geodesic distance *s*.

The geodesic is one of several curves of interest in geodesy. Other curves are: (i) normal section curves that are plane curves containing the normal at one of the terminal points; in Figure 1 there would be two normal section curves joining *A* and *B* and both would be of different lengths and also, both longer than the geodesic; (ii) curve of alignment that is the locus of all points P_k where the normal section plane through P_k contains the terminal points of the line; and (iii) great elliptic arcs that are plane curves containing the terminal points of the line and the centre of the ellipsoid. Normal section curves, curves of alignment and great elliptic arcs are all longer than the geodesic and Bowring (1972) gives equations for the differences in length between these curves and the geodesic.

Some ellipsoid relationships

The size and shape of an ellipsoid is defined by one of three pairs of parameters: (i) a, b where *a* and *b* are the *semi-major* and *semi-minor* axes lengths of an ellipsoid respectively, or (ii) a, f where *f* is the *flattening* of an ellipsoid, or (iii) a, e^2 where e^2 is the square of the first *eccentricity* of an ellipsoid. The ellipsoid parameters a, b, f, e^2 are related by the following equations

$$f = \frac{a-b}{a} = 1 - \frac{b}{a}; \quad b = a(1-f); \quad e^2 = \frac{a^2-b^2}{a^2} = 1 - \frac{b^2}{a^2} = f(2-f) \tag{1}$$

The second eccentricity e' of an ellipsoid is also of use and

$$(e')^2 = \frac{a^2-b^2}{b^2} = \frac{e^2}{1-e^2} = \frac{f(2-f)}{(1-f)^2} \tag{2}$$

In Figure 1, the normals to the surface at *A* and *B* intersect the rotational axis of the ellipsoid (*NS* line) at H_A and H_B making angles ϕ_A, ϕ_B with the equatorial plane of the ellipsoid. These are the latitudes of *A* and *B* respectively. The longitudes λ_A, λ_B are the angles between the Greenwich meridian plane and the meridian planes $ONAH_A$ and $ONBH_B$ containing the normals through *A* and *B*. ϕ and λ are *curvilinear* coordinates and meridians of longitude (curves of constant λ) and parallels of latitude (curves of constant ϕ) are parametric curves on the ellipsoidal surface. Planes containing the normal to the ellipsoid intersect the surface creating elliptical sections known as normal sections. Amongst the infinite number of possible normal sections at a point, each having a certain radius of curvature, two are of interest: (i) the *meridian* section, containing the axis of revolution of the ellipsoid and having the least radius of curvature, denoted by ρ (rho), and (ii) the *prime vertical* section, perpendicular to the meridian plane and having the greatest radius of curvature, denoted by ν (nu).

$$\rho = \frac{a\left(1-e^2\right)}{\left(1-e^2\sin^2\phi\right)^{\frac{3}{2}}} \quad \text{and} \quad \nu = \frac{a}{\left(1-e^2\sin^2\phi\right)^{\frac{1}{2}}} \tag{3}$$

In the development that follows, use will be made of relationships that can be obtained from the differential rectangle on the ellipsoid shown in Figure 2. Here *P* and *Q* are two points on the surface connected by a curve of length *ds* with azimuth α at *P*. The meridians λ and $\lambda + d\lambda$, and parallels ϕ and $\phi + d\phi$ form a differential rectangle on the surface of the ellipsoid.

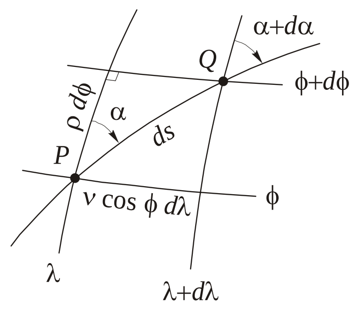

Fig. 2: Differential rectangle on ellipsoid

From Figure 2 the following relationships can be obtained

$$ds\sin\alpha = \nu\cos\phi\; d\lambda \quad \text{and} \quad ds\cos\alpha = \rho\; d\phi \tag{4}$$

Mathematical definition of a geodesic

A curve drawn on a surface so that its osculating plane at any point on the surface contains the normal to the surface is a geodesic (Lauf 1983). This definition, including a definition of the osculating plane, can be explained briefly by the following.

A point *P* on a curve (on a surface) has a position vector $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ where **i**,**j**,**k** are unit vectors in the directions of the *x*,*y*,*z* Cartesian coordinate axes and *t* is some scalar parameter. As *t* varies then the vector **r** sweeps out the curve *C* on the surface, hence the distance *s* along the curve is a function of *t*, given via $\frac{ds}{dt} = \frac{d}{dt}\mathbf{r}(t)$. Differentiating the vector **r** with respect to *s* gives a unit tangent vector **t** and differentiating **t** with respect to *s* gives the curvature vector $\kappa\mathbf{n}$, perpendicular to **t**. **n** is the principal normal vector, κ (kappa) is the curvature and $\rho = \frac{1}{\kappa}$ is the radius of curvature and also the radius of the *osculating* (kissing) circle touching *P*.

The osculating plane at *P* contains both **t** and **n** (and the osculating circle), and when this plane also contains the normal to the surface then the curvature κ is least and ρ is a maximum; this is *Meunier's theorem* (Lauf 1983), a fundamental theorem of surfaces. Therefore, if *P* and *Q* are very close and both lie on the surface and in the osculating plane, then the distance *ds* between them is the shortest possible distance on the suface.

The characteristic equation of a geodesic

The mathematical definition of a geodesic does little to help us develop solutions to the problem of computing distances of geodesics on an ellipsoid. It does lead to *the characteristic equation of a geodesic,* and this equation is the basis of all solutions to computing geodesic distances. This equation

$$\nu \cos\phi \sin\alpha = \text{constant} \tag{5}$$

is known as *Clairaut's equation* in honour of the French mathematical physicist Alexis-Claude Clairaut (1713-1765). In a paper in 1733 titled *Determination géométric de la perpendicular à la méridienne tracée par M. Cassini, ...* Clairaut made an elegant study of the geodesics of surfaces of revolution and stated his theorem embodied in the equation above (Struik 1933). His paper also included the property already pointed out by Johann Bernoulli (1667-1748): the osculating plane of the geodesic is normal to the surface (DSB 1971)

The characteristic equation of a geodesic shows that the geodesic on the ellipsoid has the intrinsic property that at any point, the product of the radius $r = \nu \cos\phi$ of the parallel of latitude and the sine of the azimuth, $\sin\alpha$, of the geodesic at that point is a constant. This means that as *r* decreases in higher latitudes, in both the northern and southern hemispheres, $\sin\alpha$ changes until it reaches a maximum or minimum of ± 1. Such a point is known as a *vertex* and the latitude ϕ will take maximum value ϕ_0.

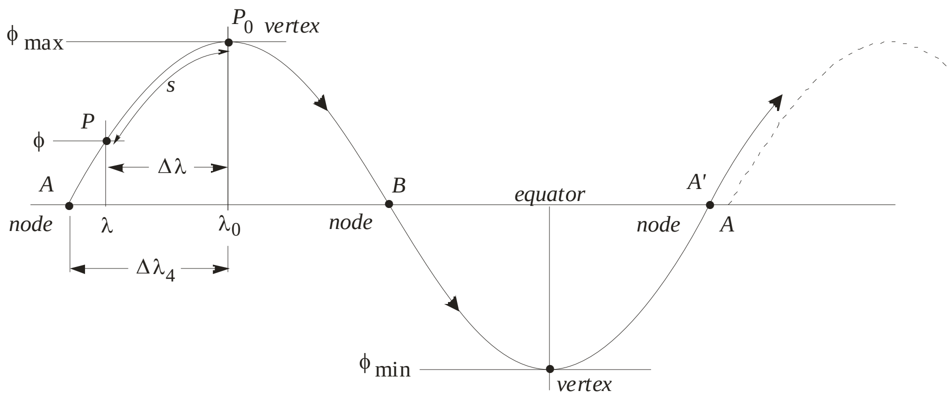

Fig. 3: Schematic diagram of the oscillation of a geodesic on an ellipsoid

Thus the geodesic oscillates over the surface of the ellipsoid between two parallels of latitude having a maximum in the Northern and Southern Hemispheres and crossing the equator at nodes. As we will demonstrate later, due to the eccentricity of the ellipsoid, the geodesic will not repeat after a complete revolution.

Figure 3 shows a schematic diagram of the oscillation of a geodesic on an ellipsoid. *P* is a point on a geodesic that crosses the equator at *A*, heading in a north-easterly direction reaching a maximum northerly latitude $\phi_{\max}$ at the vertex P_0 (north), then descends in a south-easterly direction crossing the equator at *B,* reaching a maximum southerly latitude $\phi_{\min}$ at P_0 (south), then ascends in a north-easterly direction crossing the equator again at A'. This is one complete revolution of the geodesic, but $\lambda_{A'}$ does not equal λ_A due to the eccentricity of the ellipsoid. Hence we say that the geodesic curve does not repeat after a complete revolution.

EQUATIONS FOR COMPUTATION ALONG GEODESICS

Using Clairaut's equation and simple differential relationships, expressions for distances s and longitude differences $\Delta\lambda$ (see Figure 3) between P on a geodesic and the vertex P_0 can be obtained. These expressions are in the form of elliptic integrals, which by their nature do not have exact (or closed) solutions.

Expanding the integrands into infinite series, integrating term-by-term and then truncating to a finite number of terms is the usual technique to obtain working solutions for s and $\Delta\lambda$ (e.g., Thomas 1970). In this section, we show how this method can be simplified by using recurrence relationships to generate solutions to the integrals in the series. Our relationships are slightly different from Pittman (1986) and our notation is a little different but in all other respects, we have followed his elegant approach.

Relationships between parametric latitude *ψ* and geodetic latitude *Φ*

Development of formulae is simplified if *parametric* latitude ψ is used rather than *geodetic* latitude ϕ. The connections between the two latitudes can be obtained from the following relationships.

Figure 4 shows a portion of a meridian NPE of an ellipsoid having semi-major axis $OE = a$ and semi-minor axis $ON = b$. P is a point on the ellipsoid and Q is a point on an auxiliary circle centred on O of radius a. P and Q have the same perpendicular distance from the axis of revolution ON. The normal to the ellipsoid at P cuts the major axis at an angle ϕ (the geodetic latitude) and intersects the rotational axis at H and the distance $PH = \nu$. The angle $QOE = \psi$ is the parametric latitude.

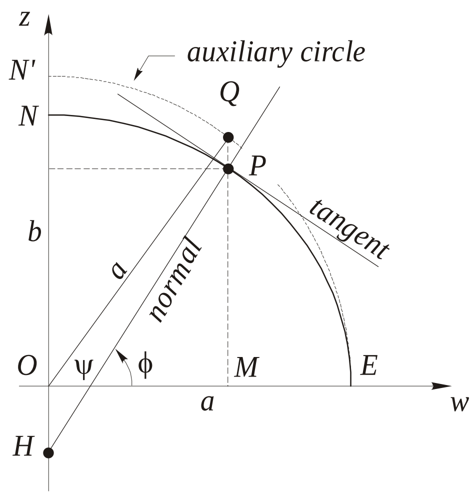

Fig. 4: Meridian section of ellipsoid

The Cartesian equation of the ellipse is $\frac{w^2}{a^2} + \frac{z^2}{b^2} = 1$ and the Cartesian equation of the auxiliary circle is $w^2 + z^2 = a^2$. We may re-arrange both equations so that w^2 is on the left-hand side of the equals sign giving $w^2 = a^2 - \frac{a^2}{b^2} z^2$ (ellipse) and $w^2 = a^2 - z^2$ (circle). Now, since the w-coordinates of P and Q are the same then $a^2 - \frac{a^2}{b^2} z_P^2 = a^2 - z_Q^2$ which leads to $z_P = \frac{b}{a} z_Q$.

Using this relationship

$$w = OM = a\cos\psi \quad \text{and} \quad z = MP = b\sin\psi \tag{6}$$

and differentiating equations (6) with respect to ψ gives $\frac{dw}{d\psi} = -a\sin\psi$, $\frac{dz}{d\psi} = b\cos\psi$ and the chain rule gives $\frac{dz}{dw} = \frac{dz}{d\psi}\frac{d\psi}{dw} = -\frac{b}{a}\cot\psi$.

Now by definition, $\frac{dz}{dw}$ is the gradient of the tangent and from Figure 4 we may write $\frac{dz}{dw} = -\tan\left(90^\circ - \phi\right) = -\cot\phi$. Equating the two expressions for dz/dw gives a relationship between ψ and ϕ as

$$\tan\psi = \frac{b}{a}\tan\phi = \left(1 - f\right)\tan\phi \tag{7}$$

From equation (6) and Figure 4, $w = a\cos\psi = \nu\cos\phi$ and using equation (3) gives

$$\cos\psi = \frac{\cos\phi}{\left(1 - e^2\sin^2\phi\right)^{1/2}} \tag{8}$$

Alternatively, using the trigonometric identity $\sin^2 A + \cos^2 A = 1$, equation (8) can be written as

$$\sin\phi = \frac{\sin\psi}{\left(1 - e^2\cos^2\psi\right)^{1/2}} \tag{9}$$

The latitudes Φ₀ and ψ₀ of the geodesic vertex

Denoting the latitude of the vertex as ϕ_0 (a maximum), Clairaut's equation (5) gives

$$\nu_0\cos\phi_0 = \text{constant} = \nu\cos\phi\sin\alpha \tag{10}$$

Denoting the parametric latitude of the vertex as ψ_0 and using $a\cos\psi = \nu\cos\phi$ from before, equation (10) becomes $a\cos\psi_0 = a\cos\psi\sin\alpha$ and ψ_0 is defined as

$$\cos\psi_0 = \cos\psi\sin\alpha \tag{11}$$

Squaring both sides of equation (11) and using again the identity $\sin^2 A + \cos^2 A = 1$ we can obtain the azimuth α of a geodesic as

$$\cos\alpha = \frac{\sqrt{\cos^2\psi - \cos^2\psi_0}}{\cos\psi} \tag{12}$$

From equation (11) we see that if the azimuth α of a geodesic is known at *P* having parametric latitude ψ, the parametric latitude ψ_0 of the vertex P_0 can be computed. Conversely, given ψ and ψ_0 of points *P* and P_0 the azimuth of the geodesic between them may be computed from equation (12).

In the following sections, two differential equations; one for $\frac{ds}{d\psi}$ and the other for $\frac{d\lambda}{d\psi}$, will be developed that will enable solutions for the geodesic distance *s* and the longitude difference $\Delta\lambda$ between *P* and the vertex P_0.

Differential equations for distance $\dfrac{ds}{d\psi}$ and longitude difference $\dfrac{d\lambda}{d\psi}$

From equation (9) we may write $\sin^2\psi = \left(1-e^2\cos^2\psi\right)\sin^2\phi$ and differentiating implicitly and re-arranging gives

$$\frac{d\phi}{d\psi} = \frac{\left(1-e^2\sin^2\phi\right)\sin\psi\cos\psi}{\left(1-e^2\cos^2\psi\right)\sin\phi\cos\phi} \tag{13}$$

Using the chain rule and equation (4) gives an expression for the derivative $\dfrac{ds}{d\psi}$ as

$$\frac{ds}{d\psi} = \frac{ds}{d\phi}\frac{d\phi}{d\psi} = \frac{\rho}{\cos\alpha}\frac{\left(1-e^2\sin^2\phi\right)\sin\psi\cos\psi}{\left(1-e^2\cos^2\psi\right)\sin\phi\cos\phi} \tag{14}$$

Using equations (7), (8), (9) and the fact that $1-e^2 = \dfrac{b^2}{a^2}$, we may write

$$\frac{ds}{d\psi} = a\cos\psi\frac{\left(1-e^2\cos^2\psi\right)^{1/2}}{\left(\cos^2\psi-\cos^2\psi_0\right)^{1/2}} \tag{15}$$

Similarly, the chain rule and equations (4) and (15) gives

$$\frac{d\lambda}{d\psi} = \frac{d\lambda}{ds}\frac{ds}{d\psi} = \frac{\sin\alpha}{\nu\cos\phi}a\cos\psi\frac{\left(1-e^2\cos^2\psi\right)^{1/2}}{\left(\cos^2\psi-\cos^2\psi_0\right)^{1/2}} \tag{16}$$

Using equation (10) and the relationship $a\cos\psi = \nu\cos\phi$, we may write

$$\frac{d\lambda}{d\psi} = \frac{\cos\psi_0}{\cos\psi}\frac{\left(1-e^2\cos^2\psi\right)^{1/2}}{\left(\cos^2\psi-\cos^2\psi_0\right)^{1/2}} \tag{17}$$

Equations (15) and (17) are the basic differential equations that will yield solutions for distance s and longitude difference $\Delta\lambda$ along the geodesic curve between P and the vertex P_0.

Formula for computing geodesic distance *s* between *P* and the vertex P_0

Equation (15) can be simplified by letting $u=\sin\psi$ and $u_0=\sin\psi_0$, so that $\dfrac{du}{d\psi}=\cos\psi$ and $\cos^2\psi-\cos^2\psi_0 = u_0^2-u^2$, hence

$$\frac{ds}{d\psi} = a\frac{du}{d\psi}\frac{\left(1-e^2\cos^2\psi\right)^{1/2}}{\left(u_0^2-u^2\right)^{1/2}} \tag{18}$$

The chain rule gives $\dfrac{ds}{du} = \dfrac{ds}{d\psi} \bigg/ \dfrac{du}{d\psi} = \dfrac{a\left(1-e^2\cos^2\psi\right)^{1/2}}{\left(u_0^2 - u^2\right)^{1/2}}$ but using $\cos^2\psi = 1-\sin^2\psi$ and equations (1) and (2) we are able to obtain, after some manipulation

$$\frac{ds}{du} = \frac{b\left(1+\varepsilon u^2\right)^{1/2}}{\left(u_0^2 - u^2\right)^{1/2}} \tag{19}$$

where $\varepsilon = \left(e'\right)^2$. The geodesic distance *s* between *P* and the vertex P_0 is given by

$$s = b\int_{p=u}^{p=u_0} \frac{\left(1+\varepsilon p^2\right)^{1/2}}{\left(u_0^2 - p^2\right)^{1/2}}\,dp \tag{20}$$

where $\sin\psi \le p \le \sin\psi_0$. Equation (20) can be simplified by use of the *binomial series* and the numerator of the integrand is given by

$$\left(1+\varepsilon p^2\right)^{1/2} = \sum_{n=0}^{\infty} B_n^{\frac{1}{2}} \left(\varepsilon p^2\right)^n \tag{21}$$

where $B_n^{\frac{1}{2}}$ are binomial coefficients computed from the recurrence relationship

$$B_n^{\frac{1}{2}} = \frac{3-2n}{2n} B_{n-1}^{\frac{1}{2}}, \quad n \ge 1 \text{ and } B_0^{\frac{1}{2}} = 1 \tag{22}$$

Equation (20) can now be written as

$$s = b\int_u^{u_0} \frac{1}{\left(u_0^2 - p^2\right)^{1/2}} \sum_{n=0}^{\infty} B_n^{\frac{1}{2}} \varepsilon^n p^{2n}\,dp = b\sum_{n=0}^{\infty} B_n^{\frac{1}{2}} \varepsilon^n \int_u^{u_0} \frac{p^{2n}}{\left(u_0^2 - p^2\right)^{1/2}}\,dp = b\sum_{n=0}^{\infty} \varepsilon^n B_n^{\frac{1}{2}} I_n \tag{23}$$

where

$$I_n = \int_u^{u_0} \frac{p^{2n}}{\left(u_0^2 - p^2\right)^{1/2}}\,dp, \quad \text{for } n \ge 0 \tag{24}$$

The solution of the integral I_n is fundamental to the computation of the distance *s* along the geodesic between *P* and P_0, and the usual technique is to find solutions for each integral I_n and expand equation (23) into a finite series; e.g. Thomas (1970, pp. 33-34). Pittman's (1986) approach, outlined below, was to developed the integral I_n as a recurrence equation having the general form $I_n = a_{n-1} + b_{n-1} I_{n-1}$ where the coefficients a_{n-1} and b_{n-1} are functions of n, ψ and ψ_0 and an initial value of I_0 is a function of ψ and ψ_0 only.

Now $$I_n = \int_u^{u_0} \frac{p^{2n}}{\left(u_0^2 - p^2\right)^{1/2}}\,dp = -\int_u^{u_0} p^{2n-1} \frac{-p}{\left(u_0^2 - p^2\right)^{1/2}}\,dp = -\int_u^{u_0} p^{2n-1} \frac{d}{dp}\left(u_0^2 - p^2\right)^{1/2}\,dp$$

and using integration by parts (e.g., Ayres 1972) the integral I_n becomes

$$\begin{aligned} I_n &= -\left[p^{2n-1}\left(u_0^2 - p^2\right)^{1/2} - \int \left(u_0^2 - p^2\right)^{1/2}\left(2n-1\right)p^{2n-2}dp \right]_{p=u}^{p=u_0} \\ &= u^{2n-1}\left(u_0^2 - u^2\right)^{1/2} + \left(2n-1\right)\int_u^{u_0}\left(u_0^2 - p^2\right)\frac{p^{2n-2}}{\left(u_0^2 - p^2\right)^{1/2}}dp \\ &= u^{2n-1}\left(u_0^2 - u^2\right)^{1/2} + \left(2n-1\right)\left[u_0^2 I_{n-1} - I_n\right] \end{aligned} \tag{25}$$

and

$$2n\,I_n = u^{2n-1}\left(u_0^2 - u^2\right)^{1/2} + \left(2n-1\right)u_0^2 I_{n-1} \qquad \text{for } n = 1,2,3,\ldots \tag{26}$$

Let $U = \dfrac{u}{u_0}$ so that $u = Uu_0$, $u_0^2 - u^2 = u_0^2\left(1 - U^2\right)$ giving

$$2n\,I_n = \left(Uu_0\right)^{2n-1}u_0\left(1 - U^2\right)^{1/2} + \left(2n-1\right)u_0^2 I_{n-1} \qquad \text{for } n = 1,2,3,\ldots \tag{27}$$

Let $J_n = \dfrac{2n\,I_n}{u_0^{2n}}$ so that $J_{n-1} = \dfrac{2\left(n-1\right)u_0^2}{u_0^{2n}}I_{n-1}$ and the recurrence formula for I_n becomes a simpler recurrence formula for J_n

$$J_n = U^{2n-1}\sqrt{1 - U^2} + \frac{2n-1}{2\left(n-1\right)}J_{n-1} \qquad \text{for } n = 2,3,\ldots \tag{28}$$

with initial condition

$$J_1 = \frac{2I_1}{u_0^2} = U\sqrt{1 - U^2} + I_0 \tag{29}$$

I_0 has a simple result derived from equation (24) as follows:

$$I_0 = \left(1/u_0\right)\int_u^{u_0}\left(1 - \left[p/u_0\right]^2\right)^{-1/2}dp \tag{30}$$

and with the transformation $p = u_0\cos\theta$, $dp/d\theta = -u_0\sin\theta$ and $1 - \left[p/u_0\right]^2 = 1 - \cos^2\theta$

$$I_0 = \int_{\theta = \arccos\left(\frac{u}{u_0}\right)}^{0}\left(-1\right)d\theta = \arccos\left(\frac{u}{u_0}\right) = \arccos U \tag{31}$$

Using these results, the distance *s* along the geodesic between *P* and the vertex P_0 is

$$\begin{aligned} s &= b\left\{ I_0 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\varepsilon^n u_0^{2n}B_n^{\frac{1}{2}}J_n \right\} \\ &= bI_0 + \frac{b}{2}\varepsilon u_0^2 B_1^{\frac{1}{2}}J_1 + \frac{b}{4}\varepsilon^2 u_0^4 B_2^{\frac{1}{2}}J_2 + \frac{b}{6}\varepsilon^3 u_0^6 B_3^{\frac{1}{2}}J_3 + \cdots \\ &= D_0 + D_1 + D_2 + D_3 + \cdots \end{aligned} \tag{32}$$

Formula for computing difference in longitude Δλ between *P* and *P*₀

Using the binomial series we may write equation (17) as

$$\frac{d\lambda}{d\psi}=\cos\psi_0\sum_{n=0}^{\infty}\left(-1\right)^n e^{2n}B_n^{\frac{1}{2}}\frac{\cos^{2n-1}\psi}{\left(\cos^2\psi-\cos^2\psi_0\right)^{1/2}} \tag{33}$$

and the difference in longitude between P and the vertex P_0 is

$$\Delta\lambda=\int_{\theta=\psi}^{\psi_0}\frac{d\lambda}{d\theta}d\theta=\cos\psi_0\sum_{n=0}^{\infty}\left(-1\right)^n e^{2n}B_n^{\frac{1}{2}}L_n \tag{34}$$

where the integral L_n is

$$L_n=\int_{\theta=\psi}^{\psi_0}\frac{\cos^{2n}\theta}{\cos\theta\left(\cos^2\theta-\cos^2\psi_0\right)^{1/2}}d\theta,\quad n\geq 0 \tag{35}$$

Again, let $u=\sin\psi$, $u_0=\sin\psi_0$ and put $p=\sin\theta$. Then $d\theta/dp=\sec\theta$, $\cos^2\theta=1-p^2$, and with

$$\frac{\cos^{2n}\theta}{\cos\theta}d\theta=\frac{\left(\cos^2\theta\right)^n}{\cos^2\theta}\cos\theta\, d\theta=\frac{\left(1-p^2\right)^n}{1-p^2}dp=\left(1-p^2\right)^{n-1}dp$$

and

$$\left(\cos^2\theta-\cos^2\psi_0\right)^{1/2}=\left(1-\sin^2\theta-\left(1-\sin^2\psi_0\right)\right)^{1/2}=\left(u_0^2-p^2\right)^{1/2}$$

giving

$$L_n=\int_u^{u_0}\frac{\left(1-p^2\right)^{n-1}}{\left(u_0^2-p^2\right)^{1/2}}dp,\quad n\geq 1 \tag{36}$$

Using the binomial series, the numerator of the integrand can be expanded into a polynomial $\left(1-p^2\right)^{n-1}=\sum_{m=0}^{n-1}\left(-1\right)^m B_m^{n-1}p^{2m}$, where the binomial coefficients B_m^{n-1} are given by

$$B_m^{n-1}=\frac{n-m}{m}B_{m-1}^{n-1}\qquad\text{for } m=2,3,4,\ldots \tag{37}$$

with an initial value $B_1^{n-1}=n-1$ and noting that $B_0^{n-1}=1$.

Using these results, equation (36) becomes

$$L_n=\sum_{m=0}^{n-1}\left(-1\right)^m B_m^{n-1}\int_u^{u_0}\frac{p^{2m}}{\left(u_0^2-p^2\right)^{1/2}}dp=\sum_{m=0}^{n-1}\left(-1\right)^m B_m^{n-1}I_m \tag{38}$$

where

$$I_m=\int_u^{u_0}\frac{p^{2m}}{\left(u_0^2-p^2\right)^{1/2}}dp,\quad\text{for}\quad m\geq 0 \tag{39}$$

and equation (39) is the same as equation (24) except for a change of index variable.

Using this similarity and the expressions above, the longitude difference given by equation (34) can be expressed as

$$\Delta\lambda = \cos\psi_0 \left\{ L_0 + \sum_{n=1}^{\infty} (-1)^n e^{2n} B_n^{\frac{1}{2}} \sum_{m=0}^{n-1} (-1)^m B_m^{n-1} I_m \right\} \tag{40}$$

Equation (40) can expanded as

$$\begin{aligned}\Delta\lambda = \cos\psi_0 \Bigg\{ & L_0 + \left[-e^2 B_1^{\frac{1}{2}} + \sum_{n=2}^{\infty} (-1)^n e^{2n} B_n^{\frac{1}{2}} \right] I_0 \\ & + \sum_{n=2}^{\infty} (-1)^n e^{2n} B_n^{\frac{1}{2}} \sum_{m=1}^{n-1} (-1)^m B_m^{n-1} I_m \Bigg\}\end{aligned} \tag{41}$$

and then simplified by use of the binomial series, where

$$\left(1-e^2\right)^{1/2} = \sum_{n=0}^{\infty} (-1)^n e^{2n} B_n^{\frac{1}{2}} = 1 + \sum_{n=1}^{\infty} (-1)^n e^{2n} B_n^{\frac{1}{2}} = 1 - e^2 B_1^{\frac{1}{2}} + \sum_{n=2}^{\infty} (-1)^n e^{2n} B_n^{\frac{1}{2}} \tag{42}$$

The terms in $[\cdots]$ of equation (41) are the last two terms on the right-hand side of equation (42) and using this equivalence gives

$$\begin{aligned}\Delta\lambda &= \cos\psi_0 \left\{ L_0 + \left(\sqrt{1-e^2} - 1\right) I_0 + \sum_{n=2}^{\infty} (-1)^n e^{2n} B_n^{\frac{1}{2}} \sum_{m=1}^{n-1} (-1)^m B_m^{n-1} I_m \right\} \\ &= \cos\psi_0 \left\{ L_0 + \left(\sqrt{1-e^2} - 1\right) I_0 + \tfrac{1}{2} \sum_{n=2}^{\infty} (-1)^n e^{2n} B_n^{\frac{1}{2}} \sum_{m=1}^{n-1} \frac{(-1)^m}{m} u_0^{2m} B_m^{n-1} J_m \right\}\end{aligned} \tag{43}$$

where I_0 is obtained from equation (31) and J_m are given by equation (28), noting that as before $J_m = \dfrac{2m}{u_0^{2m}} I_m$.

A simple expression for L_0 is obtained from equation (35) as follows

$$L_0 = \int_{\theta=\psi}^{\psi_0} \frac{1}{\cos\theta\left(\cos^2\theta - \cos^2\psi_0\right)^{1/2}} d\theta = \int_{\theta=\psi}^{\psi_0} \frac{\sec^2\theta}{\left(\sin^2\psi_0 - \tan^2\theta\cos^2\psi_0\right)^{1/2}} d\theta \tag{44}$$

Putting $x = \cot\psi_0 \tan\theta$ then $d\theta/dx = \tan\psi_0 \cos^2\theta$ and

$$\begin{aligned}\sin^2\psi_0 - \tan^2\theta\cos^2\psi_0 &= \sin^2\psi_0 \left(1 - \tan^2\theta \frac{\cos^2\psi_0}{\sin^2\psi_0}\right) \\ &= \sin^2\psi_0 \left(1 - \tan^2\theta\cot^2\psi_0\right) \\ &= \sin^2\psi_0 \left(1 - x^2\right)\end{aligned}$$

so that

$$L_0 = \frac{\tan\psi_0}{\sin\psi_0} \int_{x=\frac{\tan\psi}{\tan\psi_0}}^{1} \frac{dx}{\sqrt{1-x^2}} \tag{45}$$

since $\int \dfrac{dx}{\sqrt{1-x^2}} = \begin{cases} \arcsin x \\ \frac{\pi}{2} - \arccos x \end{cases}$, then using the second result gives

$$L_0 = \sec\psi_0 \int_{x=\frac{\tan\psi}{\tan\psi_0}}^{1} \frac{dx}{\sqrt{1-x^2}} = \sec\psi_0 \arccos\left(\frac{\tan\psi}{\tan\psi_0}\right) \tag{46}$$

Equation (40) can be simplified further to give the longitude difference $\Delta\lambda$ between P and the vertex P_0 as

$$\Delta\lambda = \cos\psi_0 \left\{M_0 + M_1 + M_2 + M_3 + \cdots\right\} \tag{47}$$

where

$$M_n = \begin{cases} L_0 & \text{for } n = 0 \\ \left(\sqrt{1-e^2} - 1\right) I_0 & \text{for } n = 1 \\ \frac{1}{2} B_n^{\frac{1}{2}} \left(-1\right)^n e^{2n} K_n & \text{for } n \geq 2 \end{cases} \tag{48}$$

and

$$K_n = \sum_{m=1}^{n-1} \frac{\left(-1\right)^m}{m} u_0^{2m} B_m^{n-1} J_m \quad \text{for } n = 2,3,4,\ldots \tag{49}$$

A GEODESIC ON AN ELLIPSOID DOES NOT REPEAT AFTER A SINGLE REVOLUTION

Earlier, it was mentioned that due to the eccentricity of the ellipsoid, the geodesic will not repeat after a complete revolution. Here is a demonstration of that fact.
When P is at the node A of Figure 3 then $\Delta\lambda = \Delta\lambda_4$ and using equation (17) we have

$$4\left(\Delta\lambda_4\right) = 4\cos\psi_0 \int_{\theta=0}^{\psi_0} \frac{\left(1 - e^2\cos^2\theta\right)^{1/2}}{\cos\theta\left(\cos^2\theta - \cos^2\psi_0\right)^{1/2}} d\theta \tag{50}$$

Since this integral is difficult to evaluate, we instead determine upper and lower bounds for the quantity $4\left(\Delta\lambda_4\right)$ by using the bounds of the integration variable θ. This allows certain terms within the integral to be disposed of and a simplified integral evaluated.

For $0 \leq \theta \leq \psi_0$, the bounds on the numerator of the integrand are
$\left(1-e^2\right)^{1/2} \leq \left(1 - e^2\cos^2\theta\right)^{1/2} \leq \left(1 - e^2\cos^2\psi_0\right)^{1/2}$ so that on the one hand

$$\begin{aligned} 4\left(\Delta\lambda_4\right) &\leq 4\cos\psi_0 \int_{\theta=0}^{\psi_0} \frac{\left(1 - e^2\cos^2\psi_0\right)^{1/2}}{\cos\theta\left(\cos^2\theta - \cos^2\psi_0\right)^{1/2}} d\theta \\ &= 4\cos\psi_0 \left(1 - e^2\cos^2\psi_0\right)^{1/2} L_0\big|_{\psi=0} \\ &= 4\cos\psi_0 \left(1 - e^2\cos^2\psi_0\right)^{1/2} \tfrac{1}{2}\pi\sec\psi_0 \\ &= 2\pi\left(1 - e^2\cos^2\psi_0\right)^{1/2} \end{aligned} \tag{51}$$

while on the other hand

$$4\left(\Delta\lambda_4\right) \ge 4\cos\psi_0 \int_{\theta=0}^{\psi_0} \frac{\left(1-e^2\right)^{1/2}}{\cos\theta\left(\cos^2\theta-\cos^2\psi_0\right)^{1/2}}\,d\theta$$

$$= 2\pi\left(1-e^2\right)^{1/2} \tag{52}$$

Combining these inequalities gives the bounds for the quantity $4\left(\Delta\lambda_4\right)$ as

$$2\pi\left(1-e^2\right)^{1/2} \le 4\left(\Delta\lambda_4\right) \le 2\pi\left(1-e^2\cos^2\psi_0\right)^{1/2} \tag{53}$$

Therefore, after a single revolution, $4\left(\Delta\lambda_4\right) < 2\pi$ when $0^\circ < \psi_0 < 90^\circ$. Note that when $\psi_0 = 0^\circ$ the geodesic is an arc of the equator (a circle) and when $\psi_0 = 90^\circ$ the geodesic is an arc of the meridian (an ellipse).

NUMERICAL RESULTS FOR DISTANCE AND LONGITUDE EQUATIONS

Equations (32) and (47) for computing distance s and longitude difference $\Delta\lambda$ between P and the vertex P_0 are relatively simple summations of terms. To test the number of terms required for accurate answers, a geodesic was chosen with an azimuth $\alpha = 43^\circ 12' 36''$ at P having latitude $\phi = 9^\circ 35' 24''$ on the ellipsoid of the Geodetic Reference System 1980 (GRS80) (Moritz 1980), defined by $a = 6378137$ metres and $f = 1/298.257\,222101$.

Numerical constants for GRS80 ellipsoid and geodesic

$b \quad = a\left(1-f\right) \qquad = 6356752.314140356$ metres

$\psi \quad = \arctan\left[\left(1-f\right)\tan\phi\right] = 0.166826262923$ radians

$\psi_0 = \arccos\left[\cos\psi\sin\alpha\right] = 0.829602797993$ radians

$u \quad = \sin\psi \qquad = 0.166053515348; \quad u_0 = \sin\psi_0 \quad = 0.737663250899$

$U \quad = \dfrac{u}{u_0} = \dfrac{\sin\psi}{\sin\psi_0} = 0.225107479796; \quad I_0 = \arccos U = 1.343742980976$ radians

$V \quad = \dfrac{\tan\psi}{\tan\psi_0} \qquad = 0.154125311675; \quad L_0 = \sec\psi_0 \arccos V = 2.097333540996$ radians

Table 1: Ellipsoid and geodesic constants and binomial coefficients for equations (32) and (47)

n	e^{2n}	ε^{n}	u_0^{2n}	$B_n^{\frac{1}{2}}$
1	6.694380022901e-003	6.739496775479e-003	0.544147071727	0.500000000000
2	4.481472389101e-005	4.542081678669e-005	0.296096035669	-0.125000000000
3	3.000067923478e-007	3.061134482735e-007	0.161119790759	0.062500000000
4	2.008359477428e-009	2.063050597570e-009	0.087672862339	-0.039062500000
5	1.344472156450e-011	1.390392284997e-011	0.047706931312	0.027343750000
6	9.000407545482e-014	9.370544321391e-014	0.025959586974	-0.020507812500
7	6.025214847044e-016	6.315275323850e-016	0.014125833235	0.016113281250
8	4.033507790574e-018	4.256177768135e-018	0.007686530791	-0.013092041016

Table 2: Recurrence formula values and distance components for equation (32)

n	J_n	D_n	
1	1.563072838216	8.541841303930e+006	8541841.303930 m
2	2.355723441968	9.109578467516e+003	9109.5784675
3	2.945217495733	-6.293571169346e+000	-6.2935712
4	3.436115617261	9.618619108010e-003	0.0096186
5	3.865631515581	-1.929070816523e-005	-0.0000193
6	4.252194740421	4.456897529564e-008	0.0000000
7	4.606544305836	-1.123696751599e-010	-0.0000000
8	4.935583185013	3.006580650377e-013	0.0000000
	sum	8.550944598425e+006	s = 8550944.598425 m

Table 3: Recurrence formula values and longitude components for equation (47)

n	J_n	M_n		
0		2.097333540996e+000		
1	1.563072838216	-4.505315819380e-003		
2	2.355723441968	2.382298926901e-006		
3	2.945217495733	1.267831357153e-008		
4	3.436115617261	6.525291638252e-011		
5	3.865631515581	3.431821056093e-013		
6	4.252194740421	1.852429353592e-015	$\Delta\lambda = \cos\psi_0\,(sum)$	$\cong 1.413013969112$ radians
7	4.606544305836	1.023576994037e-017		$= 80.959736823113$ degrees
8	4.935583185013	5.769507252421e-020		
	sum	2.092830620219e+000		$= 80^\circ\, 57'\, 35.052563''$

Inspection of these numerical values indicates than an upper limit of $N = 8$ in the summations is more than sufficient for accuracies of 0.000001 metre in distances and 0.000001 second of arc for longitude differences. [Results for s and $\Delta\lambda$ can be confirmed using Vincenty's equations (Vincenty 1975) that have been programmed in a Microsoft™ *Excel* workbook that can be downloaded from the website of Geoscience Australia at http://www.ga.gov.au/]

It should be noted here that the distance and longitude equations [equations (32) and (47)] are not themselves, solutions to the direct or inverse problems. Instead, they are the basic tools, which if used in certain ways, enable the solution to those problems.

In a computer program, equations (32) and (47) would be embedded in a function that returned s and $\Delta\lambda$ given the ellipsoid parameters (a, f), parametric latitudes (ψ, ψ_0) and the upper limit of summations (N). A brief explanation of how such a function might be used is given below.

USING THE DISTANCE AND LONGITUDE EQUATIONS TO COMPUTE THE DIRECT AND INVERSE PROBLEM

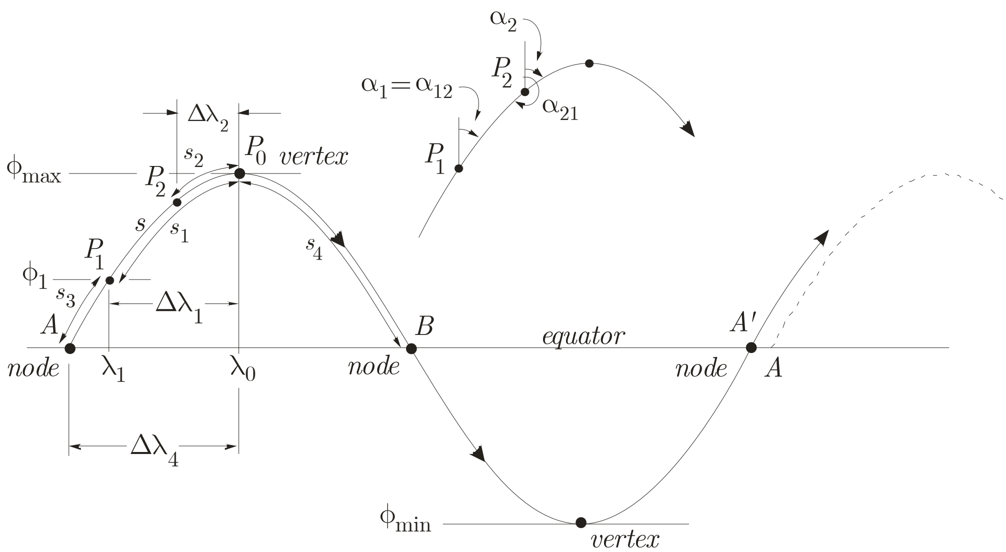

Fig. 5: Schematic diagram of a geodesic between P_1 and P_2 on an ellipsoid

Direct solution

The key here is to use the distance equation in an iterative computation of $\sin\psi_2$. Once this is known, then ϕ_2, λ_2 and α_{21} follow. The steps in the computation are:

1. Test the azimuth to determine whether the geodesic is heading towards or away from the nearest vertex P_0, noting that P_0 will be in the same hemisphere as P_1.
2. Compute ψ_1 and ψ_0; then use the distance and longitude equations to compute s_1 and $\Delta\lambda_1$ between P_1 and P_0, as well as λ_0. (see Fig. 5).
3. With $u = \sin\psi = 0$, compute s_4 and $\Delta\lambda_4$ between the node and P_0.

4. Compute $s_2 = \begin{cases} s - s_1 & \text{if geodesic is heading towards } P_0 \\ s + s_1 & \text{if geodesic is heading away from } P_0 \end{cases}$. If $s_2 > 0$ then P_2 is after P_0 and closer to another vertex P_0' in which case s_2 is reduced by multiples of $2s_4$ until $s_2 < s_4$ and the number of vertices *n* determined (vertices are $2s_4$ apart). If $s_2 < 0$ then P_2 is before P_0. (Note that in Fig. 5, $s_2 < 0$ and P_2 is before P_0)
5. Compute ψ_2 by iteration. An approximate value ψ_2' is found from equations (32) by taking the first term only; hence $\dfrac{s}{b} = I_0 = \arccos\left(\dfrac{\sin\psi}{\sin\psi_0}\right)$ and $\sin\psi_2' = \sin\psi_0 \cos\left(\dfrac{s_2}{b}\right)$.

   Now a re-arrangement of the differential equation (19) gives $du = \dfrac{ds}{b}\sqrt{\dfrac{u_0^2 - u^2}{1+\varepsilon u^2}}$ where $u = \sin\psi_2'$, $ds = s_2' - s_2$ and s_2' is computed from the distance equation with the approximate parametric latitude ψ_2'. Equation (19), linking *ds* and *du*, is the basis of the iterative solution for $\sin\psi_2$ (and hence ϕ_2).
6. After computing ψ_2 the longitude difference $\Delta\lambda_2$ is computed and depending on the number of vertices and the direction of the geodesic, λ_2 is determined. The azimuth α_2 follows from Clairaut's equation and the reverse azimuth α_{21} obtained.

Inverse solution

This is the more difficult of the two solutions since ψ_0 is unknown and must be determined by iteration, using approximations for s, α_1 and α_2 obtained by approximating the ellipsoid with a sphere and using spherical trigonometry. The steps in the computation are:

1. Convert longitudes of P_1 and P_2 to east longitudes in the range $0^\circ < \lambda_1, \lambda_2 < 360^\circ$ and determine a longitude difference $\Delta\lambda$ in the range $-180^\circ \le \Delta\lambda \le 180^\circ$. $\pm\Delta\lambda$ corresponding to east/west direction of the geodesic from P_1.
2. Compute parametric latitudes ψ_1 and ψ_2 then use these and $\Delta\lambda$ as latitudes and longitude difference on a sphere to compute spherical distance σ and spherical angles β_1 and β_2. These can be used to determine approximations of *s* and α_{12}.
3. Compute ψ_0 by iteration. Approximations $\Delta\lambda_1'$ and $\Delta\lambda_2'$ can be obtained from equation (47) noting that $M_0 = \sec\psi_0 \arccos\left(\dfrac{\tan\psi}{\tan\psi_0}\right)$ and ignoring terms $M_1, M_2, M_3, \ldots$

   This gives $\Delta\lambda_1' = \arccos\left(\dfrac{\tan\psi_1}{\tan\psi_0}\right)$ and $\Delta\lambda_2' = \arccos\left(\dfrac{\tan\psi_2}{\tan\psi_0}\right)$, and

$f\left(\psi_0\right)=\Delta\lambda'-\Delta\lambda=\left\{\pm\arccos\left(\frac{\tan\psi_1}{\tan\psi_0}\right)\pm\arccos\left(\frac{\tan\psi_2}{\tan\psi_0}\right)\pm\Delta\lambda_4'\right\}-\Delta\lambda$ where the $\pm$ signs are associated with the east/west direction of the geodesic.

ψ_0 can be found using Newton's iterative method (Williams 1972)

$$\left(\psi_0\right)_{n+1}=\left(\psi_0\right)_n-\frac{f\left(\psi_0\right)}{f'\left(\psi_0\right)} \tag{54}$$

where $f'\left(\psi_0\right)$ is the derivative of $f\left(\psi_0\right)$. An initial value of ψ_0 can be computed from equation (11).

4. Once ψ_0 is known then $s_1,\Delta\lambda_1;\, s_2,\Delta\lambda_2$ and $s_4,\Delta\lambda_4$ can be computed from the distance and longitude equations and *s* obtained. The forward and reverse azimuths can be found from Clairaut's equation (5).

CONCLUSION

Pittman's (1986) recurrence relationships for evaluating integrals allow beautifully compact equations for distance *s* and longitude difference $\Delta\lambda$ along a geodesic between *P* and the vertex P_0. These equations can be easily translated into a computer program function returning *s* and $\Delta\lambda$ given *a, f, u* and u_0. Using such a function, algorithms (as outlined above), can be constructed to solve the direct and inverse problems on the ellipsoid. Pittman's (1986) paper (which included FORTRAN computer code) has a concise development of the necessary equations and algorithms. The paper here has a more detailed development of the recurrence relationships (with a slightly different formulation) as well as additional information on the definition and properties of a geodesic.

Interestingly, Pittman's (1986) method is entirely different to other approaches that fall (roughly) into two groups: (i) numerical integration techniques and (ii) series expansion of integrals; the latter of these with a history of development extending back to Bessel's (1826) method. Numerical integration, a technique made practical with the arrival of computers in the mid to late 20th century, is relatively modern. So too is Pittman's method.

To our knowledge, this is the first paper (since the original) discussing his elegant method; a method that has much to recommend it, and one that we hope might become the object of study in undergraduate surveying courses and discussion in the geodetic literature.

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THE NORMAL SECTION CURVE ON AN ELLIPSOID

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ABSTRACT

These notes provide a detailed derivation of the equation for a normal section curve on an ellipsoid and from this equation a technique for computing the arc length along a normal section curve is developed. Solutions for the direct and inverse problems of the normal section on an ellipsoid are given and MATLAB functions are provided showing the algorithms developed.

INTRODUCTION

In geodesy, the normal section curve is a plane curve created by intersecting a plane containing the normal to the ellipsoid (a normal section plane) with the surface of the ellipsoid, and the ellipsoid is a reference surface approximating the true shape of the Earth. In general, there are two normal section curves between two points on an ellipsoid, a fact that will be explained below, so the normal section curve is not a unique curve. And the distance along a normal section curve is not the shortest distance between two points. The shortest distance is along the <u>geodesic</u>, a unique curve on the surface defining the shortest distance, but the difference in length between the normal section and a geodesic can be shown to be negligible in all practical cases.

The azimuth of a normal section plane between two points on an ellipsoid can be easily determined by coordinate geometry if the latitudes and longitudes of the points are expressed in a local Cartesian coordinate system – this will be explained in detail below.

The distance along a normal section curve can be determined by numerical integration once the polar equation of the curve is known. And the derivation of the polar equation of a normal section curve is developed in detail by first proving that normal sections of ellipsoids are in fact ellipses, then deriving Cartesian equations of the ellipsoid and the normal section in local Cartesian coordinates and finally transforming the local Cartesian coordinates to polar coordinates. The differential equation for arc length (as a function of polar coordinates) is derived and a solution using a numerical technique known as Romberg integration is developed for the arc length along a normal section curve.

The azimuth of the normal section as a function of Cartesian coordinates); the polar equation of the normal section curve; and the solution of the arc length using Romberg integration are the core components of solutions of the direct and inverse cases of the normal sections on an ellipsoid. These are fundamental geodetic operations and can be likened to the equivalent operations of plane surveying; radiations (computing coordinates of points given bearings and distances radiating from a point of known coordinates) and joins; (computing bearings and distances between points having known coordinates). The solution of the direct and inverse cases of the normal section are set out in detail and MATLAB functions are provided.

THE ELLIPSOID

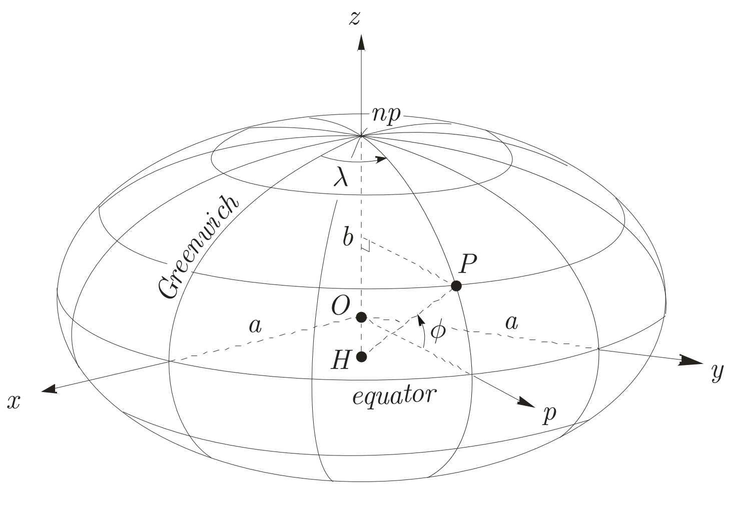


Figure 1: The reference ellipsoid

In geodesy, the ellipsoid is a surface of revolution created by rotating an ellipse (whose major and minor semi-axes lengths are a and b respectively and $a > b$) about its minor axis. The ϕ, λ curvilinear coordinate system is a set of orthogonal parametric curves on the surface – parallels of latitude ϕ and meridians of longitude λ with their respective reference planes; the equator and the Greenwich meridian.

Longitudes are measured 0° to $\pm 180^\circ$ (east positive, west negative) from the Greenwich meridian and latitudes are measured 0° to $\pm 90^\circ$ (north positive, south negative) from the equator. The x,y,z geocentric Cartesian coordinate system has an origin at O, the centre of the ellipsoid, and the z-axis is the minor axis (axis of revolution). The xOz plane is the Greenwich meridian plane (the origin of longitudes) and the xOy plane is the equatorial plane.

The positive x-axis passes through the intersection of the Greenwich meridian and the equator, the positive y-axis is advanced 90° east along the equator and the positive z-axis passes through the north pole of the ellipsoid.

The Cartesian equation of the ellipsoid is

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1 \tag{1}$$

where a and b are the semi-axes of the ellipsoid $(a > b)$.

The first-eccentricity squared e^2 and the flattening f of the ellipsoid are defined by

$$\begin{aligned} e^2 &= \frac{a^2 - b^2}{a^2} = f(2-f) \\ f &= \frac{a-b}{a} \end{aligned} \tag{2}$$

and the polar radius c, and the second-eccentricity squared e'^2 are defined by

$$\begin{aligned} c &= \frac{a^2}{b} = \frac{a}{1-f} \\ e'^2 &= \frac{a^2 - b^2}{b^2} = \frac{f(2-f)}{(1-f)^2} = \frac{e^2}{1-e^2} \end{aligned} \tag{3}$$

PROOF THAT NORMAL SECTION CURVES ARE ELLIPSES

Normal section curves are plane curves; i.e., curves on the surface of the ellipsoid created by intersecting the surface with a plane; and this plane (the normal section plane) contains the normal to the surface at one of the terminal points.

A meridian of longitude is also a normal section curve and <u>all meridians of longitude on the ellipsoid are ellipses</u> having semi-axes a and b $(a > b)$ since all meridian planes – e.g., Greenwich meridian plane xOz and the meridian plane pOz containing P – contain the z-axis of the ellipsoid and their curves of intersection are ellipses (planes intersecting surfaces create curves of intersection on the surface). This can be seen if we let $p^2 = x^2 + y^2$ in equation (1) which gives the familiar equation of the (meridian) ellipse

$$\frac{p^2}{a^2} + \frac{z^2}{b^2} = 1 \qquad (a < b) \tag{4}$$

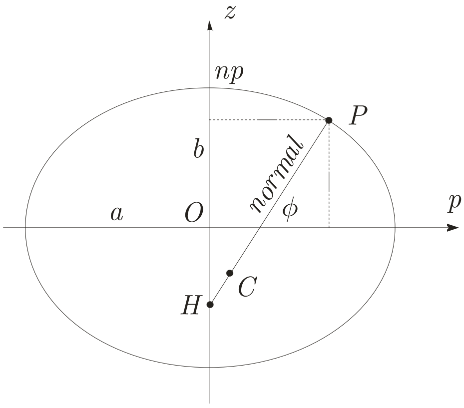


Figure 2: Meridian ellipse

In Figure 2, ϕ is the latitude of P (the angle between the equator and the normal), C is the centre of curvature and PC is the radius of curvature of the meridian ellipse at P. H is the intersection of the normal at P and the z-axis (axis of revolution).

The only parallel of latitude that is also a normal section is the equator. And in this unique case, this normal section curve (the equator) is a circle. <u>All parallels of latitude on the ellipsoid are circles</u> created by intersecting the ellipsoid with planes parallel to (or coincident with) the xOy equatorial plane. Replacing z with a constant C in equation (1) gives the equation for circular parallels of latitude

$$x^2 + y^2 = a^2\left(1 - \frac{C^2}{b^2}\right) = p^2 \qquad \left(0 \le C \le b; \;\; a > b\right) \tag{5}$$

<u>All other curves on the surface of the ellipsoid created by intersecting the ellipsoid with a plane are ellipses</u>. And this general statement covers all normal section planes that are not meridians or the equator. This can be demonstrated by using another set of coordinates x', y', z' that are obtained by a rotation of the x,y,z coordinates such that

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \mathbf{R}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \text{where} \quad \mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

where $\mathbf{R}$ is an orthogonal rotation matrix and $\mathbf{R}^{-1} = \mathbf{R}^T$ so

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{R}^{-1} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

giving

$$\begin{aligned}
x^2 &= r_{11}^2 x'^2 + r_{21}^2 y'^2 + r_{31}^2 z'^2 + 2r_{11}r_{21}x'y' + 2r_{11}r_{31}x'z' + 2r_{21}r_{31}y'z' \\
y^2 &= r_{12}^2 x'^2 + r_{22}^2 y'^2 + r_{32}^2 z'^2 + 2r_{12}r_{22}x'y' + 2r_{12}r_{32}x'z' + 2r_{22}r_{32}y'z' \\
z^2 &= r_{13}^2 x'^2 + r_{23}^2 y'^2 + r_{33}^2 z'^2 + 2r_{13}r_{23}x'y' + 2r_{13}r_{33}x'z' + 2r_{23}r_{33}y'z' \\
x^2 + y^2 &= \left(r_{11}^2 + r_{12}^2\right)x'^2 + \left(r_{21}^2 + r_{22}^2\right)y'^2 + \left(r_{31}^2 + r_{32}^2\right)z'^2 + 2\left(r_{11}r_{21} + r_{12}r_{22}\right)x'y' \\
&\quad +2\left(r_{11}r_{31} + r_{12}r_{32}\right)x'z' + 2\left(r_{21}r_{31} + r_{22}r_{32}\right)y'z'
\end{aligned}$$

Substituting into equation (1) gives the equation of the ellipsoid in x', y', z' coordinates

$$\begin{aligned}
&\frac{1}{a^2}\begin{Bmatrix} \left(r_{11}^2 + r_{12}^2\right)x'^2 + \left(r_{21}^2 + r_{22}^2\right)y'^2 + \left(r_{31}^2 + r_{32}^2\right)z'^2 + 2\left(r_{11}r_{21} + r_{12}r_{22}\right)x'y' \\ +2\left(r_{11}r_{31} + r_{12}r_{32}\right)x'z' + 2\left(r_{21}r_{31} + r_{22}r_{32}\right)y'z' \end{Bmatrix} \\
&+\frac{1}{b^2}\left\{r_{13}^2 x'^2 + r_{23}^2 y'^2 + r_{33}^2 z'^2 + 2r_{13}r_{23}x'y' + 2r_{13}r_{33}x'z' + 2r_{23}r_{33}y'z'\right\} = 1
\end{aligned} \tag{6}$$

In equation (6) let $z' = C_1$ where C_1 is a constant. The result will be the equation of a curve created by intersecting an inclined plane with the ellipsoid, i.e.,

$$\begin{aligned}
&\left\{\frac{r_{11}^2 + r_{12}^2}{a^2} + \frac{r_{13}^2}{b^2}\right\}x'^2 + 2\left\{\frac{r_{11}r_{21} + r_{12}r_{22}}{a^2} + \frac{r_{13}r_{23}}{b^2}\right\}x'y' + \left\{\frac{r_{21}^2 + r_{22}^2}{a^2} + \frac{r_{23}^2}{b^2}\right\}y'^2 \\
&+\left\{2C_1\left(r_{11}r_{31} + r_{12}r_{32} + r_{13}r_{33}\right)\right\}x' + \left\{2C_1\left(r_{21}r_{31} + r_{22}r_{32} + r_{23}r_{33}\right)\right\}y' \\
&= 1 - C_1^2\left\{r_{31}^2 + r_{32}^2 + r_{33}^2\right\}
\end{aligned} \tag{7}$$

This equation can be expressed as

$$Ax'^2 + 2Hx'y' + By'^2 + Dx' + Ey' = 1 \tag{8}$$

where it can be shown that $AB - H^2 > 0$, hence it is the general Cartesian equation of an ellipse that is offset from the coordinate origin and rotated with respect to the coordinate axes (Grossman 1981). Equations of a similar form can be obtained for inclined planes $x' = C_2$ and $y' = C_3$, hence we may say, in general, inclined planes intersecting the ellipsoid will create curves of intersection that are ellipses.

NORMAL SECTION CURVES BETWEEN P_1 AND P_2 ON THE ELLIPSOID

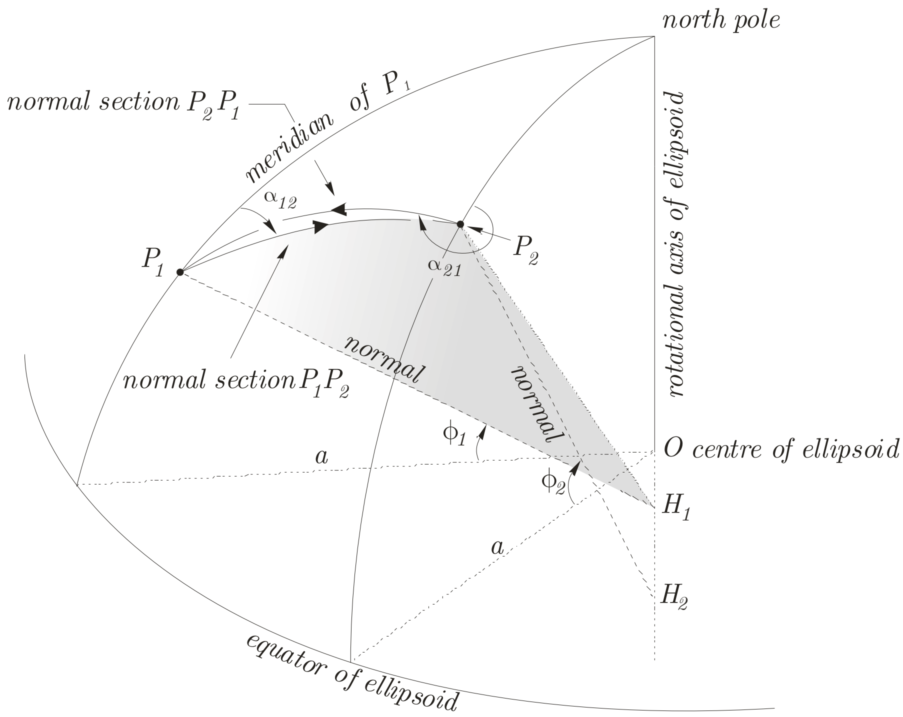


Figure 3: Normal section curves between P_1 and P_2 on the ellipsoid

Figure 3 shows P_1 and P_2 on the surface of an ellipsoid. The normals at P_1 and P_2 (that lie in the meridian planes ONP_1H_1 and ONP_2H_2 respectively) cut the rotational axis at H_1 and H_2, making angles ϕ_1, ϕ_2 with the equatorial plane of the ellipsoid. These are the latitudes of P_1 and P_2 respectively.

The plane containing the ellipsoid normal at P_1, and also the point P_2 intersects the surface of the ellipsoid along the normal section curve P_1P_2. The reciprocal normal section curve P_2P_1 (the intersection of the plane containing the normal at P_2, and also the point P_1 with the ellipsoidal surface) does not in general coincide with the normal section curve P_1P_2 although the distances along the two curves are, for all practical purposes, the same.

Hence there is not a unique normal section curve between P_1 and P_2, unless both P_1 and P_2 are on the same meridian or both are on the equator.

The azimuth α_{12}, is the clockwise angle (0° to 360°) measured at P_1 in the local horizon plane from north (the direction of the meridian) to the normal section plane containing P_2. The azimuth α_{21} is the azimuth of the normal section plane P_2P_1 measured at P_2.

LOCAL CARTESIAN COORDINATES

Figure 4 shows a local Cartesian coordinate system E,N,U with an origin at P on the reference ellipsoid with respect to the geocentric Cartesian system x,y,z whose origin is a the centre of the ellipsoid

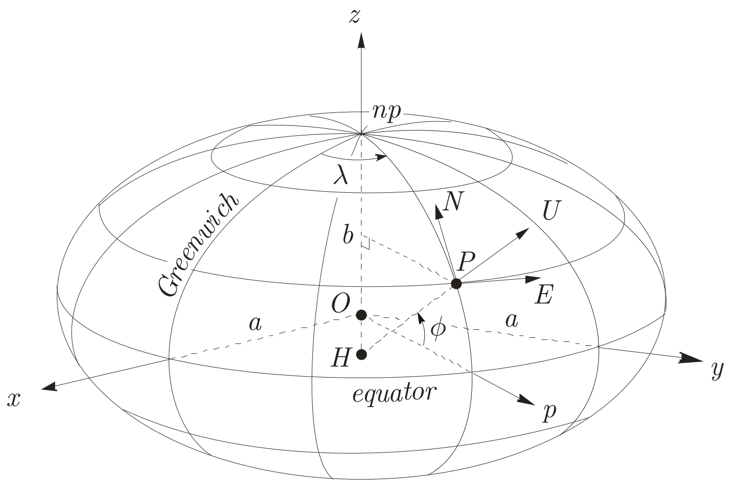


Figure 4: x,y,z geocentric Cartesian and E,N,U local Cartesian coordinates

Geocentric x,y,z Cartesian coordinates are computed from the following equations

$$\begin{aligned} x &= \nu \cos\phi \cos\lambda \\ y &= \nu \cos\phi \sin\lambda \\ z &= \nu\left(1-e^2\right)\sin\phi \end{aligned} \tag{9}$$

where $\nu = PH$ in Figure 4 is the radius of curvature in the prime vertical plane and

$$\nu = \frac{a}{\sqrt{1-e^2\sin^2\phi}} \tag{10}$$

The origin of the local E,N,U system lies at the point $P\left(\phi_0,\lambda_0\right)$. The positive *U-axis* is coincident with the normal to the ellipsoid passing through P and in the direction of increasing radius of curvature ν. The N-U plane lies in the meridian plane passing through P and the positive N-axis points in the direction of North. The E-U plane is perpendicular to the N-U plane and the positive E-axis points East. The E-N plane is often referred to as the local geodetic horizon plane.

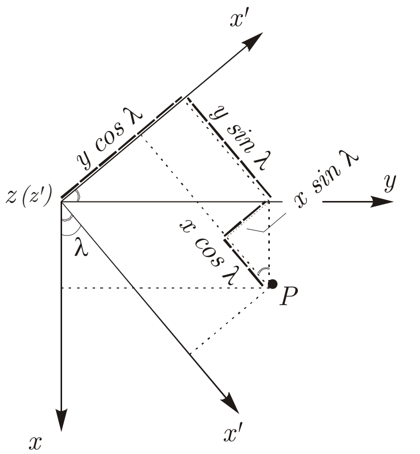
Geocentric and local Cartesian coordinates are related by the matrix equation

$$\begin{bmatrix} U \\ E \\ N \end{bmatrix} = \mathbf{R}_{\phi\lambda} \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} \tag{11}$$

where x_0, x_0, z_0 are the geocentric Cartesian coordinates of the origin of the E,N,U system and $\mathbf{R}_{\phi\lambda}$ is a rotation matrix derived from the product of two separate rotation matrices.

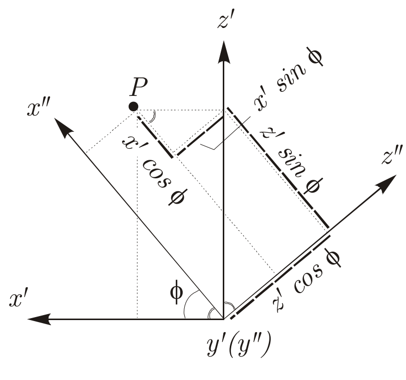
$$\mathbf{R}_{\phi\lambda} = \mathbf{R}_{\phi}\mathbf{R}_{\lambda} = \begin{bmatrix} \cos\phi_0 & 0 & \sin\phi_0 \\ 0 & 1 & 0 \\ -\sin\phi_0 & 0 & \cos\phi_0 \end{bmatrix}\begin{bmatrix} \cos\lambda_0 & \sin\lambda_0 & 0 \\ -\sin\lambda_0 & \cos\lambda_0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{12}$$

The first, $\mathbf{R}_{\lambda}$ (a positive right-handed rotation about the x-axis by λ) takes the x,y,z axes to x',y',z'. The z'-axis is coincident with the *z-axis* and the x'-y' plane is the Earth's equatorial plane. The x'-y' plane is the meridian plane passing through P and the y'-axis is perpendicular to the meridian plane and in the direction of East.



$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\lambda & \sin\lambda & 0 \\ -\sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{R}_{\lambda}}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The second $\mathbf{R}_{\phi}$ (a rotation about the y'-axis by ϕ) takes the x',y',z' axes to the x'',y'',z'' axes. The x''-axis is parallel to the U-axis, the y''-axis is parallel to the E-axis and the z''-axis is parallel to the N-axis.



$$\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}}_{\mathbf{R}_{\phi}}\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

Performing the matrix multiplication in equation (12) gives

$$\mathbf{R}_{\phi\lambda} = \begin{bmatrix} \cos\phi_0 \cos\lambda_0 & \cos\phi_0 \sin\lambda_0 & \sin\phi_0 \\ -\sin\lambda_0 & \cos\lambda_0 & 0 \\ -\sin\phi_0 \cos\lambda_0 & -\sin\phi_0 \sin\lambda_0 & \cos\phi_0 \end{bmatrix} \tag{13}$$

Rotation matrices formed from rotations about coordinate axes are often called Euler rotation matrices in honour of the Swiss mathematician Léonard Euler (1707-1783). They are orthogonal, satisfying the condition $\mathbf{R}^T\mathbf{R} = \mathbf{I}$ (i.e., $\mathbf{R}^{-1} = \mathbf{R}^T$).

A re-ordering of the rows of the matrix $\mathbf{R}_{\phi\lambda}$ gives the transformation in the more usual form E,N,U

$$\begin{bmatrix} E \\ N \\ U \end{bmatrix} = \mathbf{R} \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} \tag{14}$$

where
$$\mathbf{R} = \begin{bmatrix} -\sin\lambda_0 & \cos\lambda_0 & 0 \\ -\sin\phi_0 \cos\lambda_0 & -\sin\phi_0 \sin\lambda_0 & \cos\phi_0 \\ \cos\phi_0 \cos\lambda_0 & \cos\phi_0 \sin\lambda_0 & \sin\phi_0 \end{bmatrix} \tag{15}$$

From equation (14) we can see that coordinate differences $\Delta E = E_k - E_i$, $\Delta N = N_k - N_i$ and $\Delta U = U_k - U_i$ in the local geodetic horizon plane are given by

$$\begin{bmatrix} \Delta E \\ \Delta N \\ \Delta U \end{bmatrix} = \mathbf{R} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \tag{16}$$

where $\Delta x = x_k - x_i$, $\Delta y = y_k - y_i$ and $\Delta z = z_k - z_i$ are geocentric Cartesian coordinate differences.

NORMAL SECTION AZIMUTH ON THE ELLIPSOID

The matrix relationship given by equation (16) can be used to derive an expression for the azimuth of a normal section between two points on the reference ellipsoid. The normal section plane between points P_1 and P_2 on the Earth's terrestrial surface contains the normal at point P_1, the intersection of the normal and the rotational axis of the ellipsoid at H_1 (see Figure 3) and P_2. This plane will intersect the local geodetic horizon plane in a line having an angle with the north axis, which is the direction of the meridian at P_1.

This angle is the azimuth of the normal section plane $P_1 - P_2$ denoted as α_{12} and will have components ΔE and ΔN in the local geodetic horizon plane. From plane geometry

$$\tan \alpha_{12} = \frac{\Delta E}{\Delta N} \tag{17}$$

By inspection of equations (15) and (16) we may write the equation for normal section azimuth between points P_1 and P_2 as

$$\tan \alpha_{12} = \frac{\Delta E}{\Delta N} = \frac{-\Delta x \sin \lambda_1 + \Delta y \cos \lambda_1}{-\Delta x \sin \phi_1 \cos \lambda_1 - \Delta y \sin \phi_1 \sin \lambda_1 + \Delta z \cos \phi_1} \tag{18}$$

where $\Delta x = x_2 - x_1$, $\Delta y = y_2 - y_1$ and $\Delta z = z_2 - z_1$

EQUATION OF THE ELLIPSOID IN LOCAL CARTESIAN COORDINATES

The Cartesian equation of the ellipsoid is given by equation (1) as

$$\boxed{\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1} \tag{19}$$

and multiplying both sides of equation (19) by a^2 gives

$$x^2 + y^2 + \frac{a^2}{b^2} z^2 = a^2 \tag{20}$$

Re-arranging equation (3) gives $\dfrac{a^2}{b^2} = e'^2 + 1$ and substituting this result into equation (20) and re-arranging gives an alternative expression for the Cartesian equation of an ellipsoid as

$$\boxed{x^2 + y^2 + z^2 + e'^2 z^2 - a^2 = 0} \tag{21}$$

We now find expressions for x^2, y^2 and z^2 in terms of local Cartesian coordinates that when substituted into equation (21) and simplified will give the equation of the ellipsoid in local Cartesian coordinates. The relevant substitutions are set out below.

The relationship between geocentric and local Cartesian coordinates is given by equation (14) as

$$\begin{bmatrix} E \\ N \\ U \end{bmatrix} = \mathbf{R} \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} \tag{22}$$

where the orthogonal rotation matrix **R** is given by equation (15) as

$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} -\sin\lambda_0 & \cos\lambda_0 & 0 \\ -\sin\phi_0\cos\lambda_0 & -\sin\phi_0\sin\lambda_0 & \cos\phi_0 \\ \cos\phi_0\cos\lambda_0 & \cos\phi_0\sin\lambda_0 & \sin\phi_0 \end{bmatrix} \tag{23}$$

and

$$\begin{aligned} x_0 &= \nu_0\cos\phi_0\cos\lambda_0 \\ y_0 &= \nu_0\cos\phi_0\sin\lambda_0 \\ z_0 &= \nu_0\left(1-e^2\right)\sin\phi_0 \end{aligned} \tag{24}$$

with the radius of curvature of the prime vertical section

$$\nu_0 = \frac{a}{\sqrt{1-e^2\sin^2\phi_0}} \tag{25}$$

Re-arranging equation (22) gives

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{R}^{-1}\begin{bmatrix} E \\ N \\ U \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \tag{26}$$

where

$$\mathbf{R}^{-1} = \mathbf{R}^T = \begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix} \tag{27}$$

Expanding equation (26) gives

$$\begin{aligned} x &= r_{11}E + r_{21}N + r_{31}U + x_0 \\ y &= r_{12}E + r_{22}N + r_{32}U + y_0 \\ z &= r_{13}E + r_{23}N + r_{33}U + z_0 \end{aligned} \tag{28}$$

and

$$\begin{aligned} x^2 &= r_{11}^2E^2 + r_{21}^2N^2 + r_{31}^2U^2 + 2r_{11}r_{21}EN + 2r_{11}r_{31}EU + 2r_{21}r_{31}NU \\ &\quad + x_0^2 + 2r_{11}Ex_0 + 2r_{21}Nx_0 + 2r_{31}Ux_0 \\ y^2 &= r_{12}^2E^2 + r_{22}^2N^2 + r_{31}^2U^2 + 2r_{12}r_{22}EN + 2r_{12}r_{32}EU + 2r_{22}r_{32}NU \\ &\quad + y_0^2 + 2r_{12}Ey_0 + 2r_{22}Ny_0 + 2r_{32}Uy_0 \\ z^2 &= r_{13}^2E^2 + r_{23}^2N^2 + r_{33}^2U^2 + 2r_{13}r_{23}EN + 2r_{13}r_{33}EU + 2r_{23}r_{33}NU \\ &\quad + z_0^2 + 2r_{13}Ez_0 + 2r_{23}Nz_0 + 2r_{33}Uz_0 \end{aligned} \tag{29}$$

with

$$
\begin{aligned}
x^2+y^2+z^2 = {} & \left(r_{11}^2+r_{12}^2+r_{13}^2\right)E^2+\left(r_{21}^2+r_{22}^2+r_{23}^2\right)N^2+\left(r_{31}^2+r_{32}^2+r_{33}^2\right)U^2 \\
& +2\left(r_{11}r_{21}+r_{12}r_{22}+r_{13}r_{23}\right)EN \\
& +2\left(r_{11}r_{31}+r_{12}r_{32}+r_{13}r_{33}\right)EU \\
& +2\left(r_{21}r_{31}+r_{22}r_{32}+r_{23}r_{33}\right)NU \\
& +x_0^2+y_0^2+z_0^2 \\
& +2\left(r_{11}x_0+r_{12}y_0+r_{13}z_0\right)E \\
& +2\left(r_{21}x_0+r_{22}y_0+r_{23}z_0\right)N \\
& +2\left(r_{31}x_0+r_{32}y_0+r_{33}z_0\right)U
\end{aligned}
\tag{30}
$$

Now using the equivalences for $r_{11}, r_{12}, \text{etc}$ given in equation (23), certain terms in equation (30) can be simplified as

$$
\begin{aligned}
r_{11}^2+r_{12}^2+r_{13}^2 &= \sin^2\lambda_0+\cos^2\lambda_0 = 1 \\
r_{21}^2+r_{22}^2+r_{23}^2 &= \sin^2\phi_0\left(\cos^2\lambda_0+\sin^2\lambda_0\right)+\cos^2\phi_0 = 1 \\
r_{31}^2+r_{32}^2+r_{33}^2 &= \cos^2\phi_0\left(\cos^2\lambda_0+\sin^2\lambda_0\right)+\sin^2\phi_0 = 1
\end{aligned}
$$

and

$$
\begin{aligned}
r_{11}r_{21}+r_{12}r_{22}+r_{13}r_{23} &= \sin\lambda_0\sin\phi_0\cos\lambda_0-\cos\lambda_0\sin\phi_0\sin\lambda_0+0 \\
&= 0 \\
r_{11}r_{31}+r_{12}r_{32}+r_{13}r_{33} &= -\sin\lambda_0\cos\phi_0\cos\lambda_0+\cos\lambda_0\cos\phi_0\sin\lambda_0+0 \\
&= 0 \\
r_{21}r_{31}+r_{22}r_{32}+r_{23}r_{33} &= -\sin\phi_0\cos\phi_0\cos^2\lambda_0-\sin\phi_0\cos\phi_0\sin^2\lambda_0+\cos\phi_0\sin\phi_0 \\
&= -\sin\phi_0\cos\phi_0\left(\cos^2\lambda_0+\sin^2\lambda_0\right)+\cos\phi_0\sin\phi_0 \\
&= 0
\end{aligned}
$$

Substituting these results into equation (30) gives

$$
\begin{aligned}
x^2+y^2+z^2 = {} & E^2+N^2+U^2+x_0^2+y_0^2+z_0^2 \\
& +2\left(r_{11}x_0+r_{12}y_0+r_{13}z_0\right)E \\
& +2\left(r_{21}x_0+r_{22}y_0+r_{23}z_0\right)N \\
& +2\left(r_{31}x_0+r_{32}y_0+r_{33}z_0\right)U
\end{aligned}
\tag{31}
$$

Using equation (24) and noting that equation (25) can be re-arranged as $1-e^2\sin^2\phi_0=\dfrac{a^2}{\nu_0^2}$ we have

$$
\begin{aligned}
x_0^2 + y_0^2 + z_0^2 &= \nu_0^2 \cos^2 \phi_0 \left(\cos^2 \lambda_0 + \sin^2 \lambda_0\right) + \nu_0^2 \left(1 - e^2\right)^2 \sin^2 \phi_0 \\
&= \nu_0^2 \cos^2 \phi_0 + \nu_0^2 \sin^2 \phi_0 \left(1 - 2e^2 + e^4\right) \\
&= \nu_0^2 \cos^2 \phi_0 + \nu_0^2 \sin^2 \phi_0 - 2\nu_0^2 e^2 \sin^2 \phi_0 + \nu_0^2 e^4 \sin^2 \phi_0 \\
&= \nu_0^2 - 2\nu_0^2 e^2 \sin^2 \phi_0 + \nu_0^2 e^4 \sin^2 \phi_0 \\
&= \nu_0^2 \left(1 - e^2 \sin^2 \phi_0\right) - \nu_0^2 e^2 \sin^2 \phi_0 \left(1 - e^2\right) \\
&= a^2 - \left(\nu_0^2 - a^2\right)\left(1 - e^2\right)
\end{aligned}
$$

From equations (31), (23) and (24) we have

$$
\begin{aligned}
r_{11} x_0 + r_{12} y_0 + r_{13} z_0 &= -\nu_0 \cos\phi_0 \cos\lambda_0 \sin\lambda_0 + \nu_0 \cos\phi_0 \sin\lambda_0 \cos\lambda_0 + 0 \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
r_{21} x_0 + r_{22} y_0 + r_{23} z_0 &= -\nu_0 \cos\phi_0 \sin\phi_0 \cos^2 \lambda_0 - \nu_0 \sin\phi_0 \cos\phi_0 \sin^2 \lambda_0 \\
&\quad + \nu_0 \left(1 - e^2\right) \sin\phi_0 \cos\phi_0 \\
&= -\nu_0 \cos\phi_0 \sin\phi_0 \left(\cos^2 \lambda_0 + \sin^2 \lambda_0 - 1 + e^2\right) \\
&= -\nu_0 e^2 \cos\phi_0 \sin\phi_0
\end{aligned}
$$

and

$$
\begin{aligned}
r_{31} x_0 + r_{32} y_0 + r_{33} z_0 &= \nu_0 \cos^2 \phi_0 \cos^2 \lambda_0 + \nu_0 \cos^2 \phi_0 \sin^2 \lambda_0 + \nu_0 \left(1 - e^2\right) \sin^2 \phi_0 \\
&= \nu_0 \cos^2 \phi_0 + \nu_0 \left(1 - e^2\right) \sin^2 \phi_0 \\
&= \nu_0 \cos^2 \phi_0 + \nu_0 \sin^2 \phi_0 - \nu_0 e^2 \sin^2 \phi_0 \\
&= \nu_0 \left(1 - e^2 \sin^2 \phi_0\right) \\
&= a^2
\end{aligned}
$$

Substituting these results into equation (31) gives

$$
\begin{aligned}
x^2 + y^2 + z^2 = E^2 + N^2 + U^2 &+ \nu_0^2 \left(1 - e^2 \sin^2 \phi_0\right) - \nu_0^2 e^2 \sin^2 \phi_0 \left(1 - e^2\right) \\
&- 2\nu_0 e^2 \sin\phi_0 \cos\phi_0 N + 2\nu_0 \left(1 - e^2 \sin^2 \phi_0\right) U
\end{aligned}
\tag{32}
$$

Using the expression for z^2 given in equation (29), the term $e'^2 z^2$ in equation (21) can be expressed as

$$
\begin{aligned}
e'^2 z^2 = e'^2 \Big\{ & r_{13}^2 E^2 + r_{23}^2 N^2 + r_{33}^2 U^2 + 2 r_{13} r_{23} EN + 2 r_{13} r_{33} EU + 2 r_{23} r_{33} NU \\
& + z_0^2 + 2 r_{13} E z_0 + 2 r_{23} N z_0 + 2 r_{33} U z_0 \Big\}
\end{aligned}
\tag{33}
$$

where

$$
\begin{aligned}
& r_{13}^2 = 0; \quad r_{23}^2 = \cos^2 \phi; \quad r_{33}^2 = \sin^2 \phi; \\
& 2 r_{13} r_{23} = 0; \quad 2 r_{13} r_{33} = 0; \quad 2 r_{23} r_{33} = 2 \cos\phi_0 \sin\phi_0; \\
& z_0^2 = \nu_0^2 \left(1 - e^2\right)^2 \sin^2 \phi_0; \\
& 2 r_{13} z_0 = 0; \quad 2 r_{23} z_0 = 2\nu_0 \left(1 - e^2\right) \cos\phi_0 \sin\phi_0; \quad 2 r_{33} z_0 = 2\nu_0 \left(1 - e^2\right) \sin^2 \phi_0
\end{aligned}
$$

and equation (33) can be expressed as

$$
\begin{aligned}
e'^2 z^2 &= e'^2\left(\cos^2\phi_0 N^2 + \sin^2\phi_0 U^2 + 2\cos\phi_0\sin\phi_0 NU\right)\\
&\quad + e'^2\left(\nu_0^2\left(1-e^2\right)^2\sin^2\phi_0 + 2\nu_0\left(1-e^2\right)\cos\phi_0\sin\phi_0 N + 2\nu_0\left(1-e^2\right)\sin^2\phi_0 U\right)
\end{aligned}
$$

But $e'^2 = \dfrac{e^2}{1-e^2}$ so we may write

$$
\begin{aligned}
e'^2 z^2 &= e'^2\left(\cos\phi_0\, N + \sin\phi_0\, U\right)^2\\
&\quad + \frac{e^2}{1-e^2}\left(\nu_0^2\left(1-e^2\right)^2\sin^2\phi_0 + 2\nu_0\left(1-e^2\right)\cos\phi_0\sin\phi_0 N + 2\nu_0\left(1-e^2\right)\sin^2\phi_0 U\right)\\
&= e'^2\left(\cos\phi_0\, N + \sin\phi_0\, U\right)^2\\
&\quad + \nu_0^2\left(1-e^2\right)^2 e^2\sin^2\phi_0 + 2\nu_0 e^2\cos\phi_0\sin\phi_0 N + 2\nu_0 e^2\sin^2\phi_0 U
\end{aligned}
\tag{34}
$$

Substituting equations (32) and (34) into equation (21) gives

$$
\begin{aligned}
E^2 + N^2 + U^2 &+ e'^2\left(\cos\phi_0\, N + \sin\phi_0\, U\right)^2 - a^2\\
&+ \nu_0^2\left(1-e^2\sin^2\phi_0\right) - \nu_0^2 e^2\sin^2\phi_0\left(1-e^2\right)\\
&- 2\nu_0 e^2\sin\phi_0\cos\phi_0\, N + 2\nu_0\left(1-e^2\sin^2\phi_0\right)U\\
&+ \nu_0^2 e^2\sin^2\phi_0\left(1-e^2\right) + 2\nu_0 e^2\sin\phi_0\cos\phi_0\, N + 2\nu_0 e^2\sin^2\phi_0\, U = 0
\end{aligned}
$$

And simplifying and noting that $\nu_0^2\left(1-e^2\sin^2\phi_0\right) = a^2$ gives the Cartesian equation of the ellipsoid in local coordinates E, N, U as

$$
\boxed{E^2 + N^2 + U^2 + e'^2\left(\cos\phi_0\, N + \sin\phi_0\, U\right)^2 + 2\nu_0 U = 0}
\tag{35}
$$

The origin of the E, N, U system is at P_1 with coordinates ϕ_0, λ_0 where the radius of curvature of the prime vertical section is $\nu_0 = \dfrac{a}{\left(1-e^2\sin^2\phi_0\right)^{\frac{1}{2}}}$ and the first and second eccentricities of the ellipsoid $\left(a, f\right)$ are obtained from $e^2 = f\left(2-f\right)$ and $e'^2 = \dfrac{e^2}{1-e^2}$

Equation (35) is similar to an equation given by Bowring (1978, p. 363, equation (10) with $x \equiv N$ $\;y \equiv -U$, $z \equiv E$). Bowring does not give a derivation, but notes that his equation is taken from Tobey (1928).

CARTESIAN EQUATION OF THE NORMAL SECTION CURVE

The Cartesian equation of the normal section curve is developed as a function of local Cartesian coordinates ζ, η, ξ which are rotated from the local E,N,U system by the azimuth α of the normal section plane.

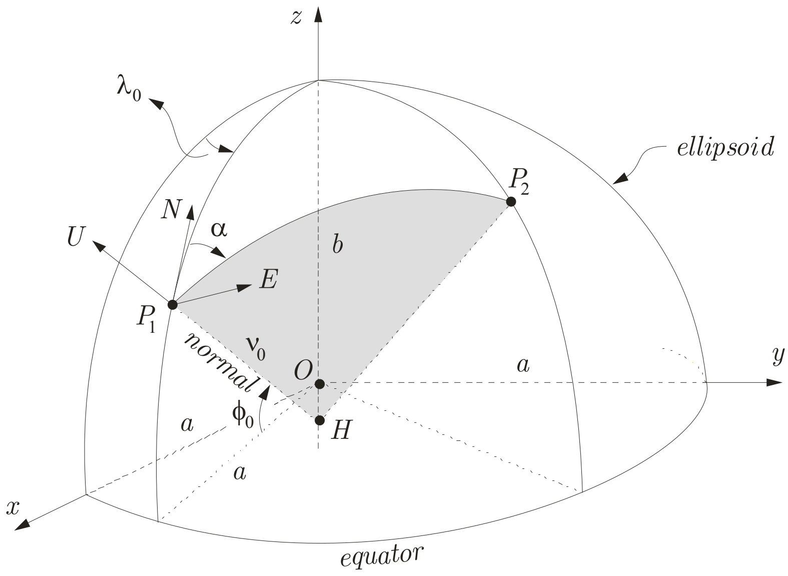


Figure 5: Normal section plane between P_1 and P_2 on the ellipsoid

Figure 5 shows a normal section plane having an azimuth α between P_1 and P_2 on the ellipsoid and a local Cartesian coordinate system E,N,U with an origin at P_1.

Cartesian equations of the ellipsoid in geocentric and local coordinates given by equations (1), (21) and (35) are:

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1$$

$$x^2 + y^2 + z^2 + e'^2 z^2 - a^2 = 0$$

$$E^2 + N^2 + U^2 + e'^2 \left(\cos\phi_0 \, N + \sin\phi_0 \, U \right)^2 + 2\nu_0 U = 0$$

Consider a rotation of the E,N,U system about the U-axis by the azimuth α so that the rotated N-axis lies in the normal section plane and the rotated E-axis is perpendicular to the plane. Denote this rotated E,N,U system as ζ, η, ξ shown in Figure 6

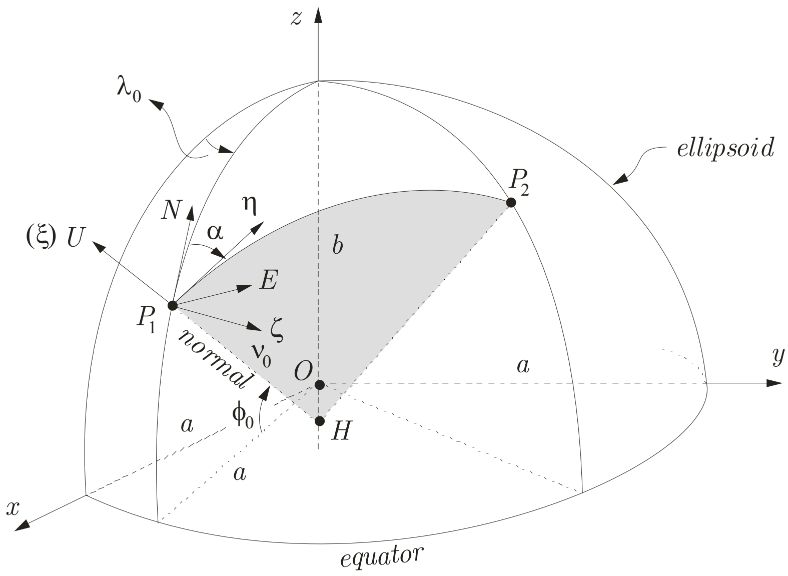
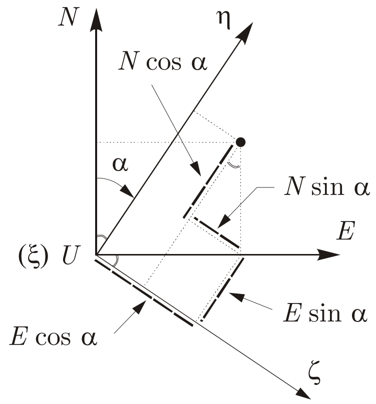


Figure 6: Rotated local coordinate system ζ, η, ξ

These two local Cartesian systems; E, N, U and ζ, η, ξ are related by



$$\begin{bmatrix} \zeta \\ \eta \\ \xi \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E \\ N \\ U \end{bmatrix}$$

and

$$\begin{bmatrix} E \\ N \\ U \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \zeta \\ \eta \\ \xi \end{bmatrix}$$

and we may write

$$\begin{aligned} E &= \zeta\cos\alpha + \eta\sin\alpha; & E^2 &= \zeta^2\cos^2\alpha + \eta^2\sin^2\alpha + 2\zeta\eta\cos\alpha\sin\alpha \\ N &= \eta\cos\alpha - \zeta\sin\alpha; & N^2 &= \zeta^2\sin^2\alpha + \eta^2\cos^2\alpha - 2\zeta\eta\cos\alpha\sin\alpha \\ U &= \xi & U^2 &= \xi^2 \end{aligned} \tag{36}$$

giving

$$E^2 + N^2 + U^2 = \zeta^2 + \eta^2 + \xi^2 \tag{37}$$

Substituting equations (36) and (37) into equation (35) gives

$$\zeta^2 + \eta^2 + \xi^2 + e'^2\left(-\zeta\sin\alpha\cos\phi_0 + \eta\cos\alpha\cos\phi_0 + \xi\sin\phi_0\right)^2 + 2\nu_0\xi = 0 \tag{38}$$

This is the Cartesian equation of an ellipsoid where the local Cartesian coordinates ζ,η,ξ have an origin at $P_1\left(\phi_0,\lambda_0\right)$ on the ellipsoid $\left(a,f\right)$ with the ξ-axis in the direction of the outward normal at P_1; the ξ-η plane is coincident with the normal section plane making an angle α with the meridian plane of P_1; and the ξ-ζ plane is perpendicular to the normal section plane. As before the radius of curvature of the prime vertical section is $\nu_0 = \dfrac{a}{\left(1-e^2\sin^2\phi_0\right)^{\frac{1}{2}}}$ and the first and second eccentricities of the ellipsoid are obtained from $e^2 = f\left(2-f\right)$ and $e'^2 = \dfrac{e^2}{1-e^2}$.

Setting $\zeta = 0$ in equation (38) will give the equation of the normal section plane as

$$\eta^2 + \xi^2 + e'^2\left(\eta\cos\alpha\cos\phi_0 + \xi\sin\phi_0\right)^2 + 2\nu_0\xi = 0 \tag{39}$$

Expanding equation (39) gives

$$\eta^2 + \eta^2 e'^2\cos^2\alpha\cos^2\phi_0 + \xi^2 + \xi^2 e'^2\sin^2\phi_0 + 2\eta\xi e'^2\cos\alpha\cos\phi_0\sin\phi_0 + 2\nu_0\xi = 0$$

which can be simplified to

$$\xi^2\left(1+g^2\right) + 2\,\xi\eta gh + \eta^2\left(1+h^2\right) + 2\nu_0\xi = 0 \tag{40}$$

where g and h are constants of the normal section and

$$\begin{aligned} g &= e'\sin\phi_0 &&= \frac{e}{\sqrt{1-e^2}}\sin\phi_0 \\ h &= e'\cos\alpha\cos\phi_0 &&= \frac{e}{\sqrt{1-e^2}}\cos\alpha\sin\phi_0 \end{aligned} \tag{41}$$

Equation (40) is similar to Clarke (1880, equation 14, p. 107) although Clarke's derivation is different and very concise; taking only 11 lines of text and diagrams.

POLAR EQUATION OF THE NORMAL SECTION CURVE

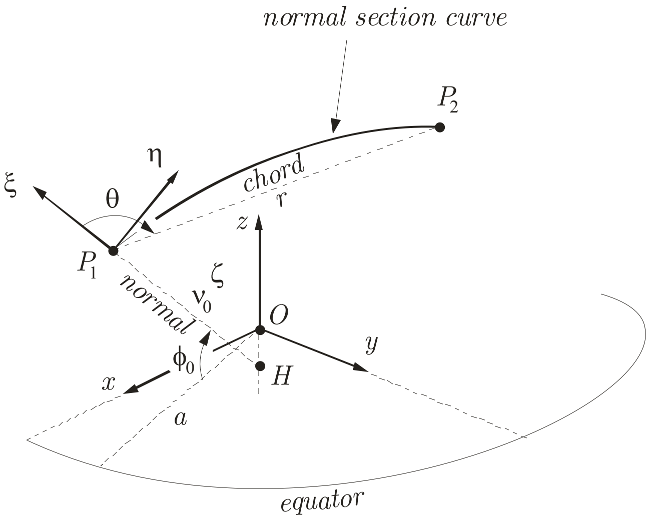


Figure 7: Normal section curve $f\left(\xi,\eta\right)$

The Cartesian equation of the normal section curve in local coordinates $\xi,\eta,\zeta=0$ is given by equations (40) and (41) given the latitude ϕ_0 of P_1, the ellipsoid constant e^2 and the azimuth α of the normal section plane.

The equation of the curve in polar coordinates r,θ; where r is a chord of the curve and θ is the zenith distance of the chord, can be obtained in the following manner.

First, from Figure 7, we may write

$$\begin{aligned}\xi &= r\cos\theta \\ \eta &= r\sin\theta\end{aligned} \tag{42}$$

And second, we may re-arrange equation (40) as

$$\xi^2+\eta^2+\left(g\xi+h\eta\right)^2=-2\nu_0\xi \tag{43}$$

Squaring equations (42) and adding gives

$$\xi^2+\eta^2=r^2\cos^2\theta+r^2\sin^2\theta=r^2 \tag{44}$$

and the third term in equation (43) can be expressed as

$$\begin{aligned}\left(g\xi+h\eta\right)^2 &= \left(g\,r\cos\theta+h\,r\sin\theta\right)^2 \\ &= g^2r^2\cos^2\theta+h^2r^2\sin^2\theta+2gh\,r^2\sin\theta\cos\theta \\ &= r^2\left(g\cos\theta+h\sin\theta\right)^2\end{aligned} \tag{45}$$

Substituting equations (44) and (45) into equation (43) and re-arranging gives the polar equation of the normal section curve

$$\boxed{r = \frac{-2\nu_0 \cos\theta}{1 + \left(g\cos\theta + h\sin\theta\right)^2} \qquad (46)}$$

ARC LENGTH ALONG A NORMAL SECTION CURVE

To evaluate the arc length s along the normal section curve, consider the following

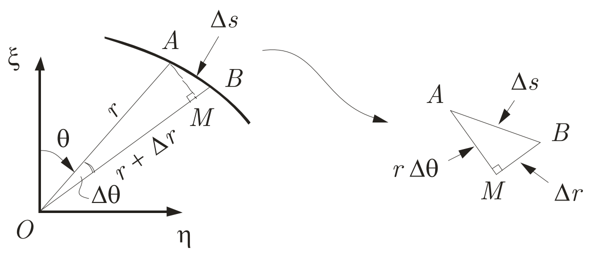


Figure 8: Small element of arc length along a normal section curve

In Figure 8, when $\Delta\theta$ is small, then $AM \simeq r\,\Delta\theta$ and the arc length Δs is approximated by the chord AB and $\left(\Delta s\right)^2 \simeq \left(r\,\Delta\theta\right)^2 + \left(\Delta r\right)^2$ or

$$\begin{aligned}\Delta s &= \sqrt{\left(r\,\Delta\theta\right)^2 + \left(\Delta r\right)^2} \\ &= \sqrt{\left(\Delta\theta\right)^2\left(r^2 + \left(\frac{\Delta r}{\Delta\theta}\right)^2\right)}\end{aligned}$$

and

$$\frac{\Delta s}{\Delta\theta} = \sqrt{r^2 + \left(\frac{\Delta r}{\Delta\theta}\right)^2}$$

Taking the limit of $\dfrac{\Delta s}{\Delta\theta}$ as $\Delta\theta \to 0$ gives

$$\lim_{\Delta\theta\to 0}\left(\frac{\Delta s}{\Delta\theta}\right) = \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \qquad (47)$$

and the arc length is given by

$$s = \int ds = \int_{\theta=\theta_A}^{\theta=\theta_B} \left\{ r^2 + \left(\frac{dr}{d\theta} \right)^2 \right\}^{\frac{1}{2}} d\theta \tag{48}$$

Referring to Figure 7 the η-axis is tangential to the normal section curve P_1P_2 at P_1 and the zenith distance $\theta = \theta_A = \dfrac{\pi}{2}$ and $r = 0$. And when $\theta = \theta_B = \theta_2$ then the chord $r = P_1P_2$ and the arc length of the normal section curve is given by

$$\boxed{s = \int ds = \int_{\theta=\frac{\pi}{2}}^{\theta=\theta_2} \left\{ r^2 + \left(\frac{dr}{d\theta} \right)^2 \right\}^{\frac{1}{2}} d\theta} \tag{49}$$

r is given by equation (46) with normal section constants g and h given by equations (41). The derivative $\dfrac{dr}{d\theta}$ can be obtained from equation (46) using the quotient rule for differential calculus

$$\frac{dr}{d\theta} = \frac{d}{d\theta}\left(\frac{u}{v} \right) = \frac{v\dfrac{du}{d\theta} - u\dfrac{dv}{d\theta}}{v^2} \tag{50}$$

where

$$\begin{aligned} u &= -2\nu_0 \cos\theta; & v &= 1 + \left(g\cos\theta + h\sin\theta \right)^2 \\ \frac{du}{d\theta} &= 2\nu_0 \sin\theta; & \frac{dv}{d\theta} &= 2\left(g\cos\theta + h\sin\theta \right)\left(h\cos\theta - g\sin\theta \right) \end{aligned} \tag{51}$$

The arc length of the normal section curve between P_1 and P_2 can be found by evaluating the integral given in equation (49). This integral cannot be solved analytically but may be evaluated by a numerical integration technique known Romberg integration. Appendix 1 contains a development of the formula used in Romberg integration as well as a MATLAB function demonstrating the algorithm.

SOLVING THE DIRECT AND INVERSE PROBLEMS ON THE ELLIPSOID USING NORMAL SECTIONS

The direct problem on an ellipsoid is: given latitude and longitude of P_1, azimuth α_{12} of the normal section P_1P_2 and the arc length s along the normal section curve; compute the latitude and longitude of P_2.

The inverse problem on an ellipsoid is: given the latitudes and longitudes of P_1 and P_2 compute the azimuth α_{12} and the arc length s along the normal section curve P_1P_2.

Note 1. In general there are two normal section curves joining P_1 and P_2. We are only dealing with the single normal section P_1P_2 (containing the normal at P_1 – see Figure 3) and so only the forward azimuth α_{12} is given or computed. The reverse azimuth α_{21} is the azimuth of the normal section P_2P_1 (containing the normal at P_2) which is a different curve from normal section curve P_1P_2.

Note 2. The usual meaning of: *solving the direct and inverse problems on the ellipsoid* would imply the use of the geodesic; the unique curve defining the shortest distance between two points. And solving these problems is usually done using Bessel's method with Vincenty's equations (Deakin & Hunter 2007) or Pittman's method (Deakin & Hunter 2007).

In the solutions of the direct and inverse problems set out in subsequent sections, the following notation and relationships are used.

- a, f semi-major axis length and flattening of ellipsoid.
- b semi-minor axis length of the ellipsoid, $b = a(1-f)$
- e^2 eccentricity of ellipsoid squared, $e^2 = f(2-f)$
- e'^2 2nd-eccentricity of ellipsoid squared, $e'^2 = \dfrac{e^2}{1-e^2}$
- ϕ, λ latitude and longitude on ellipsoid: ϕ measured 0° to $\pm 90^\circ$ (north latitudes positive and south latitudes negative) and λ measured 0° to $\pm 180^\circ$ (east longitudes positive and west longitudes negative).
- s length of the normal section curve on the ellipsoid.
- α_{12} azimuth of normal section P_1P_2
- α'_{12} azimuth of normal section P_2P_1 (measured in the local horizon plane of P_1)
- α_{21} reverse azimuth; azimuth of normal section P_2P_1
- c chord P_1P_2
- θ zenith distance of the chord c

x,y,z are geocentric Cartesian coordinates with an origin at the centre of the ellipsoid and where the z-axis is coincident with the rotational axis of the ellipsoid, the x-z plane is the Greenwich meridian plane and the x-y plane is the equatorial plane of the ellipsoid.

x',y',z' are geocentric Cartesian coordinates with an origin at the centre of the ellipsoid and where the z'-axis is coincident with the rotational axis of the ellipsoid, the x'-z' plane is the meridian plane of P_1 and the x'-y' plane is the equatorial plane of the ellipsoid. The x',y',z' system is rotated from the x,y,z system by an angle λ_1 about the z-axis.

vectors a vector **a** defining the length and direction of a line from point 1 to point 2 is given by the formula $\mathbf{a} = a_i\mathbf{i} + a_j\mathbf{j} + a_k\mathbf{k}$ where $a_i = x_2 - x_1$, $a_j = y_2 - y_1$ and $a_k = z_2 - z_1$ are the vector components and **i**, **j**, and **k** are unit vectors in the direction of the positive x, y, and z axes respectively. The components of a unit vector $\hat{\mathbf{a}} = \dfrac{\mathbf{a}}{\left|\mathbf{a}\right|}$ can be calculated by dividing each component by the magnitude of the vector $\left|\mathbf{a}\right| = \sqrt{a_i^2 + a_j^2 + a_k^2}$.

For vectors **a** and **b** the <u>vector dot product</u> is $\mathbf{a} \bullet \mathbf{b} = \left|\mathbf{a}\right|\left|\mathbf{b}\right|\cos\theta$ where θ is the angle between the vectors. For unit vectors $\hat{\mathbf{a}} \bullet \hat{\mathbf{b}} = \cos\theta$. The vector dot product is a scalar quantity $S = a_i b_i + a_j b_j + a_k b_k$, hence for unit vectors the angle between them is given by $\cos\theta = S$.

For vectors **a** and **b** the <u>vector cross product</u> is $\mathbf{a} \times \mathbf{b} = \left|\mathbf{a}\right|\left|\mathbf{b}\right|\sin\theta\,\hat{\mathbf{p}}$ where $\hat{\mathbf{p}}$ is a unit vector perpendicular to the plane containing **a** and **b** and in the direction of a right-handed screw rotated from **a** to **b**. The result of a vector cross product is another vector whose components are given by $\mathbf{a} \times \mathbf{b} = \left(a_j b_k - a_k b_j\right)\mathbf{i} - \left(a_i b_k - a_k b_i\right)\mathbf{j} + \left(a_i b_j - a_j b_i\right)\mathbf{k}$. The components of the unit vector $\hat{\mathbf{p}}$ are found by dividing each component of the cross product by the magnitudes $\left|\mathbf{a}\right|$ and $\left|\mathbf{b}\right|$, and the sine of the angle between them. For unit vectors $\hat{\mathbf{a}} \times \hat{\mathbf{b}} = \sin\theta\,\hat{\mathbf{p}}$ and for perpendicular unit vectors $\hat{\mathbf{a}} \times \hat{\mathbf{b}} = \hat{\mathbf{p}}$.

THE DIRECT PROBLEM ON THE ELLIPSOID USING A NORMAL SECTION

The direct problem is: Given latitude and longitude of P_1, azimuth α_{12} of the normal section P_1P_2 and the arc length s along the normal section curve; compute the latitude and longitude of P_2.

With the ellipsoid constants a, f, e^2 and e'^2 and given $\phi_1, \lambda_1, \alpha_{12}$ and s the problem may be solved by the following sequence.

1. Compute ν_1 the radius of curvature in the prime vertical plane of P_1 from

$$\nu_1 = \frac{a}{\left(1 - e^2 \sin^2 \phi_1\right)^{\frac{1}{2}}}$$

2. Compute the constants g and h of the normal section P_1P_2 from

$$\begin{aligned} g &= e' \sin\phi_1 &&= \frac{e}{\sqrt{1-e^2}} \sin\phi_1 \\ h &= e' \cos\alpha_{12} \cos\phi_1 &&= \frac{e}{\sqrt{1-e^2}} \cos\alpha_{12} \sin\phi_1 \end{aligned}$$

3. Compute the chord $c = P_1P_2$ and the zenith distance θ of the chord P_1P_2 by iteration using the following sequence of operations until there is negligible change in the computed chord distance

start Set the counter $k = 1$ and set the chord $c_k = s$

(i) Set the counter $n = 1$ and set the zenith distance $\theta_n = \dfrac{\pi}{2}$

(ii) Use Newton-Raphson iteration to compute the zenith distance of the chord using equation (46) rearranged as

$f\left(\theta\right) = c + c\left(g\cos\theta + h\sin\theta\right)^2 - 2\nu_1 \cos\theta = 0$ and the iterative formula

$\theta_{n+1} = \theta_n - \dfrac{f\left(\theta_n\right)}{f'\left(\theta_n\right)}$ where $f'\left(\theta_n\right)$ is the derivative of $f\left(\theta_n\right)$ and

$$\begin{aligned} f\left(\theta_n\right) &= c_k + c_k\left(g\cos\theta_n + h\sin\theta_n\right)^2 - 2\nu_1\cos\theta_n \\ f'\left(\theta_n\right) &= 2c_k\left(g\cos\theta_n + h\sin\theta_n\right)\left(h\cos\theta_n - g\sin\theta_n\right) - 2\nu_1\sin\theta_n \end{aligned}$$

Note that the iteration for θ is terminated when θ_n and θ_{n+1} differ by an acceptably small value.

(iii) Compute the arc length s_k using Romberg integration given $a, f, \phi_1, \alpha_{12}, \theta$

(iv) Compute the small change in arc length $ds = s_k - s$

(v) If $ds < 0.000001$ then go to end; else go (vi)

(vi) Increment k, compute new chord $c_k = c_{k-1} - ds$ and go to (i)

end Iteration for the chord $c = P_1P_2$ and the zenith distance θ of the chord P_1P_2 is complete.

4. Compute the x,y,z coordinates of P_1 using

$$\begin{aligned} x_1 &= \nu_1 \cos\phi_1 \cos\lambda_1 \\ y_1 &= \nu_1 \cos\phi_1 \sin\lambda_1 \\ z_1 &= \nu_1 \left(1 - e^2\right) \sin\phi_1 \end{aligned}$$

5. Compute coordinate differences $\Delta x', \Delta y', \Delta z'$ in the x', y', z' using

$$\begin{aligned} \Delta x' &= -c \sin\theta \cos\alpha_{12} \sin\phi_1 + c\cos\theta\cos\phi_1 \\ \Delta y' &= c\sin\theta\sin\alpha_{12} \\ \Delta z' &= c\sin\theta\cos\alpha_{12}\cos\phi_1 + c\cos\theta\sin\phi_1 \end{aligned}$$

6. Rotate the x',y',z' coordinate differences to x,y,z coordinate differences by a rotation of λ_1 about the z'-axis using

$$\begin{aligned} \Delta x &= \Delta x' \cos\lambda_1 - \Delta y' \sin\lambda_1 \\ \Delta y &= \Delta x' \sin\lambda_1 + \Delta y' \cos\lambda_1 \\ \Delta z &= \Delta z' \end{aligned}$$

7. Compute x,y,z coordinates of P_2 using

$$\begin{aligned} x_2 &= x_1 + \Delta x \\ y_2 &= y_1 + \Delta y \\ z_2 &= z_1 + \Delta z \end{aligned}$$

8. Compute latitude and longitude of P_2 by conversion $x, y, z \Rightarrow \phi, \lambda, h$ using Bowring's method.

Shown below is the output of a MATLAB function *nsection_ direct.m* that solves the direct problem on the ellipsoid for normal sections.
The ellipsoid is the GRS80 ellipsoid and ϕ, λ for P_1 are -10° and 110° respectively with $\alpha_{12} = 140^\circ\, 28'\, 31.981931''$ and $s = 5783\,228.924\,736$ m. ϕ, λ computed for P_2 are -45° and 155° respectively.

```
>> nsection_direct

////////////////////////////////
// Normal Section: Direct Case //
////////////////////////////////

ellipsoid parameters
a   =  6378137.000000000
f   = 1/298.257222101000
e2  =  6.694380022901e-003
ep2 =  6.694380022901e-003

Latitude  P1 =  -10  0  0.000000 (D M S)
Longitude P1 =  110  0  0.000000 (D M S)

Azimuth of normal section P1-P2
Az12  = 140 28 31.981931  (D M S)

normal section distance P1-P2
s =   5783228.924736

chord distance P1-P2
c =   5586513.169887
iterations =   13

Zenith distance of chord at P1
zd     = 116  2 20.450079  (D M S)
iterations =    5

Cartesian coordinates
            X               Y               Z
P1   -2148527.045536  5903029.542697 -1100248.547700
P2   -4094327.792179  1909216.404490 -4487348.408756
dX = -1945800.746643
dY = -3993813.138207
dZ = -3387099.861057

Latitude  P2 =  -45  0  0.000000 (D M S)
Longitude P2 =  154 59 60.000000 (D M S)

>>
```

THE INVERSE PROBLEM ON THE ELLIPSOID USING A NORMAL SECTION

The inverse problem is: Given latitudes and longitudes of P_1 and P_2 on the ellipsoid compute the azimuth α_{12} of the normal section P_1P_2 and the arc length s of the normal section curve.

With the ellipsoid constants a, f, e^2 and e'^2 and given ϕ_1, λ_1 and ϕ_2, λ_2 the problem may be solved by the following sequence.

1. Compute ν_1 and ν_2 the radii of curvature in the prime vertical plane of P_1 and P_2 from

$$\nu = \frac{a}{\left(1 - e^2 \sin^2 \phi\right)^{\frac{1}{2}}}$$

2. Compute the x,y,z coordinates of P_1, P_2, P_3 and P_4 noting that P_3 is at the intersection of the normal through P_1 and the rotational axis of the ellipsoid and P_4 is at the intersection of the normal through P_2 and the rotational axis. Coordinate of P_1 and P_2 are obtained from

$$\begin{aligned} x &= \nu \cos\phi \cos\lambda \\ y &= \nu \cos\phi \sin\lambda \\ z &= \nu\left(1 - e^2\right)\sin\phi \end{aligned}$$

The x and y coordinates of P_3 and P_4 are zero and the z coordinate is obtained from

$$z = -\nu e^2 \sin\phi$$

3. Compute the coordinate differences

$$\begin{aligned} \Delta x &= x_2 - x_1 \\ \Delta y &= y_2 - y_1 \\ \Delta z &= z_2 - z_1 \end{aligned}$$

4a. Compute vector $\mathbf{c} = \left(\Delta x\right)\mathbf{i} + \left(\Delta y\right)\mathbf{j} + \left(\Delta z\right)\mathbf{k}$ in the direction of the chord P_1P_2.

4b. Compute chord distance $c = \left|\mathbf{c}\right|$ and the unit vector $\hat{\mathbf{c}} = \dfrac{\mathbf{c}}{\left|\mathbf{c}\right|}$

5. Compute vector $\mathbf{u} = \left(x_1\right)\mathbf{i} + \left(y_1\right)\mathbf{j} + \left(z_1 - z_3\right)\mathbf{k}$ and the unit vector $\hat{\mathbf{u}} = \dfrac{\mathbf{u}}{\left|\mathbf{u}\right|}$ in the direction of the outward normal through P_1.

6. Set the unit vector $\hat{\mathbf{z}} = 0\,\mathbf{i} + 0\,\mathbf{j} + 1\,\mathbf{k}$ in the direction of the z-axis

7. Compute the zenith distance of the chord from the vector dot product

$$\cos\theta = \hat{u}_i \hat{c}_i + \hat{u}_j \hat{c}_j + \hat{u}_k \hat{c}_k$$

8. Compute the unit vector $\hat{\mathbf{e}}$ perpendicular to the meridian plane of P_1 from vector cross product ($\hat{\mathbf{e}}$ is in the direction of east)

$$\hat{\mathbf{e}} = \frac{\hat{\mathbf{z}} \times \hat{\mathbf{u}}}{\cos\phi_1} = \left(\frac{\hat{z}_j \hat{u}_k - \hat{z}_k \hat{u}_j}{\cos\phi_1} \right) \mathbf{i} - \left(\frac{\hat{z}_i \hat{u}_k - \hat{z}_k \hat{u}_i}{\cos\phi_1} \right) \mathbf{j} + \left(\frac{\hat{z}_i \hat{u}_j - \hat{z}_j \hat{u}_i}{\cos\phi_1} \right) \mathbf{k}$$

9. Compute the unit vector $\hat{\mathbf{n}}$ in the meridian plane of P_1 from vector cross product. ($\hat{\mathbf{n}}$ is in the direction of north)

$$\hat{\mathbf{n}} = \hat{\mathbf{u}} \times \hat{\mathbf{e}} = \left(\hat{u}_j \hat{e}_k - \hat{u}_k \hat{e}_j \right) \mathbf{i} - \left(\hat{u}_i \hat{e}_k - \hat{u}_k \hat{e}_i \right) \mathbf{j} + \left(\hat{u}_i \hat{e}_j - \hat{u}_j \hat{e}_i \right) \mathbf{k}$$

10. Compute the unit vector $\hat{\mathbf{p}}$ perpendicular to the normal section $P_1 P_2$ from vector cross product. ($\hat{\mathbf{p}}$ lies in the local horizon plane of P_1)

$$\hat{\mathbf{p}} = \frac{\hat{\mathbf{u}} \times \hat{\mathbf{c}}}{\sin\theta} = \left(\frac{\hat{u}_j \hat{c}_k - \hat{u}_k \hat{c}_j}{\sin\theta} \right) \mathbf{i} - \left(\frac{\hat{u}_i \hat{c}_k - \hat{u}_k \hat{c}_i}{\sin\theta} \right) \mathbf{j} + \left(\frac{\hat{u}_i \hat{c}_j - \hat{u}_j \hat{c}_i}{\sin\theta} \right) \mathbf{k}$$

11. Compute the unit vector $\hat{\mathbf{g}}$ in the local horizon plane of P_1 and in the direction of the normal section $P_1 P_2$ from vector cross product.

$$\hat{\mathbf{g}} = \hat{\mathbf{p}} \times \hat{\mathbf{u}} = \left(\hat{p}_j \hat{u}_k - \hat{p}_k \hat{u}_j \right) \mathbf{i} - \left(\hat{p}_i \hat{u}_k - \hat{p}_k \hat{u}_i \right) \mathbf{j} + \left(\hat{p}_i \hat{u}_j - \hat{p}_j \hat{u}_i \right) \mathbf{k}$$

12. Compute the azimuth α_{12} if the normal section $P_1 P_2$ using vector dot products to first compute angles α (between $\hat{\mathbf{n}}$ and $\hat{\mathbf{g}}$) and β (between $\hat{\mathbf{e}}$ and $\hat{\mathbf{g}}$) from

$$\cos\alpha = \hat{n}_i \hat{g}_i + \hat{n}_j \hat{g}_j + \hat{n}_k \hat{g}_k$$
$$\cos\beta = \hat{e}_i \hat{g}_i + \hat{e}_j \hat{g}_j + \hat{e}_k \hat{g}_k$$

If $\beta > 90^\circ$ then $\alpha_{12} = 360^\circ - \alpha$; else $\alpha_{12} = \alpha$

13. Compute the vector $\mathbf{w} = \left(x_1 \right) \mathbf{i} + \left(y_1 \right) \mathbf{j} + \left(z_1 - z_4 \right) \mathbf{k}$ and the unit vector $\hat{\mathbf{w}} = \dfrac{\mathbf{w}}{\left| \mathbf{w} \right|}$ ($\mathbf{w}$ is in the direction of the line $P_4 P_1$ and lies in the meridian plane of P_1).

14. Compute the angle γ between $\hat{\mathbf{w}}$ and $\hat{\mathbf{c}}$ from the vector dot product

$$\cos\gamma = \hat{w}_i \hat{c}_i + \hat{w}_j \hat{c}_j + \hat{w}_k \hat{c}_k$$

15. Compute the angle δ between $\hat{\mathbf{w}}$ and $\hat{\mathbf{u}}$ from the vector dot product (δ lies in the meridian plane of P_1)

$$\cos\delta = \hat{w}_i\hat{u}_i + \hat{w}_j\hat{u}_j + \hat{w}_k\hat{u}_k$$

16. Compute the unit vector $\hat{\mathbf{q}}$ perpendicular to the normal section P_2P_1 from vector cross product

$$\hat{\mathbf{q}} = \frac{\hat{\mathbf{w}}\times\hat{\mathbf{c}}}{\sin\gamma} = \left(\frac{\hat{w}_j\hat{c}_k - \hat{w}_k\hat{c}_j}{\sin\gamma}\right)\mathbf{i} - \left(\frac{\hat{w}_i\hat{c}_k - \hat{w}_k\hat{c}_i}{\sin\gamma}\right)\mathbf{j} + \left(\frac{\hat{w}_i\hat{c}_j - \hat{w}_j\hat{c}_i}{\sin\gamma}\right)\mathbf{k}$$

17. Compute the unit vector $\hat{\mathbf{h}}$ in the local horizon plane of P_1 and in the direction of the normal section P_2P_1 from vector cross product.

$$\hat{\mathbf{h}} = \frac{\hat{\mathbf{q}}\times\hat{\mathbf{u}}}{\cos\delta} = \left(\frac{\hat{q}_j\hat{u}_k - \hat{q}_k\hat{u}_j}{\cos\delta}\right)\mathbf{i} - \left(\frac{\hat{q}_i\hat{u}_k - \hat{q}_k\hat{u}_i}{\cos\delta}\right)\mathbf{j} + \left(\frac{\hat{q}_i\hat{u}_j - \hat{q}_j\hat{u}_i}{\cos\delta}\right)\mathbf{k}$$

18. Compute the azimuth α'_{12} of the normal section P_2P_1 using vector dot products to first compute angles α (between $\hat{\mathbf{n}}$ and $\hat{\mathbf{h}}$) and β (between $\hat{\mathbf{e}}$ and $\hat{\mathbf{h}}$) from

$$\begin{aligned}\cos\alpha &= \hat{n}_i\hat{h}_i + \hat{n}_j\hat{h}_j + \hat{n}_k\hat{h}_k \\ \cos\beta &= \hat{e}_i\hat{h}_i + \hat{e}_j\hat{h}_j + \hat{e}_k\hat{h}_k\end{aligned}$$

If $\beta > 90^\circ$ then $\alpha'_{12} = 360^\circ - \alpha$; else $\alpha'_{12} = \alpha$

19. Compute the small angle ε between the two normal section planes at P_1

$$\varepsilon = \left|\alpha_{12} - \alpha'_{12}\right|$$

20. Compute arc length s along the normal section curve P_1P_2 using Romberg Integration.

Shown below is the output of a MATLAB function *nsection_inverse.m* that solves the inverse problem on the ellipsoid for normal sections.

The ellipsoid is the GRS80 ellipsoid and ϕ,λ for P_1 are -10° and 110° respectively and ϕ,λ for P_2 are -45° and 155° respectively.

Computed azimuths are $\alpha_{12} = 140^\circ\, 28'\, 31.981931''$ and $\alpha'_{12} = 140^\circ\, 32'\, 18.496009''$, and $s = 5783\,228.924\,736$ m.

```
>> nsection_inverse

//////////////////////////////////
// Normal Section: Inverse Case //
//////////////////////////////////

ellipsoid parameters
a   =  6378137.000000000
f   = 1/298.257222101000
e2  =  6.694380022901e-003
ep2 =  6.694380022901e-003

Latitude  P1 =  -10  0  0.000000 (D M S)
Longitude P1 =  110  0  0.000000 (D M S)

Latitude  P2 =  -45  0  0.000000 (D M S)
Longitude P2 =  155  0  0.000000 (D M S)

Cartesian coordinates
            X               Y               Z
P1   -2148527.045536  5903029.542697 -1100248.547700
P2   -4094327.792180  1909216.404490 -4487348.408755
P3          0.000000        0.000000     7415.121539
P4          0.000000        0.000000    30242.470131
dX = -1945800.746645
dY = -3993813.138206
dZ = -3387099.861055

Chord distance P1-P2
chord   =   5586513.169886

Zenith distance of chord at P1
zd      = 116  2 20.450079  (D M S)

Azimuth of normal section P1-P2
Az12    = 140 28 31.981931  (D M S)

Azimuth of normal section P2-P1
Az21    = 297 47 44.790362  (D M S)

Azimuth of normal section P2-P1 at P1
Az'12   = 140 32 18.496009  (D M S)

Angle between normal sections at P1
epsilon =    0  3 46.514078 (D M S)

ROMBERG INTEGRATION TABLE
1  5783427.529966
2  5783278.294728  5783228.549649
3  5783241.249912  5783228.901640  5783228.925106
4  5783232.004951  5783228.923298  5783228.924742  5783228.924736
5  5783229.694723  5783228.924646  5783228.924736  5783228.924736

normal section distance P1-P2
s =   5783228.924736

>>
```

DIFFERENCE IN LENGTH BETWEEN GEODESIC AND NORMAL SECTION

There are five curves of interest in geodesy; the <u>geodesic</u>, the <u>normal section</u>, the <u>great elliptic arc</u> the <u>loxodrome</u> and the <u>curve of alignment</u>.

The geodesic between P_1 and P_2 on an ellipsoid is the unique curve on the surface defining the shortest distance; all other curves will be longer in length. The normal section curve P_1P_2 is a plane curve created by the intersection of the normal section plane containing the normal at P_1 and also P_2 with the ellipsoid surface. And as we have shown there is the other normal section curve P_2P_1. The curve of alignment is the locus of all points Q such that the normal section plane at Q also contains the points P_1 and P_2. The curve of alignment is very close to a geodesic. The great elliptic arc is the plane curve created by intersecting the plane containing P_1, P_2 and the centre O with the surface of the ellipsoid and the loxodrome is the curve on the surface that cuts each meridian between P_1 and P_2 at a constant angle.

Approximate equations for the difference in length between the geodesic, the normal section curve and the curve of alignment were developed by Clarke (1880, p. 133) and Bowring (1972, p. 283) developed an approximate equation for the difference between the geodesic and the great elliptic arc. Following Bowring (1972), let

$$\begin{aligned} s &= \text{ geodesic length} \\ L &= \text{ normal section length} \\ D &= \text{ great elliptic length} \\ S &= \text{ curve of alignment length} \end{aligned}$$

then

$$\begin{aligned} L - s &= \frac{e^4}{90} s \left(\frac{s}{R}\right)^4 \cos^4\phi_1 \sin^2\alpha_{12} \cos^2\alpha_{12} + \cdots \\ D - s &= \frac{e^4}{24} s \left(\frac{s}{R}\right)^2 \sin^2\phi_1 \cos^2\phi_1 \sin^2\alpha_{12} + \cdots \\ S - s &= \frac{e^4}{360} s \left(\frac{s}{R}\right)^4 \cos^4\phi_1 \sin^2\alpha_{12} \cos^2\alpha_{12} + \cdots \end{aligned} \tag{52}$$

where R can be taken as the radius of curvature in the prime vertical at P_1. Now for a given value of s, $L - s$ will be a maximum if $\phi_1 = 0^\circ$ (P_1 on the equator) and $\alpha_{12} = 45^\circ$ in which case $\cos^4\phi_1 \sin^2\alpha_{12} \cos^2\alpha_{12} = \dfrac{1}{4}$, thus

$$\left(L - s\right) < \frac{e^4}{360} s \left(\frac{s}{R}\right)^4 \tag{53}$$

For the GRS80 ellipsoid where $f = 1/298.257222101$, $e^2 = f\left(2 - f\right)$, and for $s = 1600000$ m and $R = 6371000$ m and equation (53) gives $L - s < 0.001$ m.

This can be verified by using two MATLAB functions: *Vincenty_Direct.m* that computes the direct case on the ellipsoid for the geodesic and *nsection_inverse.m* that computes the inverse case on the ellipsoid for the normal section. Suppose P_1 has latitude and longitude $\phi_1 = 0^\circ$, $\lambda_1 = 0^\circ$ on the GRS80 ellipsoid and that the azimuth and distance of the geodesic are $\alpha_{12} = 45^\circ$ and $s = 1600000$ m respectively. The coordinates of P_2 are obtained from *Vincenty_Direct.m* as shown below. These values are then used in *nsection_direct.m* to compute the normal section azimuth and distance P_1P_2.

The difference $L - s = 0.000789$ m.

```
>> Vincenty_Direct

//////////////////////////////////////////////
// DIRECT CASE on ellipsoid: Vincenty's method
//////////////////////////////////////////////

ellipsoid parameters
a    =  6378137.000000000
f    = 1/298.257222101000
b    =  6356752.314140356100
e2   =  6.694380022901e-003
ep2  =  6.739496775479e-003

Latitude & Longitude of P1
latP1 =   0  0  0.000000 (D M S)
lonP1 =   0  0  0.000000 (D M S)

Azimuth & Distance P1-P2
az12 =   45  0  0.000000 (D M S)
s    =    1600000.000000

Latitude and Longitude of P2
latP2 =  10 10 33.913466 (D M S)
lonP2 =  10 16 16.528718 (D M S)

Reverse azimuth
alpha21 = 225 55  1.180693 (D M S)

>>
```

```
>> nsection_inverse

//////////////////////////////////
// Normal Section: Inverse Case //
//////////////////////////////////

ellipsoid parameters
a   =  6378137.000000000
f   = 1/298.257222101000
e2  =  6.694380022901e-003
ep2 =  6.694380022901e-003

Latitude  P1 =    0  0  0.000000 (D M S)
Longitude P1 =    0  0  0.000000 (D M S)

Latitude  P2 =   10 10 33.913466 (D M S)
Longitude P2 =   10 16 16.528718 (D M S)

Azimuth of normal section P1-P2
Az12    =  45  0  7.344646  (D M S)

ROMBERG INTEGRATION TABLE
1 1600010.313769
2 1600002.577521 1599999.998771
3 1600000.644877 1600000.000663 1600000.000789
4 1600000.161805 1600000.000781 1600000.000789
  1600000.000789

normal section distance P1-P2
s =   1600000.000789

>>
```

Differences in length between the geodesic and normal section exceed 0.001 m for distances greater than 1,600 km. At 5,800 km the difference is approximately 0.380 m.

MATLAB FUNCTIONS

Shown below are two MATLAB functions *nsection_ direct.m* and *nsection_ inverse.m* that have been written to demonstrate the use of Romberg integration in the solution of the direct and inverse case on the ellipsoid using normal sections. These functions call other functions; *DMS.m, Cart2Geo.m* and *romberg.m* that are also shown.

MATLAB function *nsection_ direct.m*

```
function nsection_direct
%
% nsection_direct:  This function computes the direct case for a normal
% section on the reference ellipsoid.  That is, given the latitude and
% longitude of P1 and the azimuth of the normal section P1-P2 and distance
% along the normal section curve, compute the latitude and longitude of P2.

%--------------------------------------------------------------------------
% Function:  nsection_direct
%
% Usage:     nsection_direct
%
% Author:    R.E.Deakin,
%            School of Mathematical & Geospatial Sciences, RMIT University
%            GPO Box 2476V, MELBOURNE, VIC 3001, AUSTRALIA.
%            email: rod.deakin@rmit.edu.au
%            Version  1.0  23 September 2009
%            Version  1.1  16 December  2009
%
% Purpose:   nsection_inverse:  This function computes the direct case for
%  a normal section on the reference ellipsoid.  That is, given the
%  latitude and longitude of P1 and the azimuth of the normal section P1-P2
%  and distance along the normal section curve, compute the latitude and
%  longitude of P2.
%
% Functions required:
%  [D,M,S] = DMS(DecDeg)
%  s = romberg(a,f,lat1,Az12,zd)
%  [lat,lon,h] = Cart2Geo(a,flat,X,Y,Z)
%
% Variables:
%  Az12     - azimuth of normal section P1-P2
%  a        - semi-major axis of spheroid
%  d2r      - degree to radian conversion factor 57.29577951...
%  e2       - eccentricity of ellipsoid squared
%  eps      - 2nd-eccentricity squared
%  f        - f = 1/flat is the flattening of ellipsoid
%  flat     - denominator of flattening of ellipsoid
%  f_zd     - function of the zenith distance
%  fdash_zd - derivative of the function of the zenith distance
%  g,h      - constants of normal section
%  lat1     - latitude of P1 (radians)
%  lat2     - latitude of P2 (radians)
%  lon1     - longitude of P1 (radians)
%  lon2     - longitude of P2 (radians)
%  nu1      - radius of curvature in prime vertical plane at P1
%  pion2    - pi/2
%  s        - arc length of normal section P1-P2
%  s2       - sin-squared(latitude)
%  x,y      - local variables in newton-Raphson iteration for zenith
%             distance of chord P1-P2
%  X1,Y1,Z1 - Cartesian coordinates of P1
```

```
%  X2,Y2,Z2 - Cartesian coordinates of P2
%  X3,Y3,Z3 - Cartesian coordinates of P3
%  X4,Y4,Z4 - Cartesian coordinates of P4
%  zd       - zenith distance of chord
%
% Remarks:
%
% References:
%  [1] Deakin, R. E., (2009), "The Normal Section Curve on an Ellipsoid",
%          Lecture Notes, School of Mathematical and Geospatial Sciences,
%          RMIT University, November 2009.
%
%--------------------------------------------------------------------------

% Set degree to radian conversion factor and pi/2
d2r   = 180/pi;
pion2 = pi/2;

% Set ellipsoid parameters
a    = 6378137;    % GRS80
flat = 298.257222101;

% Compute ellipsoid constants
f   = 1/flat;
e2  = f*(2-f);
ep2 = e2/(1-e2);

% Set lat and long of P1 on ellipsoid
lat1 = -10/d2r;
lon1 = 110/d2r;

% Set azimuth of normal section P1-P2 and arc length of normal section
Az12 = (140 + 28/60 + 31.981931/3600)/d2r;
s    = 5783228.924736;

% [1] Compute radius of curvature in the prime vertical plane at P1
s2   = sin(lat1)^2;
nu1  = a/sqrt(1-e2*s2);

% [2] Compute constants g and h of the normal section P1-P2
ep = sqrt(ep2);
g  = ep*sin(lat1);
h  = ep*cos(lat1)*cos(Az12);

% [3] Compute the chord and the zenith distance of the chord of the normal
%     section curve P1-P2 by iteration.

% Set the chord equal to the arc length
c = s;
iter_1 = 1;
while 1
    % Set the zenith distance to 90 degrees
    zd = pion2;
    % Compute the zenith distance of the chord using Newton-Raphson iteration
    iter_2 = 1;
    while 1
        x = g*cos(zd)+h*sin(zd);
        y = h*cos(zd)-g*sin(zd);
        f_zd = c+c*x*x+2*nu1*cos(zd);
        fdash_zd = 2*c*x*y-2*nu1*sin(zd);
        new_zd = zd-(f_zd/fdash_zd);
        if abs(new_zd - zd) < 1e-15
            break;
        end
        zd = new_zd;
        if iter_2 > 10
            fprintf('Iteration for zenith distance failed to converge after 10
iterations');
            break;
```

```
        end
        iter_2 = iter_2 + 1;
    end;
    % Compute normal section arc length for zenith distance
    s_new = romberg(a,f,lat1,Az12,zd);
    ds = s_new-s;
    if abs(ds) < 1e-6
        break;
    end
    c = c - ds;
    if iter_1 > 15
        fprintf('Iteration for chord distance failed to converge after 15 iterations');
        break;
    end
    iter_1 = iter_1 + 1;
end;

% [4] Compute X,Y,Z Cartesian coordinates of P1
X1 = nu1*cos(lat1)*cos(lon1);
Y1 = nu1*cos(lat1)*sin(lon1);
Z1 = nu1*(1-e2)*sin(lat1);

% [5] Compute X',Y',Z' coord differences with Z'-X' plane coincident with meridian
%     plane of P1
dXp = -c*sin(zd)*cos(Az12)*sin(lat1) + c*cos(zd)*cos(lat1);
dYp =  c*sin(zd)*sin(Az12);
dZp =  c*sin(zd)*cos(Az12)*cos(lat1) + c*cos(zd)*sin(lat1);

% [6] Rotate X',Y',Z' coord differences by lon1 about Z'-axis
dX = dXp*cos(lon1) - dYp*sin(lon1);
dY = dXp*sin(lon1) + dYp*cos(lon1);
dZ = dZp;

% [7] Compute X,Y,Z coords of P2
X2 = X1 + dX;
Y2 = Y1 + dY;
Z2 = Z1 + dZ;

% [8] Compute lat, lon and ellipsoidal height of P2 using Bowring's method
[lat2,lon2,h2] = Cart2Geo(a,flat,X2,Y2,Z2);

%-----------------------
% Print result to screen
%-----------------------

fprintf('\n////////////////////////////////');
fprintf('\n// Normal Section: Direct Case //');
fprintf('\n////////////////////////////////');
fprintf('\n\nellipsoid parameters');
fprintf('\na   = %18.9f',a);
fprintf('\nf   = 1/%16.12f',flat);
fprintf('\ne2  = %20.12e',e2);
fprintf('\nep2 = %20.12e',e2);

% Print lat and lon of P1
[D,M,S] = DMS(lat1*d2r);
if D == 0 && lat1 < 0
    fprintf('\n\nLatitude  P1 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\n\nLatitude  P1 = %4d %2d %9.6f (D M S)',D,M,S);
end
[D,M,S] = DMS(lon1*d2r);
if D == 0 && lon1 < 0
    fprintf('\nLongitude P1 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLongitude P1 = %4d %2d %9.6f (D M S)',D,M,S);
end

% Print azimuth of normal section
```

```
fprintf('\n\nAzimuth of normal section P1-P2');
[D,M,S] = DMS(Az12*d2r);
fprintf('\nAz12  = %3d %2d %9.6f  (D M S)',D,M,S);

% Print normal section distance P1-P2
fprintf('\n\nnormal section distance P1-P2');
fprintf('\ns =  %15.6f',s);

% Print chord distance P1-P2
fprintf('\n\nchord distance P1-P2');
fprintf('\nc =  %15.6f',c);
fprintf('\niterations = %4d',iter_1);

% Print zenith distance of chord at point 1
fprintf('\n\nZenith distance of chord at P1');
[D,M,S] = DMS(zd*d2r);
fprintf('\nzd     = %3d %2d %9.6f  (D M S)',D,M,S);
fprintf('\niterations = %4d',iter_2);


% Print Coordinate table
fprintf('\n\nCartesian coordinates');
fprintf('\n            X               Y               Z');
fprintf('\nP1   %15.6f %15.6f %15.6f',X1,Y1,Z1);
fprintf('\nP2   %15.6f %15.6f %15.6f',X2,Y2,Z2);
fprintf('\ndX = %15.6f',dX);
fprintf('\ndY = %15.6f',dY);
fprintf('\ndZ = %15.6f',dZ);

% Print lat and lon of P2
[D,M,S] = DMS(lat2*d2r);
if D == 0 && lat2 < 0
    fprintf('\n\nLatitude  P2 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\n\nLatitude  P2 = %4d %2d %9.6f (D M S)',D,M,S);
end
[D,M,S] = DMS(lon2*d2r);
if D == 0 && lon2 < 0
    fprintf('\nLongitude P2 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLongitude P2 = %4d %2d %9.6f (D M S)',D,M,S);
end

fprintf('\n\n');
```

MATLAB function *nsection_ inverse.m*

```
function nsection_inverse
%
% nsection_inverse:  This function computes the inverse case for a normal
% section on the reference ellipsoid.  That is, given the latitudes and
% longitudes of two points on the ellipsoid, compute the azimuth and the
% arc length of the normal section.

%----------------------------------------------------------------------------
% Function:  nsection_inverse()
%
% Usage:     nsection_inverse
%
% Author:    R.E.Deakin,
%            School of Mathematical & Geospatial Sciences, RMIT University
%            GPO Box 2476V, MELBOURNE, VIC 3001, AUSTRALIA.
%            email: rod.deakin@rmit.edu.au
%            Version  1.0  21 September 2009
%            Version  1.1  16 December  2009
%
% Purpose:   nsection_inverse:  This function computes the inverse case for
%  a normal section on the reference ellipsoid.  That is, given the
%  latitudes and longitudes of two points on the ellipsoid, compute the
%  azimuth and the arc length of the normal section.
%
% Functions required:
%  [D,M,S] = DMS(DecDeg)
%
% Variables:
%  alpha       - angle in the local horizon plane measured from north
%  Az12        - azimuth of normal section P1-P2
%  Azdash12    - azimuth of normal section plane P2-P1 measured at P1
%  Az21        - azimuth of normal section P2-P1
%  a           - semi-major axis of spheroid
%  beta        - angle in the local horizon plane measured from east
%  chord       - chord distance between P1 and P2
%  ci,cj,ck    - components of unit vector c in the direction of the chord
%                P1-P2
%  delta       - angle in the meridian plane of P1 between w and u vectors
%  diff        - difference between successive value of integral in Romber
%                Integration
%  du,dv,dr    - derivatives in Romberg Integration
%  dX,dY,dZ    - Cartesian components of chord between between P1 and P2
%  d2r         - degree to radian conversion factor 57.29577951...
%  ei,ej,ek    - components of unit vector e in the direction of east in
%                local horizon system
%  epsilon     - small angle between azimuths of normal section planes
%  ep2         - 2nd-eccentricity squared
%  e2          - 1st-eccentricity squared
%  f           - f = 1/flat is the flattening of ellipsoid
%  finish      - integer flag (1 or 0) to test for end of Romberg
%                Integration
%  first       - first value in trapezoidal rule in Romberg Integration
%  flat        - denominator of flattening of ellipsoid
%  gamma       - angle between unit vectors w and c
%  g,h         - constants of normal section curve
%  hi,hj,hk    - components of unit vector h in the local horizon plane and
%                direction of the plane P1-P2-P4
%  Integral    - value of integral from trapezoidal rule in Romberg
%                Integration
%  inc         - interval width in trapezoidal rule
%  int         - number of intervals in trapezoidal rule where int = 2^k
%                and k = 1:m
%  j,k         - integer counters in Romberg Integration
%  last        - last value in trapezoidal rule in Romberg Integration
%  lat1        - latitude of P1 (radians)
%  lat2        - latitude of P2 (radians)
```

```
%  lon1        - longitude of P1 (radians)
%  lon2        - longitude of P2 (radians)
%  m           - maximum power of 2 to determine number of intervals in
%                trapezoidal rule
%  norm        - length of vector
%  nu1, nu2    - radii of curvature in prime vertical plane at P1 and P2
%  ni,nj,nk    - components of unit vector n
%  pion2       - pi/2
%  qi,qj,qk - components of unit vector q perpendicular to plane
%                P1-P2-P4
%  r           - polar coordinate in polar equation of normal section
%  S           - n,n array of Integrals in Romberg Integration
%  sum         - summation in trapezoidal rule
%  s2          - sin-squared(latitude)
%  ui,uj,uk    - components of unit vector u
%  wi,wj,wk    - components of unit vector w
%  x,y         - variables in Romberg Integration
%  X1,Y1,Z1    - Cartesian coordinates of P1
%  X2,Y2,Z2    - Cartesian coordinates of P2
%  X3,Y3,Z3    - Cartesian coordinates of P3
%  X4,Y4,Z4    - Cartesian coordinates of P4
%  zd          - zenith distance of chord
%
% Remarks:
%  P1 and P2 are two point on the ellipsoid and in general there are two
%  normal section curves between them.  P3 is at the intersection of the
%  rotational axis of the ellipsoid and the normal through P1.  P4 is at
%  the intersection of the rotational axis of the ellipsoid and the normal
%  through P2.  The normal section P1-P2 is the plane P1-P2-P3.  The normal
%  section P2-P1 is the plane P1-P2-P4 and since P3 and P4 are not
%  coincident (in general) then the two planes create two lines on the
%  ellipsoid and two lines on the local horizon plane at P1.
%  The necessary equations for the solution of the inverse problem (normal
%  sections) on the ellipsoid are described in [1].  The vector
%  manipulations to determine the difference between the two normal section
%  plane azimuths (measuered in the local horizon at P1) follows a vector
%  method of calculating azimuth given in [2].
%  This function uses Romberg Integration to compute the arc length along
%  the normal section curve.  This technique of numerical integration is
%  described in detail in [1].
%
% References:
%  [1] Deakin, R. E., (2009), "The Normal Section Curve on an Ellipsoid",
%          Lecture Notes, School of Mathematical and Geospatial Sciences,
%          RMIT University, November 2009.
%  [2] Deakin, R. E., (1988), "The Determination of the Instantaneous
%          Position of the NIMBUS-7 CZCS Satellite", Symposium on Remote
%          Sensing of the Coastal Zone, Queensland, 1988.
%
%--------------------------------------------------------------------------

% Degree to radian conversion factor
d2r   = 180/pi;
pion2 = pi/2;

% Set ellipsoid parameters
a    = 6378137;    % GRS80
flat = 298.257222101;

% Compute ellipsoid constants
 f   = 1/flat;
e2  = f*(2-f);
ep2 = e2/(1-e2);

% Set lat and long of P1 and P2 on ellipsoid
lat1 = -10/d2r;
lon1 =  110/d2r;
lat2 = -45/d2r;
lon2 =  155/d2r;
```

```
% [1] Compute radii of curvature in the prime vertical plane at P1 & P2
s2  = sin(lat1)^2;
nu1 = a/sqrt(1-e2*s2);
s2  = sin(lat2)^2;
nu2 = a/sqrt(1-e2*s2);

% [2] Compute Cartesian coordinates of points P1, P2, P3 and P4
% Note that P3 is at the intesection of the normal through P1 and
% the rotational axis and P4 is at the intersection of the normal
% through P2 and the rotational axis.
X1 = nu1*cos(lat1)*cos(lon1);
Y1 = nu1*cos(lat1)*sin(lon1);
Z1 = nu1*(1-e2)*sin(lat1);

X2 = nu2*cos(lat2)*cos(lon2);
Y2 = nu2*cos(lat2)*sin(lon2);
Z2 = nu2*(1-e2)*sin(lat2);

X3 = 0;
Y3 = 0;
Z3 = -nu1*e2*sin(lat1);

X4 = 0;
Y4 = 0;
Z4 = -nu2*e2*sin(lat2);

% [3] Compute coordinate differences that are the components of the chord
% P1-P2
dX = X2 - X1;
dY = Y2 - Y1;
dZ = Z2 - Z1;

% [4a] Compute the vector c in the direction of the chord between P1 and P2
ci = dX;
cj = dY;
ck = dZ;

% [4b] Compute the chord distance and the unit vector c
chord = sqrt(ci*ci + cj*cj + ck*ck);
ci = ci/chord;
cj = cj/chord;
ck = ck/chord;

% [5] Compute the unit vector u in the direction of the normal through P1
ui = X1;
uj = Y1;
uk = Z1-Z3;
norm = sqrt(ui*ui + uj*uj + uk*uk);
ui = ui/norm;
uj = uj/norm;
uk = uk/norm;

% [6] Set unit vector for the z-axis of ellipsoid
zi = 0;
zj = 0;
zk = 1;

% [7] Compute zenith distance of chord at P1 from dot product
zd = acos(ui*ci + uj*cj + uk*ck);

% [8] Compute unit vector e perpendicular to meridian plane using vector cross
% product e = (z x u)/cos(lat1).  e is in the direction of east.
ei =  (zj*uk - zk*uj)/cos(lat1);
ej = -(zi*uk - zk*ui)/cos(lat1);
ek =  (zi*uj - zj*ui)/cos(lat1);

% [9] Compute unit vector n in the meridian plane using vector cross
% product n = u x e.  n is in the direction of north.
```

```
ni =  (uj*ek - uk*ej);
nj = -(ui*ek - uk*ei);
nk =  (ui*ej - uj*ei);

% [10] Compute unit vector p perpendicular to normal section P1-P2 using
% vector cross product q = (u x c)/sin(zd)
pii =  (uj*ck - uk*cj)/sin(zd);
pj = -(ui*ck - uk*ci)/sin(zd);
pk =  (ui*cj - uj*ci)/sin(zd);

% [11] Compute unit vector g in the local horizon plane of P1 and in the
% direction of the normal section P1-P2 using vector cross product
% g = p x u
gi =  (pj*uk - pk*uj);
gj = -(pii*uk - pk*ui);
gk =  (pii*uj - pj*ui);

% [12] Compute azimuth of normal section P1-P2-P3 using vector dot product
alpha = acos(ni*gi + nj*gj + nk*gk);
beta  = acos(ei*gi + ej*gj + ek*gk);

if beta > pi/2
    Az12 = 2*pi - alpha;
else
    Az12 = alpha;
end

% [13] Compute unit vector w in direction of line P4-P1.  w will lie in the
% meridian plane of P1.
wi = X1;
wj = Y1;
wk = Z1-Z4;
norm = sqrt(wi*wi + wj*wj + wk*wk);
wi = wi/norm;
wj = wj/norm;
wk = wk/norm;

% [14] Compute the angle gamma between unit vectors w and c using vector
% dot product gamma = acos(w . c)
gamma = acos(wi*ci + wj*cj + wk*ck);

% [15] Compute the angle delta between unit vectors w and u using vector
% dot product delta = acos(w . u)
delta = acos(wi*ui + wj*uj + wk*uk);

% [16] Compute unit vector q perpendicular to plane P2-P1-P4 using vector
% cross product q = (w x c)/sin(gamma)
qi =  (wj*ck - wk*cj)/sin(gamma);
qj = -(wi*ck - wk*ci)/sin(gamma);
qk =  (wi*cj - wj*ci)/sin(gamma);

% [17] Compute unit vector h in the direction of P2 and in the local horizon
% plane using vector cross product h = (q x u)/cos(delta)
hi =  (qj*uk - qk*uj)/cos(delta);
hj = -(qi*uk - qk*ui)/cos(delta);
hk =  (qi*uj - qj*ui)/cos(delta);

% [18] Compute azimuth of section P1-P2-P4 using vector dot product
alpha = acos(ni*hi + nj*hj + nk*hk);
beta  = acos(ei*hi + ej*hj + ek*hk);

if beta > pi/2
    Azdash12 = 2*pi - alpha;
else
    Azdash12 = alpha;
end

% [19] Compute angle between normal section planes at P1
epsilon = abs(Az12-Azdash12);
```

```
% Compute normal section azimuth P2 to P1
numerator   = dX*sin(lon2) - dY*cos(lon2);
denominator = dX*sin(lat2)*cos(lon2)  + dY*sin(lat2)*sin(lon2) - dZ*cos(lat2);
Az21 = atan2(numerator,denominator);
if Az21 < 0
    Az21 = 2*pi+Az21;
end

%-----------------------
% Print result to screen
%-----------------------
fprintf('\n//////////////////////////////////');
fprintf('\n// Normal Section: Inverse Case //');
fprintf('\n//////////////////////////////////');
fprintf('\n\nellipsoid parameters');
fprintf('\na   = %18.9f',a);
fprintf('\nf   = 1/%16.12f',flat);
fprintf('\ne2  = %20.12e',e2);
fprintf('\nep2 = %20.12e',e2);

% Print lat and lon of Point 1
[D,M,S] = DMS(lat1*d2r);
if D == 0 && lat1 < 0
    fprintf('\n\nLatitude  P1 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\n\nLatitude  P1 = %4d %2d %9.6f (D M S)',D,M,S);
end
[D,M,S] = DMS(lon1*d2r);
if D == 0 && lon1 < 0
    fprintf('\nLongitude P1 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLongitude P1 = %4d %2d %9.6f (D M S)',D,M,S);
end

% Print lat and lon of point 2
[D,M,S] = DMS(lat2*d2r);
if D == 0 && lat1 < 0
    fprintf('\n\nLatitude  P2 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\n\nLatitude  P2 = %4d %2d %9.6f (D M S)',D,M,S);
end
[D,M,S] = DMS(lon2*d2r);
if D == 0 && lon2 < 0
    fprintf('\nLongitude P2 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLongitude P2 = %4d %2d %9.6f (D M S)',D,M,S);
end

% Print Coordinate table
fprintf('\n\nCartesian coordinates');
fprintf('\n           X               Y               Z');
fprintf('\nP1   %15.6f %15.6f %15.6f',X1,Y1,Z1);
fprintf('\nP2   %15.6f %15.6f %15.6f',X2,Y2,Z2);
fprintf('\nP3   %15.6f %15.6f %15.6f',X3,Y3,Z3);
fprintf('\nP4   %15.6f %15.6f %15.6f',X4,Y4,Z4);
fprintf('\ndX = %15.6f',dX);
fprintf('\ndY = %15.6f',dY);
fprintf('\ndZ = %15.6f',dZ);

% Print chord distance 1-2
fprintf('\n\nChord distance P1-P2');
fprintf('\nchord   =  %15.6f',chord);

% Print zenith distance of chord at point 1
fprintf('\n\nZenith distance of chord at P1');
[D,M,S] = DMS(zd*d2r);
fprintf('\nzd      = %3d %2d %9.6f  (D M S)',D,M,S);
```

```
% Print azimuths of normal sections
fprintf('\n\nAzimuth of normal section P1-P2');
[D,M,S] = DMS(Az12*d2r);
fprintf('\nAz12    = %3d %2d %9.6f  (D M S)',D,M,S);

fprintf('\n\nAzimuth of normal section P2-P1');
[D,M,S] = DMS(Az21*d2r);
fprintf('\nAz21    = %3d %2d %9.6f  (D M S)',D,M,S);

fprintf('\n\nAzimuth of normal section P2-P1 at P1');
[D,M,S] = DMS(Azdash12*d2r);
fprintf('\nAz''12   = %3d %2d %9.6f  (D M S)',D,M,S);

fprintf('\n\nAngle between normal sections at P1');
[D,M,S] = DMS(epsilon*d2r);
fprintf('\nepsilon = %4d %2d %9.6f (D M S)',D,M,S);

% [20] Compute arc length of normal section using ROMBERG INTEGRATION

fprintf('\n\nROMBERG INTEGRATION TABLE');

% Compute constants of normal section curve P1-P2
ep = sqrt(ep2);
g  = ep*sin(lat1);
h  = ep*cos(lat1)*cos(Az12);
m = 15;
S = zeros(m,m);
finish = 0;
for k = 1:m
    int = 2^k;
    inc = (zd-pion2)/int;
    sum = 0;
    for t = pion2:inc:zd
        x = g*cos(t)+h*sin(t);
        y = h*cos(t)-g*sin(t);
        u = -2*nu1*cos(t);
        v = 1+x*x;
        r = u/v;
        du = 2*nu1*sin(t);
        dv = 2*x*y;
        dr = (v*du-u*dv)/(v*v);
        y = sqrt(r*r + dr*dr);
        sum = sum+2*y;
        if t == pion2
            first = y;
        end
        last = y;
    end
    sum = sum-first-last;
    Integral = inc/2*sum;
    S(k,1) = Integral;
    fprintf('\n%d %15.6f',k,S(k,1));
    for j = 2:k
        S(k,j) = 1/(4^(j-1)-1)*(4^(j-1)*S(k,j-1)-S(k-1,j-1));
        fprintf(' %15.6f',S(k,j));
        diff = abs(S(k,j-1)-S(k,j));
        if diff < 1e-6
            finish = 1;
            s = S(k,j);
            break;
        end
    end
    if finish == 1
        break;
    end
end

% Print normal section distance P1-P2
fprintf('\n\nnormal section distance P1-P2');
```

```
fprintf('\ns =  %15.6f',s);

fprintf('\n\n');
```

MATLAB function *Cart2Geo.m*

```
function [lat,lon,h] = Cart2Geo(a,flat,X,Y,Z)
%
% [lat,lon,h] = Cart2Geo(a,flat,X,Y,Z)
%   Function computes the latitude (lat), longitude (lon) and height (h)
%   of a point related to an ellipsoid defined by semi-major axis (a)
%   and denominator of flattening (flat) given Cartesian coordinates
%   X,Y,Z.  Latitude and longitude are returned as radians.

%--------------------------------------------------------------------------
% Function:  Cart2Geo()
%
% Usage:     [lat,lon,h] = Cart2Geo(a,flat,X,Y,Z);
%
% Author:    R.E.Deakin,
%            School of Mathematical & Geospatial Sciences, RMIT University
%            GPO Box 2476V, MELBOURNE, VIC 3001, AUSTRALIA.
%            email: rod.deakin@rmit.edu.au
%            Version  1.0   6 April  2006
%            Version  1.1  20 August 2007
%
% Functions required:
%    radii()
%
% Purpose:
%    Function Cart2geo() will compute latitude, longitude
%    (both in radians) and height of a  point related to
%    an ellipsoid defined by semi-major axis (a) and
%    denominator of flattening (flat) given Cartesian coordinates
%    X,Y,Z.
%
% Variables:
%    a       - semi-major axis of ellipsoid
%    b       - semi-minor axis of ellipsoid
%    c       - cos(psi)
%    c3      - cos(psi) cubed
%    e2      - 1st eccentricity squared
%    ep2     - 2nd eccentricity squared
%    f       - flattening of ellipsoid
%    flat    - denominator of flattening f = 1/flat
%    h       - height above ellipsoid
%    lat     - latitude (radians)
%    lon     - longitude (radians)
%    p       - perpendicular distance from minor-axis of ellipsoid
%    psi     - parametric latitude (radians)
%    rm      - radius of curvature of meridian section of ellipsoid
%    rp      - radius of curvature of prime vertical section of ellipsoid
%    s       - sin(psi)
%    s3      - sin(psi) cubed
%
% Remarks:
%    This function uses Bowring's method, see Ref [1].
%    Bowring's method is also explained in Ref [2].
%
% References:
% [1] Bowring, B.R., 1976, 'Transformation from spatial to
%     geographical coordinates', Survey Review, Vol. XXIII,
%     No. 181, pp. 323-327.
% [2] Gerdan, G.P. & Deakin, R.E., 1999, 'Transforming Cartesian
%     coordinates X,Y,Z to geogrpahical coordinates phi,lambda,h', The
%     Australian Surveyor, Vol. 44, No. 1, pp. 55-63, June 1999.
%--------------------------------------------------------------------------
```

```
% calculate flattening f and ellipsoid constants e2, ep2 and b
f   = 1/flat;
e2  = f*(2-f);
ep2 = e2/(1-e2);
b   = a*(1-f);

% compute 1st approximation of parametric latitude psi
p   = sqrt(X*X + Y*Y);
psi = atan((Z/p)/(1-f));

% compute latitude from Bowring's equation
s   = sin(psi);
s3  = s*s*s;
c   = cos(psi);
c3  = c*c*c;
lat = atan((Z+b*ep2*s3)/(p-a*e2*c3));

% compute radii of curvature for the latitude
[rm,rp] = radii(a,flat,lat);

% compute longitude and height
lon = atan2(Y,X);
h   = p/cos(lat) - rp;


function [D,M,S] = DMS(DecDeg)
% [D,M,S] = DMS(DecDeg)  This function takes an angle in decimal degrees and returns
%   Degrees, Minutes and Seconds

val = abs(DecDeg);
D = fix(val);
M = fix((val-D)*60);
S = (val-D-M/60)*3600;
if(DecDeg<0)
  D = -D;
end
return
```

MATLAB function *romberg.m*

```
function s = romberg(a,f,lat1,Az12,zd)
%
% s = romberg(a,f,lat,az,zd)
% This function cumputes the arc length of a normal section using Romberg
% Integration, a numerical integration technique using the trapezoidal rule
% and Richardson Extrapolation.  The function requires ellipsoid parameters
% a (semi-major axis) and f (flattening of ellipsoid), lat1 (latitude of P1
% in radians), Az12 (azimuth of normal section plane P1-P2 in radians) and
% zd (zenith distance of the chord of the normal section arc P1-P2).  The
% function returns the arc length s.

%--------------------------------------------------------------------------
% Function:  romberg
%
% Usage:     s = romberg(a,f,lat1,Az12,zd);
%
% Author:    R.E.Deakin,
%            School of Mathematical & Geospatial Sciences, RMIT University
%            GPO Box 2476V, MELBOURNE, VIC 3001, AUSTRALIA.
%            email: rod.deakin@rmit.edu.au
%            Version  1.0  24 September 2009
%
% Purpose:   This function cumputes the arc length of a normal section
%  using Romberg Integration, a numerical integration technique using the
%  trapezoidal rule and Richardson Extrapolation.  The function requires
```

```
%  ellipsoid parameters a,f and lat1 (latitude of P1 in radians), Az12
%  (azimuth of normal section plane P1-P2 in radians) and zd (zenith
%  distance of the chord of the normal section arc P1-P2).
%
% Functions required:
%
% Variables:
%  Az12     - azimuth of normal section P1-P2
%  a        - semi-major axis of spheroid
%  chord    - chord distance between P1 and P2
%  d2r      - degree to radian conversion factor 57.29577951...
%  e2       - eccentricity of ellipsoid squared
%  eps      - 2nd-eccentricity squared
%  f        - f = 1/flat is the flattening of ellipsoid
%  g,h      - constants of normal section curve
%  lat1     - latitude of P1 (radians)
%  nu1      - radius of curvature in prime vertical plane at P1
%  pion2    - pi/2
%  S        - array of normal section arc lengths
%  s        - arc length of normal section P1-P2
%  s2       - sin-squared(latitude)
%  zd       - zenith distance of chord
%
% Remarks:
%
% References:
%  [1] Deakin, R. E., (2009), "The Normal Section Curve on an Ellipsoid",
%          Lecture Notes, School of Mathematical and Geospatial Sciences,
%          RMIT University, November 2009.
%
%--------------------------------------------------------------------------

% Degree to radian conversion factor
d2r   = 180/pi;
pion2 = pi/2;

% Compute ellipsoid constants
e2  = f*(2-f);
ep2 = e2/(1-e2);

% Compute radius of curvature in the prime vertical plane at P1
s2  = sin(lat1)^2;
nu1 = a/sqrt(1-e2*s2);

%---------------------------------------------------------------
% Compute arc length of normal section using ROMBERG INTEGRATION
%---------------------------------------------------------------

% fprintf('\n\nROMBERG INTEGRATION TABLE');

% Compute constants of normal section curve P1-P2
ep = sqrt(ep2);
g  = ep*sin(lat1);
h  = ep*cos(lat1)*cos(Az12);

% Set array of arc lengths
n = 15;
S = zeros(n,n);
finish = 0;
for k = 1:15
    % set the number of intervals and the increment
    int = 2^k;
    inc = (zd-pion2)/int;
    sum = 0;
    % evaluate the integral using the Trapezoidal Rule
    for t = pion2:inc:zd
        x = g*cos(t)+h*sin(t);
        y = h*cos(t)-g*sin(t);
        u = -2*nu1*cos(t);
```

```
        v = 1+x*x;
        r = u/v;
        du = 2*nu1*sin(t);
        dv = 2*x*y;
        dr = (v*du-u*dv)/(v*v);
        y = sqrt(r*r + dr*dr);
        sum = sum+2*y;
        if t == pion2
            first = y;
        end
        last = y;
    end
    sum = sum-first-last;
    Integral = inc/2*sum;
    S(k,1) = Integral;
%     fprintf('\n%d %15.6f',k,S(k,1));
    % Use Richardson extrapolation
    for j = 2:k
        S(k,j) = 1/(4^(j-1)-1)*(4^(j-1)*S(k,j-1)-S(k-1,j-1));
%         fprintf(' %15.6f',S(k,j));
        diff = abs(S(k,j-1)-S(k,j));
        if diff < 1e-6
            finish = 1;
            s = S(k,j);
            break;
        end
    end
    if finish == 1
        break;
    end
end
```

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Bowring, B. R., (1972), 'Distance and the spheroid', Correspondence, *Survey Review,* Vol. XXI, No. 164, April 1972, pp. 281-284.

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APPENDIX 1: ROMBERG INTEGRATION

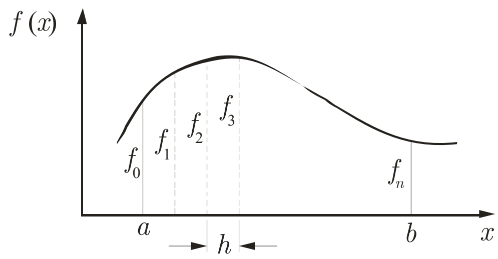
Romberg integration (Romberg 1955) is a numerical technique for evaluating a definite integral and discussions of the technique can be found in most textbooks on numerical analysis; e.g. Williams (1972). A concise treatment of the technique and a study of the historical development of methods of integration (quadrature) can be found in Dutka (1984). A development of Romberg's method – and the extrapolation formula that is at the heart of it – is given below and is followed by a MATLAB function that demonstrates the use of the technique.

Romberg integration is a method for estimating the numerical value of the definite integral

$$I = \int_a^b f(x)\,dx \tag{54}$$

It is based on the trapezoidal rule – the simplest of the Newton-Cotes integration formula for equally spaced data on the interval a,b

$$I = \int_a^b f(x)\,dx = \frac{h}{2}\left(f_0 + 2f_1 + 2f_2 + \cdots + 2f_{n-1} + f_n\right) + E \tag{55}$$



where

n is the number of intervals of width h,

$h = \dfrac{b-a}{n}$ is the common interval width or spacing,

$f_0, f_1, f_2, \ldots$ are values of the function evaluated at $x = \left[a,\ a+h,\ a+2h,\ \ldots\right]$,

E is the error term

When the function $f(x)$ has continuous derivatives the error term E can be expressed as a convergent power series and we may write

$$I = \int_a^b f(x)\,dx = \frac{h}{2}\left(f_0 + 2f_1 + 2f_2 + \cdots + 2f_{n-1} + f_n\right) + E = T + \sum_{j=1}^{\infty} a_j h^{2j} \tag{56}$$

where a_j are coefficients.

As the error term E is a convergent power series in h a technique known as Richardson extrapolation[1] may be employed to improve the accuracy of the result.

Richardson extrapolation can be explained as follows.

Let the value of n be a power of 2; say 2^k i.e., the number of intervals $n = 2, 4, 8, 16, \ldots, 2^k$

Denote an evaluation of the integral I given by equation (56) as

$$S_{k,1} = T + \sum_{j=1}^{\infty} a_j h^{2j} = T + a_1 h^2 + a_2 h^4 + a_3 h^6 + \cdots \tag{57}$$

If the interval width is halved, then

$$S_{k+1,1} = T + \sum_{j=1}^{\infty} a_j \left(\frac{h}{2}\right)^{2j} = T + a_1 \frac{1}{2^2} h^2 + a_2 \frac{1}{2^4} h^4 + a_3 \frac{1}{2^6} h^6 + \cdots \tag{58}$$

The first term of the error series can be eliminated by taking suitable combinations of equations (57) and (58); i.e., multiplying equation (58) by 4 and then subtracting equation (57) will eliminate the first term of the error series

$$\begin{aligned} 4S_{k+1,1} - S_{k,1} &= 4T - T + a_2 \left(\frac{4h^4}{2^4} - h^4\right) + a_3 \left(\frac{4h^6}{2^6} - h^6\right) + \cdots \\ &= 3T + \sum_{j=2}^{\infty} a_j \left(\frac{4h^{2j}}{2^{2j}} - h^{2j}\right) \end{aligned}$$

and

$$T = \frac{4S_{k+1,1} - S_{k,1}}{3} - \sum_{j=2}^{\infty} \frac{a_j h^{2j}}{3} \left(\frac{4}{2^{2j}} - 1\right) \tag{59}$$

The first term on the right-hand-side of equation (59) will be designated

$$S_{k,2} = \frac{4S_{k+1,1} - S_{k,1}}{3}$$

and the leading error term is now of order h^4.

[1] A technique named after Lewis Fry Richardson (1881–1953) a British applied mathematician, physicist, meteorologist, psychologist and pacifist who developed the numerical methods used in weather forecasting and also applied his mathematical techniques to the analysis of the causes and prevention of wars. He was also a pioneer in the study of fractals. Richardson extrapolation is also known as Richardson's deferred approach to the limit.

Successive halvings of the interval will give a sequence of values $S_{1,1}, S_{2,1}, S_{3,1}, \ldots, S_{k,1}$ and each successive pair $(S_{1,1}, S_{2,1}), (S_{2,1}, S_{3,1}), \ldots$ can be combined to give values $S_{2,2}, S_{3,2}, \ldots$; and this next sequence can be combined in a similar manner to remove the leading error term of order h^4; and so on.

By using the formula

$$S_{k,j} = \frac{1}{4^{j-1} - 1}\left(4^{j-1} S_{k,j-1} - S_{k-1,j-1}\right) \qquad \begin{array}{l} k = 1,2,3,4,\ldots \\ j = 2,3,4,5,\ldots \end{array} \tag{60}$$

the process of Richardson extrapolation leads to a triangular sequence of columns with error terms of increasing order.

n	$k \backslash j$	1	2	3	4	
2	1	$S_{1,1}$				
4	2	$S_{2,1}$	$S_{2,2}$			
16	3	$S_{3,1}$	$S_{3,2}$	$S_{3,3}$		
32	4	$S_{4,1}$	$S_{4,2}$	$S_{4,3}$	$S_{4,4}$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
error term		h^2	h^4	h^6	h^8	

The entries $S_{k,2}$ in the second column have eliminated the terms involving h^2, the entries in the third column have eliminated the terms involving h^4, etc, and as the interval $h = \dfrac{b-a}{2^k}$ the error term of the approximation $S_{k,j}$ is of the order $\left(\dfrac{b-a}{2^k}\right)^{2j}$ with each successive value in a particular row converging more rapidly to the true value of the integral.

Testing between particular values will determine when the process has converged to a suitable result.

MATLAB FUNCTION *romberg_ test.m*

This function uses Romberg Integration for the calculation of the integral $\int \sec\left(x\right)dx$

This integral has the known result $\int \sec\left(x\right)dx = \ln\left[\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right]$

MATLAB function *romberg_ test.m*

```
function romberg_test
%
% This function computes the numerical value of the integral of sec(x)
% which is known to equal ln[tan(x/2+pi/4)].
% For x = 45 degrees the integral sec(x) = 0.881373587020.
% An integration table is produced that shows the convergence to the true
% value of the integral.

%--------------------------------------------------------------------------
% Function:  romberg_test
%
% Author:    R.E.Deakin,
%            School of Mathematical & Geospatial Sciences, RMIT University
%            GPO Box 2476V, MELBOURNE, VIC 3001, AUSTRALIA.
%            email: rod.deakin@rmit.edu.au
%            Version  1.0  09 December 2009
%
% Purpose:   This function computes the numerical value of the integral of
%  sec(x) which is known to equal ln[tan(x/2+pi/4)].
%  For x = 45 degrees the integral sec(x) = 0.881373587.
%  An integration table is produced that shows the convergence to the true
%  value of the integral.
%
% Variables:
%  diff     - difference between successive approximations of the integral
%  d2r      - degree to radian conversion factor 57.29577951...
%  first    - first value of f(x)
%  fx       - value of f(x)
%  h        - interval width
%  Integral - numerical value of integral from trapezoidal rule
%  k,j      - integer counters
%  last     - last value of f(x)
%  m        - maximum number of intervals
%  n        - number of intervals
%  S        - array of integral values
%  sum      - sum of function values
%  x        - the variable
%
% References:
%  Williams, P. W., (1972), "Numerical Computation", Nelson, London.
%
%--------------------------------------------------------------------------

% Degree to radian conversion factor
d2r   = 180/pi;

fprintf('\n\nRomberg Integration Table for the integral of sec(x) for x = 45 degrees');

% Set array of values S(k,j)
m = 15;
S = zeros(m,m);
finish = 0;
for k = 1:m
    % set the number of intervals and the increment
```

```
    n = 2^k;
    h = 45/n;
    sum = 0;
    % evaluate the integral using the Trapezoidal Rule
    for x = 0:h:45
        fx = 1/cos(x/d2r);
        sum = sum+2*fx;
        if x == 0
            first = fx;
        end
        last = fx;
    end
    sum = sum-first-last;
    Integral = h/d2r/2*sum;
    S(k,1) = Integral;
    fprintf('\n%d %15.12f',k,S(k,1));
    % Use Richardson extrapolation
    for j = 2:k
        S(k,j) = 1/(4^(j-1)-1)*(4^(j-1)*S(k,j-1)-S(k-1,j-1));
        fprintf(' %15.12f',S(k,j));
        diff = abs(S(k,j-1)-S(k,j));
        if diff < 1e-12
            finish = 1;
            break;
        end
    end
    if finish == 1
        break;
    end
end
fprintf('\n\n');
```

MATLAB Command Window

```
>> help romberg_test

  This function cumputes the numerical value of the integral of sec(x)
  which is known to equal ln[tan(x/2+pi/4)].
  For x = 45 degrees the integral sec(x) = 0.881373587020.
  An integration table is produced that shows the convergence to the true
  value of the integral.


>> romberg_test


Romberg Integration Table for the integral of sec(x) for x = 45 degrees
1  0.899084147577
2  0.885885914440  0.881486503395
3  0.882507477613  0.881381332003  0.881374320577
4  0.881657432521  0.881374084157  0.881373600967  0.881373589544
5  0.881444571861  0.881373618307  0.881373587251  0.881373587033  0.881373587023
6  0.881391334699  0.881373588978  0.881373587023  0.881373587020  0.881373587020

>>
```

The output from the function *Romberg_ test.m* that is evaluating the integral

$$I = \int_{x=0^\circ}^{x=45^\circ} \sec\left(x\right) dx$$

is shown in the Romberg Integration Table and the elements are obtained as follows:

- For $k = 1$, there are $n = 2^k = 2$ intervals (or strips) of width h where

$h = \dfrac{b-a}{n} = \dfrac{45^\circ - 0^\circ}{2} = 22.50^\circ$ and the integral $I \simeq \dfrac{h}{2}\left(f_0 + 2f_1 + f_2\right)$. The function $f\left(x\right) = \sec x = \dfrac{1}{\cos x}$ evaluated at $x = 0^\circ, 22.5^\circ, 45^\circ$ gives

$$\begin{aligned} f_0 &= 1 \\ f_1 &= 1.082392200 \\ f_2 &= 1.414213562 \end{aligned}$$

and

$$S_{1,1} = I = \frac{22.5}{2}\left(\frac{\pi}{180}\right)\left(1 + 2\left(1.082392200\right) + 1.414213562\right) = 0.899084148$$

- For $k = 2$, there are $n = 2^k = 4$ intervals (or strips) of width h where

$h = \dfrac{b-a}{n} = \dfrac{45^\circ - 0^\circ}{4} = 11.25^\circ$ and the integral $I \simeq \dfrac{h}{2}\left(f_0 + 2f_1 + 2f_3 + f_4\right)$. The function $f\left(x\right) = \sec x = \dfrac{1}{\cos x}$ evaluated at $x = 0^\circ, 11.25^\circ, 22.5^\circ, 33.75^\circ, 45^\circ$ gives

$$\begin{aligned} f_0 &= 1 \\ f_1 &= 1.019591158 \\ f_2 &= 1.082392200 \\ f_3 &= 1.202689774 \\ f_4 &= 1.414213562 \end{aligned}$$

and

$$S_{2,1} = I = \frac{11.25}{2}\left(\frac{\pi}{180}\right)\left(1 + 2\left(1.019\ldots\right) + 2\left(1.082\ldots\right) + 2\left(1.202\ldots\right) + 1.414\ldots\right) = 0.885885914$$

The element $S_{2,2}$ is obtained from equation (60)

$$S_{2,2} = \frac{1}{4^1 - 1}\left(4^1 S_{2,1} - S_{1,1}\right) = \frac{1}{3}\left(4 \times 0.885885914 - 0.899084148\right) = 0.881486503$$

- For $k=3$, there are $n=2^k=8$ intervals (or strips) of width $h=5.625^\circ$ and the integral $I \simeq \frac{h}{2}\left(f_0+2f_1+2f_3+\cdots+2f_7+f_8\right)$. The function $f\left(x\right)=\sec x=\frac{1}{\cos x}$ evaluated at $x=0^\circ, 5.625^\circ, 11.25^\circ, \ldots, 39.375^\circ, 45^\circ$ gives

$$\begin{aligned}
f_0 &= 1\\
f_1 &= 1.004838572\\
f_2 &= 1.019591158\\
&\vdots\\
f_7 &= 1.293643567\\
f_8 &= 1.414213562
\end{aligned}$$

and

$$S_{3,1}=I=\frac{5.625}{2}\left(\frac{\pi}{180}\right)\left(1+2\left(1.004\ldots\right)+\cdots+2\left(1.293\ldots\right)+1.414\ldots\right)=0.882507478$$

The elements $S_{3,2}$ and $S_{3,3}$ are obtained from equation (60)

$$\begin{aligned}
S_{3,2} &= \frac{1}{4^1-1}\left(4^1 S_{3,1}-S_{2,1}\right)=\frac{1}{3}\left(4\times 0.882507478-0.885885914\right) &&= 0.881381333\\
S_{3,3} &= \frac{1}{4^2-1}\left(4^2 S_{3,2}-S_{2,2}\right)=\frac{1}{15}\left(16\times 0.881381333-0.881486503\right) &&= 0.8813374322
\end{aligned}$$

And so on for increasing values of k

Testing between successive values $S_{k,j-1}$ and $S_{k,j}$ can be used to determine when the iterative procedure is terminated.

THE CURVE OF ALIGNMENT
ON AN ELLIPSOID

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December 2009

ABSTRACT

These notes provide a detailed derivation of the equation for the curve of alignment on an ellipsoid. Using this equation and knowing the terminal points of the curve, a technique is developed for computing the location of points along the curve. A MATLAB function is provided that demonstrates the algorithm developed.

INTRODUCTION

In geodesy, the <u>curve of alignment</u> between P_1 and P_2 on the ellipsoid is the locus of a point P on the surface that moves so that a normal section plane at P contains the terminal points P_1 and P_2.

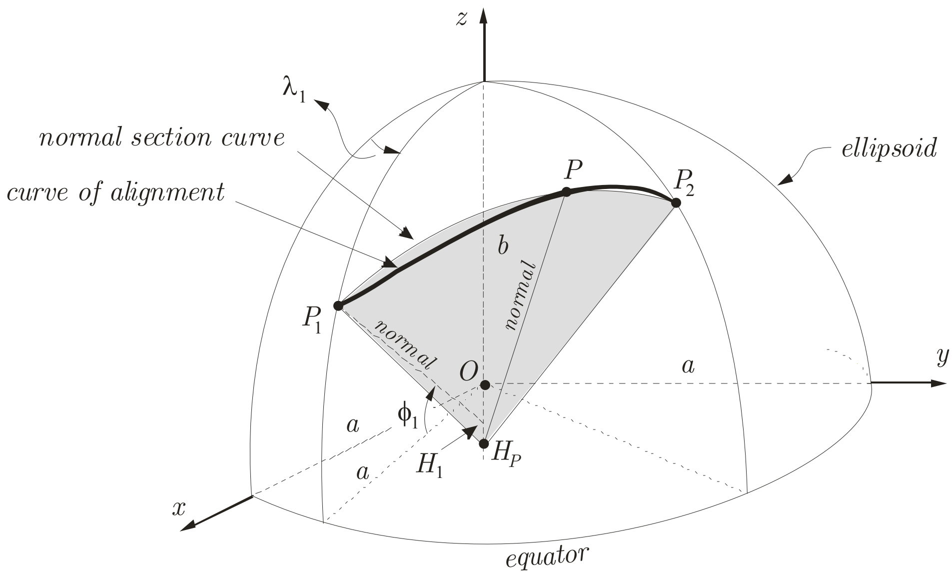


Figure 1: Curve of alignment on ellipsoid

Figure 1 shows P on the curve of alignment between P_1 and P_2. The normal to the ellipsoid at P intersects the z-axis of the ellipsoid at H_P and is contained in the plane $P_1PP_2H_P$. This normal section plane cuts the ellipsoid along the normal section curve P_1PP_2. As P moves from P_1 to P_2 – maintaining the condition that a normal section plane contains P_1 and P_2 – it traces out the curve of alignment. This is a curve on the surface having both curvature and torsion, i.e., it twists across the surface between P_1 and P_2. Note that in Figure 1, the normal at P_1 intersects the z-axis at H_1 and is not contained in the plane $P_1PP_2H_P$, unless P is at P_1.

The curve of alignment can also be described physically in the following way. Imagine a theodolite, in adjustment, that is setup on the surface of the ellipsoid somewhere between P_1 and P_2, and whose vertical axis is coincident with the ellipsoid normal. The theodolite is pointed to the backsight P_1 and the horizontal circle is clamped; then the telescope is rotated in the vertical plane and pointed towards the forsight P_2. Unless there is some fluke of positioning, it is unlikely that the theodolite cross-hairs will bisect the target P_2. So the theodolite is repositioned by moving appropriate amounts perpendicular to the line until the vertical plane of the theodolite at P contains both the backsight P_1 and the forsight P_2. A peg is place on the surface at this point. This process of "jiggling in" or "middling in" between P_1 and P_2 is repeated a short distance further along the line and another peg placed. After the last peg has been placed the curve of alignment is now defined by the pegged line on the surface.

The curve of alignment follows a path very similar to that of the geodesic and it is slightly longer; although the difference is practicably negligible at distances less than 5,000 km. This will be demonstrated below using equations developed by Clarke (1880) and Bowring (1972).

The equation for the curve developed below is similar to that derived by Thomas (1952) although the method of development is different; and it is not in a form suitable for computing the distance or azimuth of the curve. But, as it contains functions of both the latitude and longitude of a point on the curve, it is suitable for computing the latitude of a point (by iteration) given a certain longitude. Alternatively, by choosing suitable functions of given latitude, the longitude of a point on the curve can be computed directly (by solving a trigonometric equation).

EQUATION OF CURVE OF ALIGNMENT

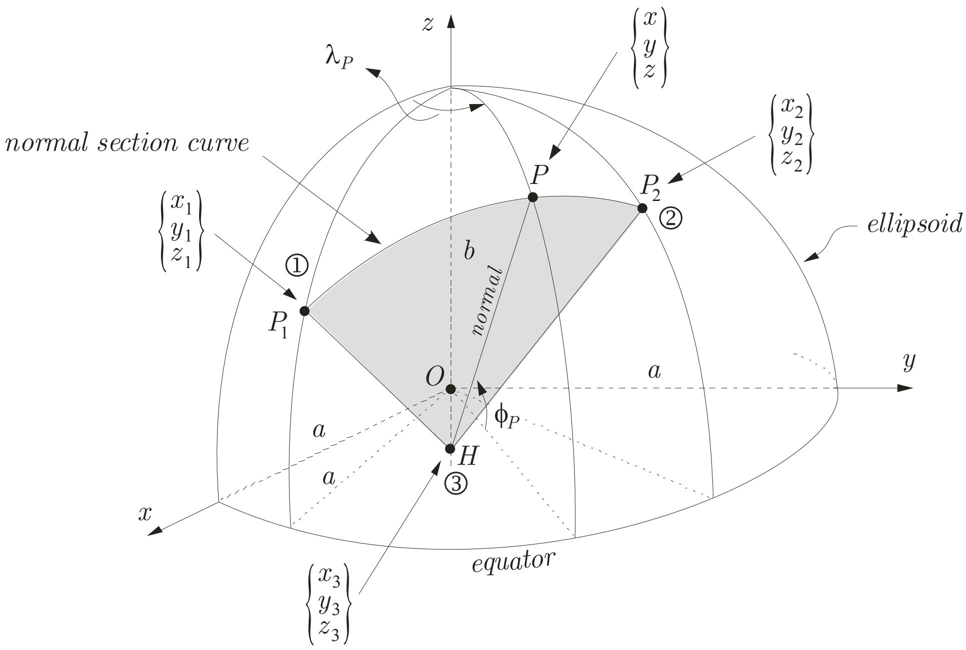


Figure 2: Normal section plane containing P_1 and P_2

Figure 2 shows a normal section plane of P on an ellipsoid that passes through P_1 and P_2. The semi-axes of the ellipsoid are a and b $(a > b)$ and the first-eccentricity squared e^2, second-eccentricity squared e'^2 and the flattening f of the ellipsoid are defined by

$$\begin{aligned}
e^2 &= \frac{a^2 - b^2}{a^2} = f(2-f) \\
e'^2 &= \frac{a^2 - b^2}{b^2} = \frac{f(2-f)}{(1-f)^2} = \frac{e^2}{1-e^2} \\
f &= \frac{a-b}{a}
\end{aligned} \tag{1}$$

Parallels of latitude ϕ and meridians of longitude λ have their respective reference planes; the equator and the Greenwich meridian, and Longitudes are measured 0° to $\pm 180^\circ$ (east positive, west negative) from the Greenwich meridian and latitudes are measured 0° to $\pm 90^\circ$ (north positive, south negative) from the equator. The x,y,z geocentric Cartesian coordinate system has an origin at O, the centre of the ellipsoid, and the z-axis is the minor axis (axis of revolution). The xOz plane is the Greenwich meridian plane (the origin of longitudes) and the xOy plane is the equatorial plane. The positive x-axis passes through the intersection of the Greenwich meridian and the equator, the positive y-axis is advanced 90° east along the equator and the positive z-axis passes through the north pole of the ellipsoid.

The normal section plane in Figure 2 is defined by points ①, ② and ③ that are P_1, P_2 and H respectively where H is at the intersection of the normal through P and the z-axis. Cartesian coordinates of ① and ② are computed from the following equations

$$\begin{aligned} x &= \nu\cos\phi\cos\lambda \\ y &= \nu\cos\phi\sin\lambda \\ z &= \nu\left(1-e^2\right)\sin\phi \end{aligned} \tag{2}$$

where $\nu = PH$ is the radius of curvature in the prime vertical plane and

$$\nu = \frac{a}{\sqrt{1-e^2\sin^2\phi}} \tag{3}$$

The distance $OH = \nu e^2 \sin\phi$ and the Cartesian coordinates of point ③ are

$$\begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\nu e^2\sin\phi \end{bmatrix} \tag{4}$$

The General equation of a plane may be written as

$$Ax + By + Cz + D = 0 \tag{5}$$

And the equation of the plane passing through points ①, ② and ③ is given in the form of a 3rd-order determinant

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_2 & y_3-y_2 & z_3-z_2 \end{vmatrix} = 0 \tag{6}$$

or expanded into 2nd-order determinants

$$\begin{vmatrix} y_2-y_1 & z_2-z_1 \\ y_3-y_2 & z_3-z_2 \end{vmatrix}\left(x-x_1\right) - \begin{vmatrix} x_2-x_1 & z_2-z_1 \\ x_3-x_2 & z_3-z_2 \end{vmatrix}\left(y-y_1\right) + \begin{vmatrix} x_2-x_1 & y_2-y_1 \\ x_3-x_2 & y_3-y_2 \end{vmatrix}\left(z-z_1\right) = 0 \tag{7}$$

Expanding the determinants in equation (7) gives

$$\begin{aligned} &\left(x-x_1\right)\left\{\left(y_2-y_1\right)\left(z_3-z_2\right)-\left(z_2-z_1\right)\left(y_3-y_2\right)\right\} \\ &-\left(y-y_1\right)\left\{\left(x_2-x_1\right)\left(z_3-z_2\right)-\left(z_2-z_1\right)\left(x_3-x_2\right)\right\} \\ &+\left(z-z_1\right)\left\{\left(x_2-x_1\right)\left(y_3-y_2\right)-\left(y_2-y_1\right)\left(x_3-x_2\right)\right\} = 0 \end{aligned} \tag{8}$$

Now from equation (4) $x_3 = y_3 = 0$ and equation (8) becomes

$$\begin{aligned} &\left(x-x_1\right)\left(y_2-y_1\right)\left(z_3-z_2\right)-\left(x-x_1\right)\left(z_2-z_1\right)\left(-y_2\right) \\ &-\left(y-y_1\right)\left(x_2-x_1\right)\left(z_3-z_2\right)+\left(y-y_1\right)\left(z_2-z_1\right)\left(-x_2\right) \\ &+\left(z-z_1\right)\left(x_2-x_1\right)\left(-y_2\right) \quad +\left(z-z_1\right)\left(y_2-y_1\right)\left(-x_2\right)=0 \end{aligned} \tag{9}$$

Expanding and simplifying equation (9) gives

$$\begin{aligned} &xz_3\left(y_2-y_1\right)+x\left(y_1z_2-y_2z_1\right)+z_3\left(x_2y_1-x_1y_2\right) \\ &+yz_3\left(x_1-x_2\right)+y\left(x_2z_1-x_1z_2\right)+z\left(x_1y_2-x_2y_1\right)=0 \end{aligned} \tag{10}$$

Now from equations (2) and (4) $x = \nu\cos\phi\cos\lambda$, $y = \nu\cos\phi\sin\lambda$, $z = \nu\left(1-e^2\right)\sin\phi$ and $z_3 = -\nu e^2\sin\phi$, and substituting these into equation (10) gives

$$\begin{aligned} &\nu e^2\left\{\left(x_2-x_1\right)\sin\lambda-\left(y_2-y_1\right)\cos\lambda\right\}\sin\phi-\left(y_2z_1-y_1z_2\right)\cos\lambda \\ &+\left(x_2z_1-x_1z_2\right)\sin\lambda-\left(x_2y_1-x_1y_2\right)\tan\phi=0 \end{aligned}$$

that is equivalent to

$$\begin{aligned} &\nu\left(1-e^2\right)\left\{e^2\left(y_2-y_1\right)\cos\lambda-e^2\left(x_2-x_1\right)\sin\lambda\right\}\sin\phi-\left(1-e^2\right)\left(y_1z_2-y_2z_1\right)\cos\lambda \\ &-\left(1-e^2\right)\left(x_1z_2-x_2z_1\right)\sin\lambda-\left(1-e^2\right)\left(x_1y_2-x_2y_1\right)\tan\phi=0 \end{aligned}$$

or, following Thomas (1952, p. 67, eq. 183); the equation of the curve of alignment is

$$\boxed{\nu\left(1-e^2\right)\left\{C\cos\lambda-H\sin\lambda\right\}\sin\phi-U\cos\lambda-V\sin\lambda-W\left(1-e^2\right)\tan\phi=0} \tag{11}$$

where

$$\begin{aligned} C&=e^2\left(y_2-y_1\right) \qquad U=\left(1-e^2\right)\left(y_1z_2-y_2z_1\right) \qquad W=x_1y_2-x_2y_1 \\ H&=e^2\left(x_2-x_1\right) \qquad V=\left(1-e^2\right)\left(x_2z_1-x_1z_2\right) \end{aligned} \tag{12}$$

Equation (11) is not suitable for computing the distance along a curve of alignment, nor is it suitable for computing the azimuth of the curve, but by certain re-arrangements it is possible to solve (iteratively) for the latitude of a point on the curve given a longitude somewhere between the longitudes of the terminal points of the curve. Or alternatively, solve (a trigonometric equation) for the longitude of a point given a latitude somewhere between the latitudes of the terminal points.

SOLVING FOR THE LATITUDE

Equation (11) can be re-arranged as

$$\boxed{A\nu\sin\phi - B\tan\phi - D = 0 \qquad (13)}$$

where A and D are functions of longitude alone and B is a constant for the curve, and

$$A = \left(1-e^2\right)\left(C\cos\lambda - H\sin\lambda\right); \quad B = W\left(1-e^2\right); \quad D = U\cos\lambda + V\sin\lambda \tag{14}$$

C, H, U, V and W are constants for the particular curve and are given by equation (12). ν is a function of the latitude of P on the curve and is given by equation (3).

The latitude ϕ can be evaluated using Newton-Raphson iteration for the real roots of the equation $f\left(\phi\right)=0$ given in the form of an iterative equation

$$\phi_{(n+1)} = \phi_{(n)} - \frac{f\left(\phi_{(n)}\right)}{f'\left(\phi_{(n)}\right)} \tag{15}$$

where n denotes the n^{th} iteration and $f\left(\phi\right)$ is given by equation (13) as

$$f\left(\phi\right) = A\nu\sin\phi - B\tan\phi - D \tag{16}$$

and the derivative $f'\left(\phi\right) = \dfrac{d}{d\phi}\left\{f\left(\phi\right)\right\}$ is given by

$$f'\left(\phi\right) = \frac{d\nu}{d\phi}A\sin\phi + \nu A\cos\phi - B\sec^2\phi \tag{17}$$

where, from equation (3)

$$\frac{d\nu}{d\phi} = \frac{\nu^3}{a^2}e^2\sin\phi\cos\phi \tag{18}$$

An initial value of $\phi_{(1)}$ (ϕ for $n=1$) can be taken as the latitude of P_1 and the functions $f\left(\phi_{(1)}\right)$ and $f'\left(\phi_{(1)}\right)$ evaluated from equations (16) and (17) using ϕ_1. $\phi_{(2)}$ (ϕ for $n=2$) can now be computed from equation (15) and this process repeated to obtain values $\phi_{(3)}, \phi_{(4)}, \ldots$. This iterative process can be concluded when the difference between $\phi_{(n+1)}$ and $\phi_{(n)}$ reaches an acceptably small value.

SOLVING FOR THE LONGITUDE

Equation (11) can also be re-arranged as

$$P\cos\lambda - Q\sin\lambda = S \tag{19}$$

where P, Q and S are functions of latitude alone and

$$P = C\nu\left(1-e^2\right)\sin\phi - U; \quad Q = H\nu\left(1-e^2\right)\sin\phi + V; \quad S = W\left(1-e^2\right)\tan\phi \tag{20}$$

C, H, U, V and W are constants for the particular curve and are given by equation (12). ν is a function of the latitude of P on the curve and is given by equation (3).

The longitude can be evaluated using Newton-Raphson iteration where

$$\lambda_{(n+1)} = \lambda_{(n)} - \frac{f\left(\lambda_{(n)}\right)}{f'\left(\lambda_{(n)}\right)} \tag{21}$$

and

$$\begin{aligned} f\left(\lambda\right) &= P\cos\lambda - Q\sin\lambda - S \\ f'\left(\lambda\right) &= -P\sin\lambda - Q\cos\lambda \end{aligned} \tag{22}$$

An initial value of $\lambda_{(1)}$ (λ for $n=1$) can be taken as the longitude of P_1.

Alternatively, the longitude can be evaluated by a trigonometric equation derived as follows. Equation (19) can be expressed as a trigonometric addition of the form

$$\begin{aligned} S &= R\cos\left(\lambda-\theta\right) \\ &= R\cos\lambda\cos\theta + R\sin\lambda\sin\theta \end{aligned} \tag{23}$$

Now, equating the coefficients of $\cos\lambda$ and $\sin\lambda$ in equations (19) and (23) gives

$$P = R\cos\theta; \quad Q = -R\sin\theta \tag{24}$$

and using these relationships

$$R = \sqrt{P^2+Q^2}; \quad \tan\theta = \frac{-Q}{P} \tag{25}$$

Substituting these results into equation (23) gives

$$\lambda = \arccos\left\{\frac{S}{\sqrt{P^2+Q^2}}\right\} + \arctan\left\{\frac{-Q}{P}\right\} \tag{26}$$

DIFFERENCE IN LENGTH BETWEEN A GEODESIC AND CURVE OF ALIGNMENT

There are five curves of interest in geodesy; the geodesic, the normal section, the great elliptic arc the loxodrome and the curve of alignment.

The geodesic between P_1 and P_2 on an ellipsoid is the unique curve on the surface defining the shortest distance; all other curves will be longer in length. The normal section curve P_1P_2 is a plane curve created by the intersection of the normal section plane containing the normal at P_1 and also P_2 with the ellipsoid surface. And as we have shown (Deakin 2009) there is the other normal section curve P_2P_1. The curve of alignment is the locus of all points P such that the normal section plane at P also contains the points P_1 and P_2. The curve of alignment is very close to a geodesic. The great elliptic arc is the plane curve created by intersecting the plane containing P_1, P_2 and the centre O with the surface of the ellipsoid and the loxodrome is the curve on the surface that cuts each meridian between P_1 and P_2 at a constant angle.

Approximate equations for the difference in length between the geodesic, the normal section curve and the curve of alignment were developed by Clarke (1880, p. 133) and Bowring (1972, p. 283) developed an approximate equation for the difference between the geodesic and the great elliptic arc. Following Bowring (1972), let

$$\begin{aligned} s &= \text{ geodesic length} \\ L &= \text{ normal section length} \\ D &= \text{ great elliptic length} \\ S &= \text{ curve of alignment length} \end{aligned}$$

then

$$\begin{aligned} L - s &= \frac{e^4}{90} s \left(\frac{s}{R}\right)^4 \cos^4\phi_1 \sin^2\alpha_{12} \cos^2\alpha_{12} + \cdots \\ D - s &= \frac{e^4}{24} s \left(\frac{s}{R}\right)^2 \sin^2\phi_1 \cos^2\phi_1 \sin^2\alpha_{12} + \cdots \\ S - s &= \frac{e^4}{360} s \left(\frac{s}{R}\right)^4 \cos^4\phi_1 \sin^2\alpha_{12} \cos^2\alpha_{12} + \cdots \end{aligned} \tag{27}$$

where R can be taken as the radius of curvature in the prime vertical at P_1. Now for a given value of s, $S - s$ will be a maximum if $\phi_1 = 0^\circ$ (P_1 on the equator) and $\alpha_{12} = 45^\circ$ in which case $\cos^4\phi_1 \sin^2\alpha_{12} \cos^2\alpha_{12} = \dfrac{1}{4}$, thus

$$\left(S-s\right)<\frac{e^4}{1440}s\left(\frac{s}{R}\right)^4 \tag{28}$$

For the GRS80 ellipsoid where $f=1/298.257222101$, $e^2=f\left(2-f\right)$, and for $s=2000000$ m (2,000 km) and $R=6371000$ m , equation (28) gives $S-s<0.001$ m .

MATLAB FUNCTIONS

Two MATLAB functions are shown below; they are: *curve_ of_ alignment_ lat.m* and *curve_ of_ alignment_ lon.m* Assuming that the terminal points of the curve are known, the first function computes the latitude of a point on the curve given a longitude and the second function computes the longitude of a point given the latitude.

Output from the two functions is shown below for points on a curve of alignment between the terminal points of the straight-line section of the Victorian–New South Wales border. This straight-line section of the border, between Murray Spring and Wauka 1978, is known as the Black-Allan Line in honour of the surveyors Black and Allan who set out the border line in 1870-71. Wauka 1978 (Gabo PM 4) is a geodetic concrete border pillar on the coast at Cape Howe and Murray Spring (Enamo PM 15) is a steel pipe driven into a spring of the Murray River that is closest to Cape Howe. The straight line is a normal section curve on the reference ellipsoid of the Geocentric Datum of Australia (GDA94) that contains the normal to the ellipsoid at Murray Spring. The GDA94 coordinates of Murray Spring and Wauka 1978 are:

Murray Spring: $\phi\ -37^\circ\, 47'\, 49.2232''$ $\lambda\ 148^\circ\, 11'\, 48.3333''$
Wauka 1978: $\phi\ -37^\circ\, 30'\, 18.0674''$ $\lambda\ 149^\circ\, 58'\, 32.9932''$

The normal section azimuth and distance are:

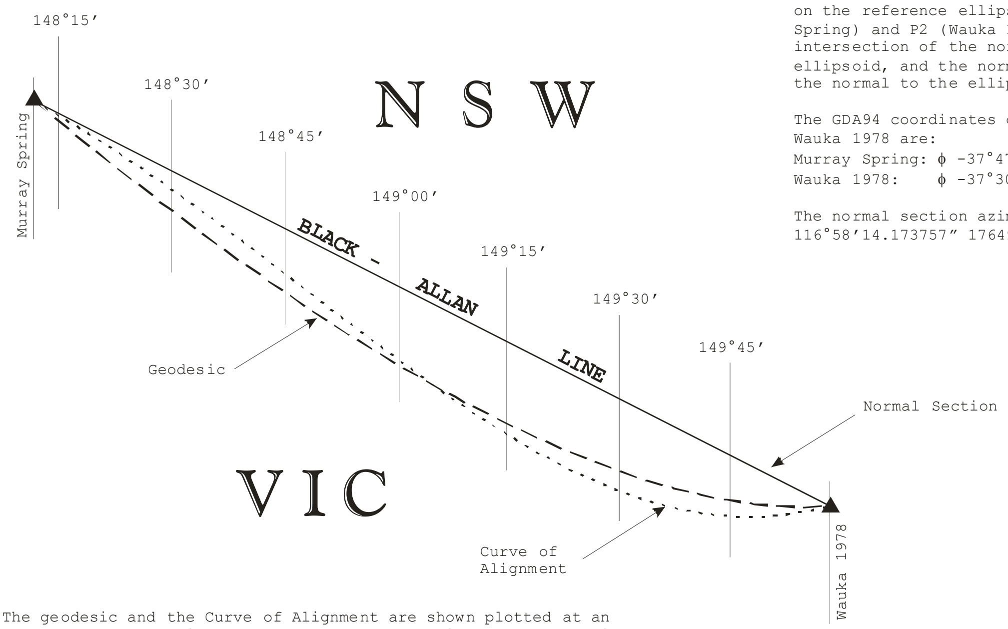
$116^\circ\, 58'\, 14.173757''$ 176495.243760 m

The geodesic azimuth and distance are:

$116^\circ\, 58'\, 14.219146''$ 176495.243758 m

Figure 3 shows a schematic view of the Black-Allan line (normal section) and the geodesic and curve of alignment. The relationships between these two curves and the normal section have been computed at seven locations along the line (A, B, C, etc.) where meridians of longitude at $0^\circ\, 15'$ intervals cut the line. The relationships are shown in Table 1.

BLACK-ALLAN LINE: VICTORIA/NSW BORDER



The Black-Allan Line is a normal section curve on the reference ellipsoid between P1 (Murray Spring) and P2 (Wauka 1978). This curve is the intersection of the normal section plane and the ellipsoid, and the normal section contains P1, the normal to the ellipsoid at P1, and P2.

The GDA94 coordinates of Murray Spring and Wauka 1978 are:
Murray Spring: ϕ -37°47'49.2232" λ 148°11'48.3333"
Wauka 1978: ϕ -37°30'18.0674" λ 149°58'32.9932"

The normal section azimuth and distance are:
116°58'14.173757" 176495.243760 m.

The geodesic and the Curve of Alignment are shown plotted at an exaggerated scale with respect to the Border Line (normal section).
At longitude 149°00'E. the Geodesic is 0.016 m south of the Border Line and the Curve of Alignment is 0.015 m south.
At longitude 149°30'E. the Geodesic is 0.015 m south of the Border Line and the Curve of Alignment is 0.019 m south.

Figure 3

BLACK-ALLAN LINE: VICTORIA/NSW BORDER

NAME	GDA94		Ellipsoid values		
	LATITUDE	LONGITUDE	dφ	ρ	dm = ρ×dφ
Murray Spring	-36°47´49.223200″	148°11´48.333300″			
A	-36°49´07.598047″ N -36°49´07.598090″ G -36°49´07.598051″ CoA	148°15´00.000000″	-00´00.000043″ -00´00.000004″	6358356.102	-0.0013 -0.0001
B	-36°55´13.876510″ N -36°55´13.876745″ G -36°55´13.876614″ CoA	148°30´00.000000″	-00´00.000235″ -00´00.000104″	6358465.209	-0.0072 . .-0.0032
C	-37°01´17.289080″ N -37°01´17.289478″ G -37°01´17.289366″ CoA	148°45´00.000000″	-00´00.000398″ -00´00.000286″	6358573.577	-0.0123 -0.0088
D	-37°07´17.845554″ N -37°07´17.846060″ G -37°07´17.846030″ CoA	149°00´00.000000″	-00´00.000506″ -00´00.000476″	6358681.204	-0.0156 -0.0147
E	-37°13´15.555723″ N -37°13´15.556262″ G -37°13´15.556326″ CoA	149°15´00.000000″	-00´00.000539″ -00´00.000603″	6358788.089	-0.0166 -0.0186
F	-37°19´10.429372″ N -37°19´10.429845″ G -37°19´10.429972″ CoA	149°30´00.000000″	-00´00.000473″ -00´00.000600″	6358894.232	-0.0146 -0.0185
G	-37°25´02.476276″ N -37°25´02.476564″ G -37°25´02.476677″ CoA	149°45´00.000000″	-00´00.000288″ -00´00.000401″	6358999.632	-0.0089 -0.0124
Wauka 1978	-37°30´18.067400″	149°58´32.993200″			

TABLE 1: Points where curves cut meridians of A, B, C, etc at 0°15´ intervals of longitude along Border Line
N = Normal Section, G = Geodesic, CoA = Curve of Alignment

```
>> curve_of_alignment_lat

==================
Curve of Alignment
==================
Ellipsoid parameters
a  = 6378137.0000
f  = 1/298.257222101

Terminal points of curve
Latitude  P1 =  -36 47 49.223200 (D M S)
Longitude P1 =  148 11 48.333300 (D M S)

Latitude  P2 =  -37 30 18.067400 (D M S)
Longitude P2 =  149 58 32.993200 (D M S)

Cartesian coordinates
            X               Y               Z
P1   -4345789.609716  2694844.030716 -3799378.032024
P2   -4386272.668061  2534883.268540 -3862005.992252

Given longitude of P3
Longitude P3 =  149 30  0.000000 (D M S)

Latitude of P3 computed from Newton-Raphson iteration
Latitude  P3 =  -37 19 10.429972 (D M S)
iterations   =    4

>>
```

```
>> curve_of_alignment_lon

==================
Curve of Alignment
==================
Ellipsoid parameters
a  = 6378137.0000
f  = 1/298.257222101

Terminal points of curve
Latitude  P1 =  -36 47 49.223200 (D M S)
Longitude P1 =  148 11 48.333300 (D M S)

Latitude  P2 =  -37 30 18.067400 (D M S)
Longitude P2 =  149 58 32.993200 (D M S)

Cartesian coordinates
            X               Y               Z
P1   -4345789.609716  2694844.030716 -3799378.032024
P2   -4386272.668061  2534883.268540 -3862005.992252

Given latitude of P3
Latitude  P3 =  -37 19 10.429972 (D M S)

Longitude of P3 computed from Newton-Raphson iteration
Longitude P3 =  149 29 60.000000 (D M S)
iterations   =    5

Longitude of P3 computed from trigonometric equation
Longitude P3 =  149 29 60.000000 (D M S)
theta P3     =    8 32 44.447661 (D M S)

>>
```

MATLAB function *curve_ of_ alignment_ lat.m*

```
function curve_of_alignment_lat
%
% curve_of_alignment_lat: Given the terminal points P1 and P2 of a curve of
% alignment on an ellipsoid, and the longitude of a point P3 on the curve,
% this function computes the latitude of P3.

%--------------------------------------------------------------------------
% Function:  curve_of_alignment_lat
%
% Usage:     curve_of_alignment_lat
%
% Author:    R.E.Deakin,
%            School of Mathematical & Geospatial Sciences, RMIT University
%            GPO Box 2476V, MELBOURNE, VIC 3001, AUSTRALIA.
%            email: rod.deakin@rmit.edu.au
%            Version  1.0  3 October  2009
%            Version  1.1 31 December 2009
%
% Purpose:   Given the terminal points P1 and P2 of a curve of alignment on
%  an ellipsoid, and the longitude of a point P3 on the curve, this
%  function computes the latitude of P3.
%
% Functions required:
%      [D,M,S] = DMS(DecDeg)
%      [X,Y,Z] = Geo2Cart(a,flat,lat,lon,h)
%      [rm,rp] = radii(a,flat,lat);
%
% Variables:
%  A,D            - curve of alignment functions of longitude
%  a              - semi-major axis of ellipsoid
%  b              - semi-minor axis of ellipsoid
%  B,C,H,W,U,V    - constants of curve of alignment
%  d2r            - degree to radian conversion factor 57.29577951...
%  d_nu           - derivative of nu w.r.t latitude
%  e2             - eccentricity of ellipsoid squared
%  f              - f = 1/flat is the flattening of ellipsoid
%  flat           - denominator of flattening of ellipsoid
%  f_lat3         - function of latitude of P3
%  fdash_lat3     - derivative of function of latitude of P3
%  h1,h2          - ellipsoidal heights of P1 and P2 (Note: h1 = h2 = 0)
%  iter           - number of iterations
%  lat1,lat2,lat3 - latitude of P1, P1, P3 (radians)
%  lon1,lon2,lon3 - longitude of P1, P2, P3 (radians)
%  new_lat3       - next latiude in Newton-Raphson iteration
%  nu             - radius of curvature in prime vertical plane
%  rho            - radius of curvature in meridain plane
%  X1,Y1,Z1       - Cartesian coordinates of P1
%  X2,Y2,Z2       - Cartesian coordinates of P2
%
% Remarks:
%  Given the terminal points P1 and P2 of a curve of alignment on an
%  ellipsoid, and the longitude of a point P3 on the curve, this function
%  computes the latitude of P3.
%
% References:
%  [1] Deakin, R.E., 2009, 'The Curve of Alignment on an Ellipsoid',
%         Lecture Notes, School of Mathematical and Geospatial Sciences,
%         RMIT University, December 2009
%  [2] Thomas, P.D., 1952, Conformal Projections in Geodesy and
%         Cartography, Special Publication No. 251, Coast and Geodetic
%         Survey, U.S. Department of Commerce, Washington, DC: U.S.
%         Government Printing Office, pp. 66-67.
%
%--------------------------------------------------------------------------

% Degree to radian conversion factor
```

```
d2r   = 180/pi;

% Set ellipsoid parameters
a    = 6378137;    % GRS80
flat = 298.257222101;

% Compute ellipsoid constants
f   = 1/flat;
e2  = f*(2-f);

% Set lat, lon and height of P1 and P2 on ellipsoid
lat1 = -(36  + 47/60 + 49.2232/3600)/d2r;   % Spring
lon1 =  (148 + 11/60 + 48.3333/3600)/d2r;
lat2 = -(37  + 30/60 + 18.0674/3600)/d2r;   % Wauka 1978
lon2 =  (149 + 58/60 + 32.9932/3600)/d2r;
h1 = 0;
h2 = 0;

% Compute Cartesian coords of P1 and P2
[X1,Y1,Z1] = Geo2Cart(a,flat,lat1,lon1,h1);
[X2,Y2,Z2] = Geo2Cart(a,flat,lat2,lon2,h2);

% Compute constants of Curve of Alignment
C = e2*(Y2-Y1);
H = e2*(X2-X1);
W = X1*Y2-X2*Y1;
U = (1-e2)*(Y1*Z2-Y2*Z1);
V = (1-e2)*(X2*Z1-X1*Z2);
B = (1-e2)*W;

% Set longitude of P3
lon3 = (149 + 30/60)/d2r;

% Set constants A and D that are functions of longitude only
A = (1-e2)*(C*cos(lon3)-H*sin(lon3));
D = U*cos(lon3)+V*sin(lon3);

%----------------------------------------------------------
% Compute the latitude of P3 using Newton-Raphson iteration
%----------------------------------------------------------
% Set starting value of phi = latitude
lat3 = lat1;
iter = 1;
while 1
    % Compute radii of curvature
    [rho,nu]   = radii(a,flat,lat3);
    d_nu       = nu^3/(a*a)*e2*sin(lat3)*cos(lat3);
    f_lat3     = A*nu*sin(lat3)-B*tan(lat3)-D;
    fdash_lat3 = d_nu*A*sin(lat3)+nu*A*cos(lat3)-B/(cos(lat3)^2);
    new_lat3   = lat3-(f_lat3/fdash_lat3);
    if abs(new_lat3 - lat3) < 1e-15
        break;
    end
    lat3 = new_lat3;
    if iter > 100
        fprintf('Iteration for latitude failed to converge after 100 iterations');
        break;
    end
    iter = iter + 1;
end;

%-----------------------
% Print result to screen
%-----------------------

fprintf('\n==================');
fprintf('\nCurve of Alignment');
fprintf('\n==================');
fprintf('\nEllipsoid parameters');
```

```
fprintf('\na  = %12.4f',a);
fprintf('\nf  = 1/%13.9f',flat);

fprintf('\n\nTerminal points of curve');
% Print lat and lon of P1
[D,M,S] = DMS(lat1*d2r);
if D == 0 && lat1 < 0
    fprintf('\nLatitude  P1 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLatitude  P1 = %4d %2d %9.6f (D M S)',D,M,S);
end
[D,M,S] = DMS(lon1*d2r);
if D == 0 && lon1 < 0
    fprintf('\nLongitude P1 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLongitude P1 = %4d %2d %9.6f (D M S)',D,M,S);
end
% Print lat and lon of P2
[D,M,S] = DMS(lat2*d2r);
if D == 0 && lat2 < 0
    fprintf('\n\nLatitude  P2 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\n\nLatitude  P2 = %4d %2d %9.6f (D M S)',D,M,S);
end
[D,M,S] = DMS(lon2*d2r);
if D == 0 && lon2 < 0
    fprintf('\nLongitude P2 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLongitude P2 = %4d %2d %9.6f (D M S)',D,M,S);
end

% Print Coordinate table
fprintf('\n\nCartesian coordinates');
fprintf('\n            X               Y               Z');
fprintf('\nP1   %15.6f %15.6f %15.6f',X1,Y1,Z1);
fprintf('\nP2   %15.6f %15.6f %15.6f',X2,Y2,Z2);

% Print lat and lon of P3
fprintf('\n\nGiven longitude of P3');
[D,M,S] = DMS(lon3*d2r);
if D == 0 && lon3 < 0
    fprintf('\nLongitude P3 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLongitude P3 = %4d %2d %9.6f (D M S)',D,M,S);
end
fprintf('\n\nLatitude of P3 computed from Newton-Raphson iteration');
[D,M,S] = DMS(lat3*d2r);
if D == 0 && lat3 < 0
    fprintf('\nLatitude  P3 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLatitude  P3 = %4d %2d %9.6f (D M S)',D,M,S);
end
fprintf('\niterations   = %4d',iter);

fprintf('\n\n');
```

MATLAB function *curve_ of_ alignment_ lon.m*

```
function curve_of_alignment_lon
%
% curve_of_alignment_lon: Given the terminal points P1 and P2 of a curve of
% alignment on an ellipsoid, and the latitude of a point P3 on the curve,
% this function computes the longitude of P3.

%--------------------------------------------------------------------------
% Function:  curve_of_alignment_lon
%
% Usage:     curve_of_alignment_lon
%
% Author:    R.E.Deakin,
%            School of Mathematical & Geospatial Sciences, RMIT University
%            GPO Box 2476V, MELBOURNE, VIC 3001, AUSTRALIA.
%            email: rod.deakin@rmit.edu.au
%            Version  1.0 31 December 2009
%
% Purpose:   Given the terminal points P1 and P2 of a curve of alignment on
%  an ellipsoid, and the latitude of a point P3 on the curve, this function
%  computes the longitude of P3.
%
% Functions required:
%      [D,M,S] = DMS(DecDeg)
%      [X,Y,Z] = Geo2Cart(a,flat,lat,lon,h)
%      [rm,rp] = radii(a,flat,lat);
%
% Variables:
%  a              - semi-major axis of ellipsoid
%  b              - semi-minor axis of ellipsoid
%  C,H,W,U,V      - constants of curve of alignment
%  d2r            - degree to radian conversion factor 57.29577951...
%  d_nu           - derivative of nu w.r.t latitude
%  e2             - eccentricity of ellipsoid squared
%  f              - f = 1/flat is the flattening of ellipsoid
%  flat           - denominator of flattening of ellipsoid
%  f_lon3         - function of longitude of P3
%  fdash_lon3     - derivative of function of longitude of P3
%  h1,h2          - ellipsoidal heights of P1 and P2 (Note: h1 = h2 = 0)
%  iter           - number of iterations
%  lambda         - longitude of P3 computed from trigonometric equation
%  lat1,lat2,lat3 - latitude of P1, P1, P3 (radians)
%  lon1,lon2,lon3 - longitude of P1, P2, P3 (radians)
%  new_lon3       - next longitude in Newton-Raphson iteration
%  nu             - radius of curvature in prime vertical plane
%  P,Q,S          - functions of latitude of a point on the curve of
%                 - alignment
%  rho            - radius of curvature in meridain plane
%  theta          - auxiliary angle in the computation of lambda
%  X1,Y1,Z1       - Cartesian coordinates of P1
%  X2,Y2,Z2       - Cartesian coordinates of P2
%
% Remarks:
%  Given the terminal points P1 and P2 of a curve of alignment on an
%  ellipsoid, and the latitude of a point P3 on the curve, this function
%  computes the longitude of P3.
%
% References:
%  [1] Deakin, R.E., 2009, 'The Curve of Alignment on an Ellipsoid',
%         Lecture Notes, School of Mathematical and Geospatial Sciences,
%         RMIT University, December 2009
%  [2] Thomas, P.D., 1952, Conformal Projections in Geodesy and
%         Cartography, Special Publication No. 251, Coast and Geodetic
%         Survey, U.S. Department of Commerce, Washington, DC: U.S.
%         Government Printing Office, pp. 66-67.
%
%--------------------------------------------------------------------------
```

```
% Degree to radian conversion factor
d2r   = 180/pi;

% Set ellipsoid parameters
a    = 6378137;    % GRS80
flat = 298.257222101;

% Compute ellipsoid constants
f   = 1/flat;
e2  = f*(2-f);

% Set lat, lon and height of P1 and P2 on ellipsoid
lat1 = -(36  + 47/60 + 49.2232/3600)/d2r;   % Spring
lon1 =  (148 + 11/60 + 48.3333/3600)/d2r;
lat2 = -(37  + 30/60 + 18.0674/3600)/d2r;   % Wauka 1978
lon2 =  (149 + 58/60 + 32.9932/3600)/d2r;
h1 = 0;
h2 = 0;
% Compute Cartesian coords of P1 and P2
[X1,Y1,Z1] = Geo2Cart(a,flat,lat1,lon1,h1);
[X2,Y2,Z2] = Geo2Cart(a,flat,lat2,lon2,h2);

% Compute constants of Curve of Alignment
C = e2*(Y2-Y1);
H = e2*(X2-X1);
W = X1*Y2-X2*Y1;
U = (1-e2)*(Y1*Z2-Y2*Z1);
V = (1-e2)*(X2*Z1-X1*Z2);

% Set latitude of P3
lat3 = -(37 + 19/60 + 10.429972/3600)/d2r;

% Set constants P, Q, S that are functions of latitude only
[rho,nu]   = radii(a,flat,lat3);
P = C*nu*(1-e2)*sin(lat3)-U;
Q = H*nu*(1-e2)*sin(lat3)+V;
S = W*(1-e2)*tan(lat3);

%---------------------- ------------------------------------
% Compute the longitude of P3 using Newton-Raphson iteration
%---------------------- ------------------------------------
% Set starting value of lon3 = longitude of P3
lon3 = lon1;
iter = 1;
while 1
    % Compute radii of curvature
    f_lon3     = P*cos(lon3)-Q*sin(lon3)-S;
    fdash_lon3 = -P*sin(lon3)-Q*cos(lon3);
    new_lon3   = lon3-(f_lon3/fdash_lon3);
    if abs(new_lon3 - lon3) < 1e-15
        break;
    end
    lon3 = new_lon3;
    if iter > 100
        fprintf('Iteration for longitude failed to converge after 100 iterations');
        break;
    end
    iter = iter + 1;
end;

%---------------------- ----------------------------------
% Compute the longitude of P3 using trigonometric equation
%---------------------- ----------------------------------
theta  = atan2(-Q,P);
lambda = acos(S/sqrt(P^2+Q^2))+theta;

%-----------------------
% Print result to screen
```

```
%-----------------------

fprintf('\n==================');
fprintf('\nCurve of Alignment');
fprintf('\n==================');
fprintf('\nEllipsoid parameters');
fprintf('\na  = %12.4f',a);
fprintf('\nf  = 1/%13.9f',flat);

fprintf('\n\nTerminal points of curve');
% Print lat and lon of P1
[D,M,S] = DMS(lat1*d2r);
if D == 0 && lat1 < 0
    fprintf('\nLatitude  P1 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLatitude  P1 = %4d %2d %9.6f (D M S)',D,M,S);
end
[D,M,S] = DMS(lon1*d2r);
if D == 0 && lon1 < 0
    fprintf('\nLongitude P1 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLongitude P1 = %4d %2d %9.6f (D M S)',D,M,S);
end
% Print lat and lon of P2
[D,M,S] = DMS(lat2*d2r);
if D == 0 && lat2 < 0
    fprintf('\n\nLatitude  P2 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\n\nLatitude  P2 = %4d %2d %9.6f (D M S)',D,M,S);
end
[D,M,S] = DMS(lon2*d2r);
if D == 0 && lon2 < 0
    fprintf('\nLongitude P2 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLongitude P2 = %4d %2d %9.6f (D M S)',D,M,S);
end

% Print Coordinate table
fprintf('\n\nCartesian coordinates');
fprintf('\n             X               Y               Z');
fprintf('\nP1   %15.6f %15.6f %15.6f',X1,Y1,Z1);
fprintf('\nP2   %15.6f %15.6f %15.6f',X2,Y2,Z2);

% Print lat and lon of P3
fprintf('\n\nGiven latitude of P3');
[D,M,S] = DMS(lat3*d2r);
if D == 0 && lat3 < 0
    fprintf('\nLatitude  P3 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLatitude  P3 = %4d %2d %9.6f (D M S)',D,M,S);
end

fprintf('\n\nLongitude of P3 computed from Newton-Raphson iteration');
[D,M,S] = DMS(lon3*d2r);
if D == 0 && lon3 < 0
    fprintf('\nLongitude P3 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLongitude P3 = %4d %2d %9.6f (D M S)',D,M,S);
end
fprintf('\niterations   = %4d',iter);

fprintf('\n\nLongitude of P3 computed from trigonometric equation');
[D,M,S] = DMS(lambda*d2r);
if D == 0 && lambda < 0
    fprintf('\nLongitude P3 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLongitude P3 = %4d %2d %9.6f (D M S)',D,M,S);
end
[D,M,S] = DMS(theta*d2r);
```

```
if D == 0 && theta < 0
    fprintf('\ntheta P3     =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\ntheta P3     = %4d %2d %9.6f (D M S)',D,M,S);
end


fprintf('\n\n');
```

MATLAB function *Geo2Cart.m*

```
function [X,Y,Z] = Geo2Cart(a,flat,lat,lon,h)
%
% [X,Y,Z] = Geo2Cart(a,flat,lat,lon,h)
%   Function computes the Cartesian coordinates X,Y,Z of a point
%   related to an ellipsoid defined by semi-major axis (a) and the
%   denominator of the flattening (flat) given geographical
%   coordinates latitude (lat), longitude (lon) and ellipsoidal
%   height (h).  Latitude and longitude are assumed to be in radians.

%--------------------------------------------------------------------------
% Function:  Geo2Cart()
%
% Usage:     [X,Y,Z] = Geo2Cart(a,flat,lat,lon,h);
%
% Author:    R.E.Deakin,
%            School of Mathematical & Geospatial Sciences, RMIT University
%            GPO Box 2476V, MELBOURNE, VIC 3001, AUSTRALIA.
%            email: rod.deakin@rmit.edu.au
%            Version  1.0   6 April  2006
%            Version  1.0  20 August 2007
%
% Functions required:
%    radii()
%
% Purpose:
%    Function Geo2Cart() will compute Cartesian coordinates X,Y,Z
%    given geographical coordinates latitude, longitude (both in
%    radians) and height of a  point related to an ellipsoid
%    defined by semi-major axis (a) and denominator of flattening
%    (flat).
%
% Variables:
%    a       - semi-major axis of ellipsoid
%    e2      - 1st eccentricity squared
%    f       - flattening of ellipsoid
%    flat    - denominator of flattening f = 1/flat
%    h       - height above ellipsoid
%    lat     - latitude (radians)
%    lon     - longitude (radians)
%    p       - perpendicular distance from minor axis of ellipsoid
%    rm      - radius of curvature of meridian section of ellipsoid
%    rp      - radius of curvature of prime vertical section of ellipsoid
%
% References:
% [1] Gerdan, G.P. & Deakin, R.E., 1999, 'Transforming Cartesian
%     coordinates X,Y,Z to geogrpahical coordinates phi,lambda,h', The
%     Australian Surveyor, Vol. 44, No. 1, pp. 55-63, June 1999.
%--------------------------------------------------------------------------

% calculate flattening f and ellipsoid constant e2
f   = 1/flat;
e2  = f*(2-f);

% compute radii of curvature for the latitude
[rm,rp] = radii(a,flat,lat);
```

```
% compute Cartesian coordinates X,Y,Z
p = (rp+h)*cos(lat);
X = p*cos(lon);
Y = p*sin(lon);
Z = (rp*(1-e2)+h)*sin(lat);
```

MATLAB function *radii.m*

```
function [rm,rp] = radii(a,flat,lat)
%
% [rm,rp]=radii(a,flat,lat)  Function computes radii of curvature in
%   the meridian and prime vertical planes (rm and rp respectively) at a
%   point whose latitude (lat) is known on an ellipsoid defined by
%   semi-major axis (a) and denominator of flattening (flat).
%   Latitude must be in radians.
%   Example: [rm,rp] = radii(6378137,298.257222101,-0.659895044);
%            should return rm = 6359422.96233327 metres and
%                          rp = 6386175.28947842 metres
%            at latitude -37 48 33.1234 (DMS) on the GRS80 ellipsoid

%----------------------------------------------------------------------------
% Function:  radii(a,flat,lat)
%
% Syntax:    [rm,rp] = radii(a,flat,lat);
%
% Author:    R.E.Deakin,
%            School of Mathematical & Geospatial Sciences, RMIT University
%            GPO Box 2476V, MELBOURNE, VIC 3001, AUSTRALIA.
%            email: rod.deakin@rmit.edu.au
%            Version  1.0  1 August 2003
%            Version  2.0  6 April  2006
%            Version  3.0  9 February 2008
%
% Purpose:   Function radii() will compute the radii of curvature in
%            the meridian and prime vertical planes, rm and rp respectively
%            for the point whose latitude (lat) is given for an ellipsoid
%            defined by its semi-major axis (a) and denominator of
%            flattening (flat).
%
% Return value: Function radii() returns rm and rp
%
% Variables:
%  a      - semi-major axis of spheroid
%  c      - polar radius of curvature
%  c2     - cosine of latitude squared
%  ep2    - 2nd-eccentricity squared
%  f      - flattening of ellipsoid
%  lat    - latitude of point (radians)
%  rm     - radius of curvature in the meridian plane
%  rp     - radius of curvature in the prime vertical plane
%  V      - latitude function defined by V-squared = sqrt(1 + ep2*c2)
%  V2,V3  - powers of V
%
% Remarks:
%  Formulae are given in [1] (section 1.3.9, page 85) and in
%  [2] (Chapter 2, p. 2-10) in a slightly different form.
%
% References:
% [1] Deakin, R.E. and Hunter, M.N., 2008, GEOMETRIC GEODESY, School of
%        Mathematical and Geospatial Sciences, RMIT University, Melbourne,
%        AUSTRALIA, March 2008.
% [2] THE GEOCENTRIC DATUM OF AUSTRALIA TECHNICAL MANUAL, Version 2.2,
%        Intergovernmental Committee on Surveying and Mapping (ICSM),
%        February 2002 (www.anzlic.org.au/icsm/gdatum)
%----------------------------------------------------------------------------
```

```
% compute flattening f eccentricity squared e2
f   = 1/flat;
c   = a/(1-f);
ep2 = f*(2-f)/((1-f)^2);

% calculate the square of the sine of the latitude
c2 = cos(lat)^2;

% compute latitude function V
V2 = 1+ep2*c2;
V  = sqrt(V2);
V3 = V2*V;

% compute radii of curvature
rm = c/V3;
rp = c/V;
```

MATLAB function *DMS.m*

```
function [D,M,S] = DMS(DecDeg)
% [D,M,S] = DMS(DecDeg)  This function takes an angle in decimal degrees and returns
%   Degrees, Minutes and Seconds

val = abs(DecDeg);
D = fix(val);
M = fix((val-D)*60);
S = (val-D-M/60)*3600;
if(DecDeg<0)
  D = -D;
end
return
```

REFERENCES

Bowring, B. R., (1972), 'Distance and the spheroid', Correspondence, *Survey Review,* Vol. XXI, No. 164, April 1972, pp. 281-284.

Clarke, A. R., (1880), *Geodesy,* Clarendon Press, Oxford.

Deakin, R. E., (2009), 'The Normal Section Curve on an Ellipsoid', Lecture Notes, School of Mathematical & Geospatial Sciences, RMIT University, Melbourne, Australia, November 2009, 53 pages.

Thomas, P. D., (1952), *Conformal Projections in Geodesy and Cartography,* Special Publication No. 251, Coast and Geodetic Survey, United States Department of Commerce, Washington, D.C.

THE GREAT ELLIPTIC ARC
ON AN ELLIPSOID

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ABSTRACT

These notes provide a detailed derivation of the equation for the great elliptic arc on an ellipsoid. Using this equation and knowing the terminal points of the curve, a technique is developed for computing the location of points along the curve. A MATLAB function is provided that demonstrates the algorithm developed.

INTRODUCTION

In geodesy, the great elliptic arc between P_1 and P_2 on the ellipsoid is the curve created by intersecting the ellipsoid with the plane containing P_1, P_2 and O (the centre of the ellipsoid).

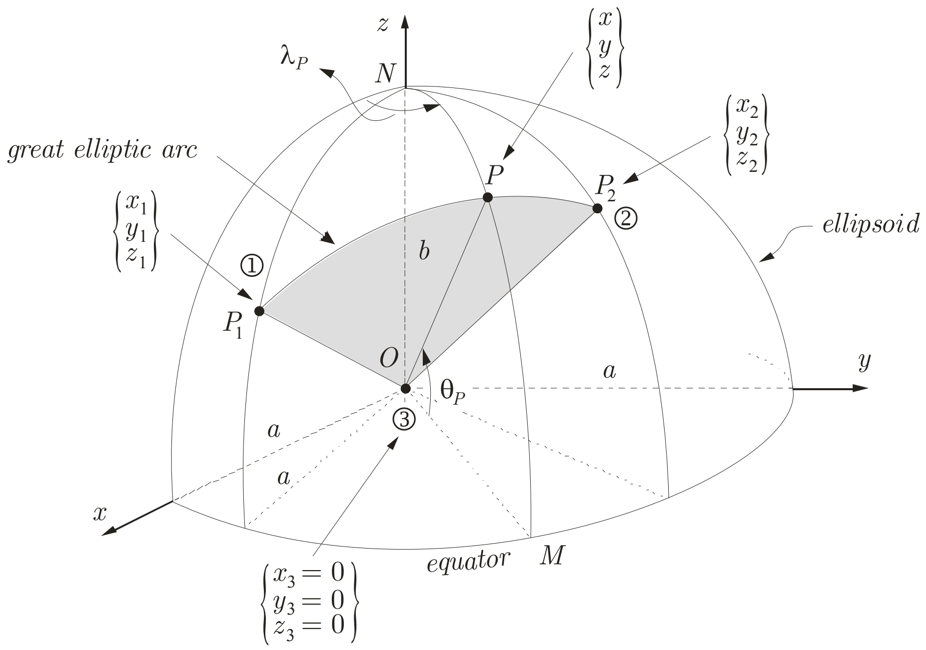


Figure 1: Great elliptic arc on ellipsoid

Figure 1 shows P on the great elliptic arc between P_1 and P_2. θ_P is the geocentric latitude of P and λ_P is the longitude of P.

There are an infinite number of planes that cut the surface of the ellipsoid and contain the chord P_1P_2 but only one of these will contain the centre O. Two other planes are the normal section plane P_1P_2 (containing the normal at P_1) and the normal section plane P_2P_1 (containing the normal at P_2). All of these curves of intersection (including the great elliptic arc and the two normal section curves) are plane curves that are arcs of ellipses (for a proof of this see Deakin, 2009a). All meridians of longitude on an ellipsoid and the ellipsoid equator are great elliptic arcs. Parallels of latitude – excepting the equator – are not great elliptic arcs. So we could say that the great elliptic arc is a <u>unique plane curve</u> on the ellipsoid – since it is created by the single plane containing P_1, P_2 and O. <u>But it is not the shortest distance</u> between P_1 and P_2; this unique property (shortest length) belongs to the geodesic.

Great elliptic arcs are not much used in geodesy as they don't have a practical connection with theodolite observations made on the surface of the earth that are approximated as observations made on an ellipsoid; e.g., normal section curves and curves of alignment. Nor are they the shortest distance between points on the ellipsoid; but, if we ignore earth rotation, they are the curves traced out on the geocentric ellipsoid by the ground point of an earth orbiting satellite or a ballistic missile moving in an orbital plane containing the earth's centre of mass. Here geocentric means O (the centre of the ellipsoid) is coincident with the centre of mass.

The equation for the curve developed below is similar to that derived for the curve of alignment in Deakin (2009b) and it is not in a form suitable for computing the distance or azimuth of the curve. But, as it contains functions of both the latitude and longitude of a point on the curve, it is suitable for computing the latitude of a point given a particular longitude; or alternatively the longitude of a point may be computed (iteratively) given a particular latitude.

EQUATION OF GREAT ELLIPTIC ARC

Figure 1 shows P on the great elliptic arc that passes through P_1 and P_2 on the ellipsoid. The semi-axes of the ellipsoid are a and b $\left(a > b\right)$ and the first-eccentricity squared e^2 and the flattening f of the ellipsoid are defined by

$$\begin{aligned} e^2 &= \frac{a^2-b^2}{a^2} = f\left(2-f\right) \\ f &= \frac{a-b}{a} \end{aligned} \tag{1}$$

Parallels of latitude ϕ and meridians of longitude λ have their respective reference planes; the equator and the Greenwich meridian, and Longitudes are measured 0° to $\pm 180^\circ$ (east positive, west negative) from the Greenwich meridian and latitudes are measured 0° to $\pm 90^\circ$ (north positive, south negative) from the equator. The x,y,z geocentric Cartesian coordinate system has an origin at O, the centre of the ellipsoid, and the z-axis is the minor axis (axis of revolution). The xOz plane is the Greenwich meridian plane (the origin of longitudes) and the xOy plane is the equatorial plane. The positive x-axis passes through the intersection of the Greenwich meridian and the equator, the positive y-axis is advanced 90° east along the equator and the positive z-axis passes through the north pole of the ellipsoid.

In Figure 1, θ_P is the geocentric latitude of P and (geodetic) latitude ϕ and geocentric latitude θ are related by

$$\tan\theta = \left(1-e^2\right)\tan\phi = \frac{b^2}{a^2}\tan\phi = \left(1-f\right)^2\tan\phi \tag{2}$$

The geometric relationship between geocentric latitude θ and (geodetic) latitude ϕ is shown in Figure 2.

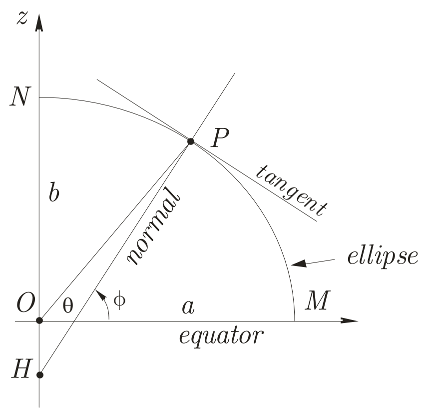


Figure 2: Meridian plane of P

The great elliptic plane in Figure 1 is defined by points ①, ② and ③ that are P_1, P_2 and the centre of the ellipsoid O respectively. Cartesian coordinates of ① and ② are computed from the following equations

$$\begin{aligned} x &= \nu \cos\phi \cos\lambda \\ y &= \nu \cos\phi \sin\lambda \\ z &= \nu\left(1 - e^2\right)\sin\phi \end{aligned} \tag{3}$$

where $\nu = PH$ (see Figure 2) is the radius of curvature in the prime vertical plane and

$$\nu = \frac{a}{\sqrt{1 - e^2 \sin^2\phi}} \tag{4}$$

The Cartesian coordinates of point ③ are all zero.

The General equation of a plane may be written as

$$Ax + By + Cz + D = 0 \tag{5}$$

And the equation of the plane passing through points ①, ② and ③ is given in the form of a 3rd-order determinant

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_2 & y_3 - y_2 & z_3 - z_2 \end{vmatrix} = 0 \tag{6}$$

or expanded into 2nd-order determinants

$$\begin{vmatrix} y_2 - y_1 & z_2 - z_1 \\ y_3 - y_2 & z_3 - z_2 \end{vmatrix}\left(x - x_1\right) - \begin{vmatrix} x_2 - x_1 & z_2 - z_1 \\ x_3 - x_2 & z_3 - z_2 \end{vmatrix}\left(y - y_1\right) + \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_2 & y_3 - y_2 \end{vmatrix}\left(z - z_1\right) = 0 \tag{7}$$

Expanding the determinants in equation (7) gives

$$\begin{aligned} &\left(x - x_1\right)\left\{\left(y_2 - y_1\right)\left(z_3 - z_2\right) - \left(z_2 - z_1\right)\left(y_3 - y_2\right)\right\} \\ -&\left(y - y_1\right)\left\{\left(x_2 - x_1\right)\left(z_3 - z_2\right) - \left(z_2 - z_1\right)\left(x_3 - x_2\right)\right\} \\ +&\left(z - z_1\right)\left\{\left(x_2 - x_1\right)\left(y_3 - y_2\right) - \left(y_2 - y_1\right)\left(x_3 - x_2\right)\right\} = 0 \end{aligned} \tag{8}$$

Now since $x_3 = y_3 = z_3 = 0$ and equation (8) becomes

$$\begin{aligned} &\left(x - x_1\right)\left\{\left(y_2 - y_1\right)\left(-z_2\right) - \left(z_2 - z_1\right)\left(-y_2\right)\right\} \\ -&\left(y - y_1\right)\left\{\left(x_2 - x_1\right)\left(-z_2\right) - \left(z_2 - z_1\right)\left(-x_2\right)\right\} \\ +&\left(z - z_1\right)\left\{\left(x_2 - x_1\right)\left(-y_2\right) - \left(y_2 - y_1\right)\left(-x_2\right)\right\} = 0 \end{aligned} \tag{9}$$

Expanding and simplifying equation (9) gives

$$x\left(y_1 z_2 - y_2 z_1\right) - y\left(x_1 z_2 - x_2 z_1\right) + z\left(x_1 y_2 - x_2 y_1\right) = 0$$

Replacing *x, y* and *z* with their equivalents, given by equations (3), gives

$$\nu\cos\phi\cos\lambda\left(y_1 z_2 - y_2 z_1\right) - \nu\cos\phi\sin\lambda\left(x_1 z_2 - x_2 z_1\right) + \nu\left(1 - e^2\right)\sin\phi\left(x_1 y_2 - x_2 y_1\right) = 0$$

and dividing both sides by $\nu \cos\phi$ gives the equation of the great elliptic arc as

$$\boxed{A\cos\lambda - B\sin\lambda + C\left(1-e^2\right)\tan\phi = 0 \qquad (10)}$$

where A, B and C are functions of the coordinates of the terminal points P_1 and P_2

$$A = y_1 z_2 - y_2 z_1 \qquad B = x_1 z_2 - x_2 z_1 \qquad C = x_1 y_2 - x_2 y_1 \tag{11}$$

Equation (10) is not suitable for computing the distance along a great elliptic arc, nor is it suitable for computing the azimuth of the curve, but by certain re-arrangements it is possible to solve (directly) for the latitude of a point on the curve given a longitude somewhere between the longitudes of the terminal points of the curve. Or alternatively, solve (iteratively) for the longitude of a point given a latitude somewhere between the latitudes of the terminal points.

SOLVING FOR THE LATITUDE

A simple re-arrangement of equation (10) allows the latitude ϕ to be evaluated from

$$\tan\phi = \frac{B\sin\lambda - A\cos\lambda}{C\left(1-e^2\right)} \tag{12}$$

where A and B and C are functions of terminal points P_1 and P_2 given by equations (11).

SOLVING FOR THE LONGITUDE

The longitude λ can be evaluated using Newton-Raphson iteration for the real roots of the equation $f\left(\lambda\right) = 0$ given in the form of an iterative equation

$$\lambda_{(n+1)} = \lambda_{(n)} - \frac{f\left(\lambda_{(n)}\right)}{f'\left(\lambda_{(n)}\right)} \tag{13}$$

where n denotes the n^{th} iteration and $f\left(\lambda\right)$ is given by equation (10) as

$$f\left(\lambda\right) = A\cos\lambda - B\sin\lambda + C\left(1-e^2\right)\tan\phi \tag{14}$$

and the derivative $f'\left(\lambda\right) = \frac{d}{d\lambda}\left\{f\left(\lambda\right)\right\}$ is given by

$$f'\left(\lambda\right) = -A\sin\lambda - B\cos\lambda \tag{15}$$

An initial value of $\lambda_{(1)}$ (λ for $n = 1$) can be taken as the longitude of P_1 and the functions $f\left(\lambda_{(1)}\right)$ and $f'\left(\lambda_{(1)}\right)$ evaluated from equations (14) and (15) using λ_1. $\lambda_{(2)}$ (λ for $n = 2$) can now be computed from equation (13) and this process repeated to obtain values $\lambda_{(3)}, \lambda_{(4)}, \ldots$. This iterative process can be concluded when the difference between $\lambda_{(n+1)}$ and $\lambda_{(n)}$ reaches an acceptably small value.

Alternatively, the longitude can be evaluated by a trigonometric equation derived as follows. Equation (10) can be expressed as

$$B \sin\lambda - A\cos\lambda = C\left(1 - e^2\right)\tan\phi \tag{16}$$

and A, B and C are given by equations (11). Equation (16) can be expressed as a trigonometric addition of the form

$$\begin{aligned} C\left(1 - e^2\right)\tan\phi &= R\cos\left(\lambda - \theta\right) \\ &= R\cos\lambda\cos\theta + R\sin\lambda\sin\theta \end{aligned} \tag{17}$$

Now, equating the coefficients of $\cos\lambda$ and $\sin\lambda$ in equations (17) and (16) gives

$$A = -R\cos\theta; \quad B = R\sin\theta \tag{18}$$

and using these relationships

$$R = \sqrt{A^2 + B^2}; \quad \tan\theta = \frac{B}{-A} \tag{19}$$

Substituting these results into equation (17) gives

$$\lambda = \arccos\left\{\frac{C\left(1 - e^2\right)\tan\phi}{\sqrt{A^2 + B^2}}\right\} + \arctan\left\{\frac{B}{-A}\right\} \tag{20}$$

DIFFERENCE IN LENGTH BETWEEN A GEODESIC AND A GREAT ELLIPTIC ARC

There are five curves of interest in geodesy; the geodesic, the normal section, the great elliptic arc the loxodrome and the curve of alignment.

The geodesic between P_1 and P_2 on an ellipsoid is the unique curve on the surface defining the shortest distance; all other curves will be longer in length. The normal section curve P_1P_2 is a plane curve created by the intersection of the normal section plane containing the normal at P_1 and also P_2 with the ellipsoid surface. And as we have shown (Deakin

2009a) there is the other normal section curve P_2P_1. The curve of alignment (Deakin 2009b, Thomas 1952) is the locus of all points P such that the normal section plane at P also contains the points P_1 and P_2. The curve of alignment is very close to a geodesic. The great elliptic arc is the plane curve created by intersecting the plane containing P_1, P_2 and the centre O with the surface of the ellipsoid and the loxodrome is the curve on the surface that cuts each meridian between P_1 and P_2 at a constant angle.

Approximate equations for the difference in length between the geodesic, the normal section curve and the curve of alignment were developed by Clarke (1880, p. 133) and Bowring (1972, p. 283) developed an approximate equation for the difference between the geodesic and the great elliptic arc. Following Bowring (1972), let

$$\begin{aligned} s &= \text{ geodesic length} \\ L &= \text{ normal section length} \\ D &= \text{ great elliptic length} \\ S &= \text{ curve of alignment length} \end{aligned}$$

then

$$\begin{aligned} L - s &= \frac{e^4}{90} s \left(\frac{s}{R}\right)^4 \cos^4\phi_1 \sin^2\alpha_{12} \cos^2\alpha_{12} + \cdots \\ D - s &= \frac{e^4}{24} s \left(\frac{s}{R}\right)^2 \sin^2\phi_1 \cos^2\phi_1 \sin^2\alpha_{12} + \cdots \\ S - s &= \frac{e^4}{360} s \left(\frac{s}{R}\right)^4 \cos^4\phi_1 \sin^2\alpha_{12} \cos^2\alpha_{12} + \cdots \end{aligned} \tag{21}$$

where R can be taken as the radius of curvature in the prime vertical at P_1. Now for a given value of s, $D - s$ will be a maximum if $\phi_1 = 45^\circ$ and $\alpha_{12} = 90^\circ$ in which case $\sin^2\phi_1 \cos^2\phi_1 \sin^2\alpha_{12} = \frac{1}{4}$, thus

$$\left(D - s\right) < \frac{e^4}{96} s \left(\frac{s}{R}\right)^4 \tag{22}$$

For the GRS80 ellipsoid where $f = 1/298.257222101$, $e^2 = f\left(2 - f\right)$, and for $s = 1200000$ m (1200 km) and $R = 6371000$ m, equation (22) gives $D - s < 0.001$ m.

MATLAB FUNCTIONS

Two MATLAB functions are shown below; they are: *great_ elliptic_ arc_ lat.m* and *great_ elliptic_ arc_ lon.m* Assuming that the terminal points of the curve are known, the first function computes the latitude of a point on the curve given a longitude and the second function computes the longitude of a point given the latitude.

Output from the two functions is shown below for points on a great elliptic arc between the terminal points of the straight-line section of the Victorian–New South Wales border. This straight-line section of the border, between Murray Spring and Wauka 1978, is known as the Black-Allan Line in honour of the surveyors Black and Allan who set out the border line in 1870-71. Wauka 1978 (Gabo PM 4) is a geodetic concrete border pillar on the coast at Cape Howe and Murray Spring (Enamo PM 15) is a steel pipe driven into a spring of the Murray River that is closest to Cape Howe. The straight line is a normal section curve on the reference ellipsoid of the Geocentric Datum of Australia (GDA94) that contains the normal to the ellipsoid at Murray Spring. The GDA94 coordinates of Murray Spring and Wauka 1978 are:

$$\begin{array}{lll} \text{Murray Spring:} & \phi\ -37^\circ\,47'\,49.2232'' & \lambda\ 148^\circ\,11'\,48.3333'' \\ \text{Wauka 1978:} & \phi\ -37^\circ\,30'18.0674'' & \lambda\ 149^\circ\,58'\,32.9932'' \end{array}$$

The normal section azimuth and distance are:

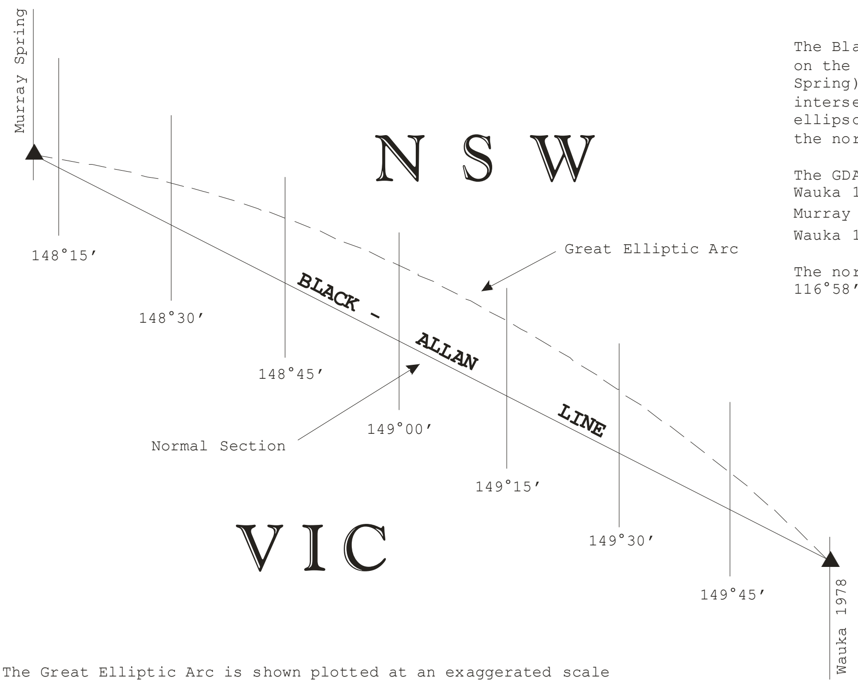
$$116^\circ\,58'14.173757'' \qquad 176495.243760 \text{ m}$$

The geodesic azimuth and distance are:

$$116^\circ\,58'14.219146'' \qquad 176495.243758 \text{ m}$$

Figure 3 shows a schematic view of the Black-Allan line (normal section) and the great elliptic arc. The relationships between the great elliptic arc and the normal section have been computed at seven locations along the line (A, B, C, etc.) where meridians of longitude at $0^\circ\,15'$ intervals cut the line. These relationships are shown in Table 1.

BLACK-ALLAN LINE: VICTORIA/NSW BORDER



The Black-Allan Line is a normal section curve on the reference ellipsoid between P1 (Murray Spring) and P2 (Wauka 1978). This curve is the intersection of the normal section plane and the ellipsoid, and the normal section contains P1, the normal to the ellipsoid at P1, and P2.

The GDA94 coordinates of Murray Spring and Wauka 1978 are:
Murray Spring: ϕ -37°47′49.2232″ λ 148°11′48.3333″
Wauka 1978: ϕ -37°30′18.0674″ λ 149°58′32.9932″

The normal section azimuth and distance are:
116°58′14.173757″ 176495.243760 m.

The Great Elliptic Arc is shown plotted at an exaggerated scale with respect to the Border Line (normal section).
At longitude 149°00′E. the Great Elliptic Arc is 1.939 m north of the Border Line.
At longitude 149°30′E. the Great Elliptic Arc is 1.522 m north of the Border Line.

Figure 3

BLACK-ALLAN LINE: VICTORIA/NSW BORDER

NAME	GDA94		Ellipsoid values		
	LATITUDE	LONGITUDE	dφ	ρ	dm = ρ×dφ
Murray Spring	-36°47´49.223200″	148°11´48.333300″			
A	-36°49´07.598047″ N -36°49´07.590584″ GEA	148°15´00.000000″	+00´00.007463″	6358356.102	+0.2301
B	-36°55´13.876510″ N -36°55´13.840305″ GEA	148°30´00.000000″	+00´00.036205″	6358465.209	+1.1161
C	-37°01´17.289080″ N -37°01´17.234433″ GEA	148°45´00.000000″	+00´00.054647″	6358573.577	+1.6846
D	-37°07´17.845554″ N -37°07´17.782643″ GEA	149°00´00.000000″	+00´00.062911″	6358681.204	+1.9394
E	-37°13´15.555723″ N -37°13´15.494607″ GEA	149°15´00.000000″	+00´00.061116″	6358788.089	+1.8841
F	-37°19´10.429372″ N -37°19´10.379991″ GEA	149°30´00.000000″	+00´00.049381″	6358894.232	+1.5224
G	-37°25´02.476276″ N -37°25´02.448453″ GEA	149°45´00.000000″	+00´00.027823″	6358999.632	+0.8578
Wauka 1978	-37°30´18.067400″	149°58´32.993200″			

TABLE 1: Points where the Great Elliptic Arc cuts meridians of A, B, C, etc at 0°15´ intervals of longitude along Border Line. N = Normal Section, GEA = Great Elliptic Arc

```
>> great_elliptic_arc_lat

==================
Great Elliptic Arc
==================
Ellipsoid parameters
a  = 6378137.0000
f  = 1/298.257222101

Terminal points of curve
Latitude  P1 =  -36 47 49.223200 (D M S)
Longitude P1 =  148 11 48.333300 (D M S)

Latitude  P2 =  -37 30 18.067400 (D M S)
Longitude P2 =  149 58 32.993200 (D M S)

Cartesian coordinates
            X               Y               Z
P1   -4345789.609716  2694844.030716 -3799378.032024
P2   -4386272.668061  2534883.268540 -3862005.992252

Given longitude of P3
Longitude P3 =  149 30  0.000000 (D M S)

Latitude of P3 computed from trigonometric equation
Latitude  P3 =  -37 19 10.379991 (D M S)

>>
```

```
>> great_elliptic_arc_lon

==================
Great Elliptic Arc
==================
Ellipsoid parameters
a  = 6378137.0000
f  = 1/298.257222101

Terminal points of curve
Latitude  P1 =  -36 47 49.223200 (D M S)
Longitude P1 =  148 11 48.333300 (D M S)

Latitude  P2 =  -37 30 18.067400 (D M S)
Longitude P2 =  149 58 32.993200 (D M S)

Cartesian coordinates
            X               Y               Z
P1   -4345789.609716  2694844.030716 -3799378.032024
P2   -4386272.668061  2534883.268540 -3862005.992252

Given latitude of P3
Latitude  P3 =  -37 19 10.379991 (D M S)

Longitude of P3 computed from Newton-Raphson iteration
Longitude P3 =  149 30  0.000001 (D M S)
iterations   =    5

Longitude of P3 computed from trigonometric equation
Longitude P3 =  149 30  0.000001 (D M S)
theta P3     =    8 39 58.683516 (D M S)

>>
```

MATLAB function *great_ elliptic_ arc_ lat.m*

```
function great_elliptic_arc_lat
%
% great_elliptic_arc_lat: Given the terminal points P1 and P2 of a great
% elliptic arc on an ellipsoid, and the longitude of a point P3 on the
% curve, this function computes the latitude of P3.

%--------------------------------------------------------------------------
% Function:  great_elliptic_arc_lat
%
% Usage:     great_elliptic_arc_lat
%
% Author:    R.E.Deakin,
%            School of Mathematical & Geospatial Sciences, RMIT University
%            GPO Box 2476V, MELBOURNE, VIC 3001, AUSTRALIA.
%            email: rod.deakin@rmit.edu.au
%            Version  1.0  3 October 2009
%            Version  1.1  5 January 2010
%
% Purpose:   Given the terminal points P1 and P2 of a great elliptic arc on
%  an ellipsoid, and the longitude of a point P3 on the curve, this
%  function computes the latitude of P3.
%
% Functions required:
%      [D,M,S] = DMS(DecDeg)
%      [X,Y,Z] = Geo2Cart(a,flat,lat,lon,h)
%      [rm,rp] = radii(a,flat,lat);
%
% Variables:
%  A,B,C          - constants of great elliptic arc
%  a              - semi-major axis of ellipsoid
%  b              - semi-minor axis of ellipsoid
%  d2r            - degree to radian conversion factor 57.29577951...
%  e2             - eccentricity of ellipsoid squared
%  f              - f = 1/flat is the flattening of ellipsoid
%  flat           - denominator of flattening of ellipsoid
%  h1,h2          - ellipsoid heights of P1 and P2
%  lat1,lat2,lat3 - latitude of P1, P1, P3 (radians)
%  lon1,lon2,lon3 - longitude of P1, P2, P3 (radians)
%  nu             - radius of curvature in prime vertical plane
%  rho            - radius of curvature in meridain plane
%  X1,Y1,Z1       - Cartesian coordinates of P1
%  X2,Y2,Z2       - Cartesian coordinates of P2
%
% Remarks:
%
% References:
%  [1] Deakin, R.E., 2010, 'The Great Elliptic Arc on an Ellipsoid',
%         Lecture Notes, School of Mathematical and Geospatial Sciences,
%         RMIT University, January 2010
%
%--------------------------------------------------------------------------

% Degree to radian conversion factor
d2r   = 180/pi;

% Set ellipsoid parameters
a    = 6378137;    % GRS80
flat = 298.257222101;
% a    = 6378160;     % ANS
% flat = 298.25;
% a = 20926062;  % CLARKE 1866
% b = 20855121;
% f = 1-(b/a);
% flat = 1/f;

% Compute ellipsoid constants
```

```
f   = 1/flat;
e2  = f*(2-f);

% Set lat, lon and height of P1 and P2 on ellipsoid
lat1 = -(36  + 47/60 + 49.2232/3600)/d2r;   % Spring
lon1 =  (148 + 11/60 + 48.3333/3600)/d2r;
lat2 = -(37  + 30/60 + 18.0674/3600)/d2r;   % Wauka 1978
lon2 =  (149 + 58/60 + 32.9932/3600)/d2r;
h1 = 0;
h2 = 0;
% Compute Cartesian coords of P1 and P2
[X1,Y1,Z1] = Geo2Cart(a,flat,lat1,lon1,h1);
[X2,Y2,Z2] = Geo2Cart(a,flat,lat2,lon2,h2);

% Compute constants of Curve of Alignment
A = Y1*Z2-Y2*Z1;
B = X1*Z2-X2*Z1;
C = X1*Y2-X2*Y1;

% Set longitude of P3
lon3 = (149 + 30/60)/d2r;

% Compute latitude of P3
lat3 = atan((B*sin(lon3)-A*cos(lon3))/(C*(1-e2)));

%-----------------------
% Print result to screen
%-----------------------

fprintf('\n==================');
fprintf('\nGreat Elliptic Arc');
fprintf('\n==================');
fprintf('\nEllipsoid parameters');
fprintf('\na  = %12.4f',a);
fprintf('\nf  = 1/%13.9f',flat);

fprintf('\n\nTerminal points of curve');
% Print lat and lon of P1
[D,M,S] = DMS(lat1*d2r);
if D == 0 && lat1 < 0
    fprintf('\nLatitude  P1 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLatitude  P1 = %4d %2d %9.6f (D M S)',D,M,S);
end
[D,M,S] = DMS(lon1*d2r);
if D == 0 && lon1 < 0
    fprintf('\nLongitude P1 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLongitude P1 = %4d %2d %9.6f (D M S)',D,M,S);
end
% Print lat and lon of P2
[D,M,S] = DMS(lat2*d2r);
if D == 0 && lat2 < 0
    fprintf('\n\nLatitude  P2 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\n\nLatitude  P2 = %4d %2d %9.6f (D M S)',D,M,S);
end
[D,M,S] = DMS(lon2*d2r);
if D == 0 && lon2 < 0
    fprintf('\nLongitude P2 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLongitude P2 = %4d %2d %9.6f (D M S)',D,M,S);
end

% Print Coordinate table
fprintf('\n\nCartesian coordinates');
fprintf('\n            X               Y               Z');
fprintf('\nP1   %15.6f %15.6f %15.6f',X1,Y1,Z1);
fprintf('\nP2   %15.6f %15.6f %15.6f',X2,Y2,Z2);
```

```
% Print lat and lon of P3
fprintf('\n\nGiven longitude of P3');
[D,M,S] = DMS(lon3*d2r);
if D == 0 && lon3 < 0
    fprintf('\nLongitude P3 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLongitude P3 = %4d %2d %9.6f (D M S)',D,M,S);
end
fprintf('\n\nLatitude of P3 computed from trigonometric equation');
[D,M,S] = DMS(lat3*d2r);
if D == 0 && lat3 < 0
    fprintf('\nLatitude  P3 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLatitude  P3 = %4d %2d %9.6f (D M S)',D,M,S);
end

fprintf('\n\n');
```

MATLAB function *great_ elliptic_ arc_ lon.m*

```
function great_elliptic_arc_lon
%
% great_elliptic_arc_lon: Given the terminal points P1 and P2 of a great
% elliptic arc on an ellipsoid, and the latitude of a point P3 on the
% curve, this function computes the longitude of P3.

%--------------------------------------------------------------------------
% Function:  great_elliptic_arc_lon
%
% Usage:     great_elliptic_arc_lon
%
% Author:    R.E.Deakin,
%            School of Mathematical & Geospatial Sciences, RMIT University
%            GPO Box 2476V, MELBOURNE, VIC 3001, AUSTRALIA.
%            email: rod.deakin@rmit.edu.au
%            Version  1.0  3 October 2009
%            Version  1.1  5 January 2010
%
% Purpose:   Given the terminal points P1 and P2 of a great elliptic arc on
%  an ellipsoid, and the latitude of a point P3 on the curve, this
%  function computes the longitude of P3.
%
% Functions required:
%      [D,M,S] = DMS(DecDeg)
%      [X,Y,Z] = Geo2Cart(a,flat,lat,lon,h)
%      [rm,rp] = radii(a,flat,lat);
%
% Variables:
%  A,B,C          - constants of great elliptic arc
%  a              - semi-major axis of ellipsoid
%  b              - semi-minor axis of ellipsoid
%  d2r            - degree to radian conversion factor 57.29577951...
%  e2             - eccentricity of ellipsoid squared
%  f              - f = 1/flat is the flattening of ellipsoid
%  flat           - denominator of flattening of ellipsoid
%  f_lat3         - function of latitude of P3
%  fdash_lat3     - derivative of function of latitude of Pp3
%  h1,h2          - ellipsoid heights of P1 and P2
%  iter           - number of iterations
%  lambda         - longitude of P3 computed from trigonometric equation
%  lat1,lat2,lat3 - latitude of P1, P1, P3 (radians)
%  lon1,lon2,lon3 - longitude of P1, P2, P3 (radians)
%  new_lat3       - next latiude in Newton-Raphson iteration
%  nu             - radius of curvature in prime vertical plane
%  rho            - radius of curvature in meridain plane
```

```
%  theta          - auxiliary angle in the computation of lambda
%  X1,Y1,Z1       - Cartesian coordinates of P1
%  X2,Y2,Z2       - Cartesian coordinates of P2
%
% Remarks:
%
% References:
%  [1] Deakin, R.E., 2010, 'The Great Elliptic Arc on an Ellipsoid',
%         Lecture Notes, School of Mathematical and Geospatial Sciences,
%         RMIT University, January 2010
%
%---------------------------------------------------------------------------

% Degree to radian conversion factor
d2r   = 180/pi;

% Set ellipsoid parameters
a    = 6378137;    % GRS80
flat = 298.257222101;
% a    = 6378160;     % ANS
% flat = 298.25;
% a = 20926062;  % CLARKE 1866
% b = 20855121;
% f = 1-(b/a);
% flat = 1/f;

% Compute ellipsoid constants
f   = 1/flat;
e2  = f*(2-f);

% Set lat, lon and height of P1 and P2 on ellipsoid
lat1 = -(36  + 47/60 + 49.2232/3600)/d2r;   % Spring
lon1 =  (148 + 11/60 + 48.3333/3600)/d2r;
lat2 = -(37  + 30/60 + 18.0674/3600)/d2r;   % Wauka 1978
lon2 =  (149 + 58/60 + 32.9932/3600)/d2r;
h1 = 0;
h2 = 0;
% Compute Cartesian coords of P1 and P2
[X1,Y1,Z1] = Geo2Cart(a,flat,lat1,lon1,h1);
[X2,Y2,Z2] = Geo2Cart(a,flat,lat2,lon2,h2);

% Compute constants of Curve of Alignment
A = Y1*Z2-Y2*Z1;
B = X1*Z2-X2*Z1;
C = X1*Y2-X2*Y1;

% Set latitude of P3
lat3 = -(37 + 19/60 + 10.379991/3600)/d2r;

%------------------------------------------------------------
% Compute the longitude of P3 using Newton-Raphson iteration
%------------------------------------------------------------
% Set starting value of lon3 = longitude of P1
lon3 = lon1;
iter = 1;
while 1
    % Compute radii of curvature
    f_lon3     = A*cos(lon3)-B*sin(lon3)+C*(1-e2)*tan(lat3);
    fdash_lon3 = -A*sin(lon3)-B*cos(lon3);
    new_lon3   = lon3-(f_lon3/fdash_lon3);
    if abs(new_lon3 - lon3) < 1e-15
        break;
    end
    lon3 = new_lon3;
    if iter > 100
        fprintf('Iteration for longitude failed to converge after 100 iterations');
        break;
    end
    iter = iter + 1;
```

```
end;

%----------------------------------------------------------
% Compute the longitude of P3 using trigonometric equation
%----------------------------------------------------------
theta  = atan2(B,-A);
lambda = acos(C*(1-e2)*tan(lat3)/sqrt(A^2+B^2))+theta;

%------------------------
% Print result to screen
%------------------------

fprintf('\n==================');
fprintf('\nGreat Elliptic Arc');
fprintf('\n==================');
fprintf('\nEllipsoid parameters');
fprintf('\na  = %12.4f',a);
fprintf('\nf  = 1/%13.9f',flat);

fprintf('\n\nTerminal points of curve');
% Print lat and lon of P1
[D,M,S] = DMS(lat1*d2r);
if D == 0 && lat1 < 0
    fprintf('\nLatitude  P1 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLatitude  P1 = %4d %2d %9.6f (D M S)',D,M,S);
end
[D,M,S] = DMS(lon1*d2r);
if D == 0 && lon1 < 0
    fprintf('\nLongitude P1 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLongitude P1 = %4d %2d %9.6f (D M S)',D,M,S);
end
% Print lat and lon of P2
[D,M,S] = DMS(lat2*d2r);
if D == 0 && lat2 < 0
    fprintf('\n\nLatitude  P2 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\n\nLatitude  P2 = %4d %2d %9.6f (D M S)',D,M,S);
end
[D,M,S] = DMS(lon2*d2r);
if D == 0 && lon2 < 0
    fprintf('\nLongitude P2 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLongitude P2 = %4d %2d %9.6f (D M S)',D,M,S);
end

% Print Coordinate table
fprintf('\n\nCartesian coordinates');
fprintf('\n            X               Y               Z');
fprintf('\nP1   %15.6f %15.6f %15.6f',X1,Y1,Z1);
fprintf('\nP2   %15.6f %15.6f %15.6f',X2,Y2,Z2);

% Print lat and lon of P3
fprintf('\n\nGiven latitude of P3');
[D,M,S] = DMS(lat3*d2r);
if D == 0 && lat3 < 0
    fprintf('\nLatitude  P3 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLatitude  P3 = %4d %2d %9.6f (D M S)',D,M,S);
end

fprintf('\n\nLongitude of P3 computed from Newton-Raphson iteration');
[D,M,S] = DMS(lon3*d2r);
if D == 0 && lon3 < 0
    fprintf('\nLongitude P3 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLongitude P3 = %4d %2d %9.6f (D M S)',D,M,S);
end
```

```
fprintf('\niterations   = %4d',iter);

fprintf('\n\nLongitude of P3 computed from trigonometric equation');
[D,M,S] = DMS(lambda*d2r);
if D == 0 && lambda < 0
    fprintf('\nLongitude P3 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLongitude P3 = %4d %2d %9.6f (D M S)',D,M,S);
end
[D,M,S] = DMS(theta*d2r);
if D == 0 && theta < 0
    fprintf('\ntheta P3     =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\ntheta P3     = %4d %2d %9.6f (D M S)',D,M,S);
end

fprintf('\n\n');
```

MATLAB function *Geo2Cart.m*

```
function [X,Y,Z] = Geo2Cart(a,flat,lat,lon,h)
%
% [X,Y,Z] = Geo2Cart(a,flat,lat,lon,h)
%   Function computes the Cartesian coordinates X,Y,Z of a point
%   related to an ellipsoid defined by semi-major axis (a) and the
%   denominator of the flattening (flat) given geographical
%   coordinates latitude (lat), longitude (lon) and ellipsoidal
%   height (h).  Latitude and longitude are assumed to be in radians.

%--------------------------------------------------------------------------
% Function:  Geo2Cart()
%
% Usage:     [X,Y,Z] = Geo2Cart(a,flat,lat,lon,h);
%
% Author:    R.E.Deakin,
%            School of Mathematical & Geospatial Sciences, RMIT University
%            GPO Box 2476V, MELBOURNE, VIC 3001, AUSTRALIA.
%            email: rod.deakin@rmit.edu.au
%            Version  1.0   6 April  2006
%            Version  1.0  20 August 2007
%
% Functions required:
%    radii()
%
% Purpose:
%    Function Geo2Cart() will compute Cartesian coordinates X,Y,Z
%    given geographical coordinates latitude, longitude (both in
%    radians) and height of a  point related to an ellipsoid
%    defined by semi-major axis (a) and denominator of flattening
%    (flat).
%
% Variables:
%    a       - semi-major axis of ellipsoid
%    e2      - 1st eccentricity squared
%    f       - flattening of ellipsoid
%    flat    - denominator of flattening f = 1/flat
%    h       - height above ellipsoid
%    lat     - latitude (radians)
%    lon     - longitude (radians)
%    p       - perpendicular distance from minor axis of ellipsoid
%    rm      - radius of curvature of meridian section of ellipsoid
%    rp      - radius of curvature of prime vertical section of ellipsoid
%
% References:
% [1] Gerdan, G.P. & Deakin, R.E., 1999, 'Transforming Cartesian
```

```
%     coordinates X,Y,Z to geogrpahical coordinates phi,lambda,h', The
%     Australian Surveyor, Vol. 44, No. 1, pp. 55-63, June 1999.
%--------------------------------------------------------------------------

% calculate flattening f and ellipsoid constant e2
f   = 1/flat;
e2  = f*(2-f);

% compute radii of curvature for the latitude
[rm,rp] = radii(a,flat,lat);

% compute Cartesian coordinates X,Y,Z
p = (rp+h)*cos(lat);
X = p*cos(lon);
Y = p*sin(lon);
Z = (rp*(1-e2)+h)*sin(lat);
```

MATLAB function *radii.m*

```
function [rm,rp] = radii(a,flat,lat)
%
% [rm,rp]=radii(a,flat,lat)  Function computes radii of curvature in
%   the meridian and prime vertical planes (rm and rp respectively) at a
%   point whose latitude (lat) is known on an ellipsoid defined by
%   semi-major axis (a) and denominator of flattening (flat).
%   Latitude must be in radians.
%   Example: [rm,rp] = radii(6378137,298.257222101,-0.659895044);
%            should return rm = 6359422.96233327 metres and
%                          rp = 6386175.28947842 metres
%            at latitude -37 48 33.1234 (DMS) on the GRS80 ellipsoid

%--------------------------------------------------------------------------
% Function:  radii(a,flat,lat)
%
% Syntax:    [rm,rp] = radii(a,flat,lat);
%
% Author:    R.E.Deakin,
%            School of Mathematical & Geospatial Sciences, RMIT University
%            GPO Box 2476V, MELBOURNE, VIC 3001, AUSTRALIA.
%            email: rod.deakin@rmit.edu.au
%            Version  1.0  1 August 2003
%            Version  2.0  6 April  2006
%            Version  3.0  9 February 2008
%
% Purpose:   Function radii() will compute the radii of curvature in
%            the meridian and prime vertical planes, rm and rp respectively
%            for the point whose latitude (lat) is given for an ellipsoid
%            defined by its semi-major axis (a) and denominator of
%            flattening (flat).
%
% Return value: Function radii() returns rm and rp
%
% Variables:
%  a      - semi-major axis of spheroid
%  c      - polar radius of curvature
%  c2     - cosine of latitude squared
%  ep2    - 2nd-eccentricity squared
%  f      - flattening of ellipsoid
%  lat    - latitude of point (radians)
%  rm     - radius of curvature in the meridian plane
%  rp     - radius of curvature in the prime vertical plane
%  V      - latitude function defined by V-squared = sqrt(1 + ep2*c2)
%  V2,V3  - powers of V
%
% Remarks:
%  Formulae are given in [1] (section 1.3.9, page 85) and in
```

```
%  [2] (Chapter 2, p. 2-10) in a slightly different form.
%
% References:
% [1] Deakin, R.E. and Hunter, M.N., 2008, GEOMETRIC GEODESY, School of
%        Mathematical and Geospatial Sciences, RMIT University, Melbourne,
%        AUSTRALIA, March 2008.
% [2] THE GEOCENTRIC DATUM OF AUSTRALIA TECHNICAL MANUAL, Version 2.2,
%        Intergovernmental Committee on Surveying and Mapping (ICSM),
%        February 2002 (www.anzlic.org.au/icsm/gdatum)
%--------------------------------------------------------------------------

% compute flattening f eccentricity squared e2
f   = 1/flat;
c   = a/(1-f);
ep2 = f*(2-f)/((1-f)^2);

% calculate the square of the sine of the latitude
c2 = cos(lat)^2;

% compute latitude function V
V2 = 1+ep2*c2;
V  = sqrt(V2);
V3 = V2*V;

% compute radii of curvature
rm = c/V3;
rp = c/V;
```

MATLAB function *DMS.m*

```
function [D,M,S] = DMS(DecDeg)
% [D,M,S] = DMS(DecDeg)  This function takes an angle in decimal degrees and returns
%   Degrees, Minutes and Seconds

val = abs(DecDeg);
D = fix(val);
M = fix((val-D)*60);
S = (val-D-M/60)*3600;
if(DecDeg<0)
  D = -D;
end
return
```

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Bowring, B. R., (1972), 'Distance and the spheroid', Correspondence, *Survey Review,* Vol. XXI, No. 164, April 1972, pp. 281-284.

Clarke, A. R., (1880), *Geodesy,* Clarendon Press, Oxford.

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Thomas, P. D., (1952), *Conformal Projections in Geodesy and Cartography,* Special Publication No. 251, Coast and Geodetic Survey, United States Department of Commerce, Washington, D.C.

THE LOXODROME ON AN ELLIPSOID

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ABSTRACT

These notes provide a detailed explanation of the geometry of the loxodrome on the ellipsoid. Equations are derived for azimuth and distance of a loxodrome between two points on an ellipsoid and these equations enable the development of algorithms for the solution of the direct and inverse problems of the loxodrome. A MATLAB function is provided that demonstrates an algorithm for the inverse problem.

INTRODUCTION

The loxodrome between P_1 and P_2 on the ellipsoid is a curved line such that every element of the curve ds intersects a meridian at a constant azimuth α. Unless $\alpha = 0°, 90°, 180°$ or $270°$ the loxodrome will spiral around the ellipsoid and terminate at one of the poles. In other cases the loxodrome will lie along a meridian of longitude $\left(\alpha = 0°, 180°\right)$ or a parallel of latitude $\left(\alpha = 90°, 270°\right)$.

Figure 1: Loxodrome on the earth's surface

In marine and air navigation, aircraft and ships sailing or flying on fixed compass headings are moving along loxodromes, hence knowledge of loxodromes is important in navigation. Mercator's projection – a normal aspect cylindrical conformal projection – has the unique property that loxodromes on the earth's surface are projected as straight lines on the map.

In geodesy the direct problem (computing position given azimuth and distance from a known location) and the inverse problem (computing azimuth and distance between known positions) are fundamental operations and can be likened to the equivalent operations of plane surveying; radiations (computing coordinates of points given bearings and distances radiating from a point of known coordinates) and joins; (computing bearings and distances between points having known coordinates). The direct and inverse problems in geodesy are usually associated with the geodesic which is the unique curve defining the shortest path on the ellipsoid but they can also be associated with other curves. So;

> The direct problem of the loxodrome on the ellipsoid is: given latitude and longitude of P_1 and the azimuth α and distance s of a loxodrome between P_1 and P_2; compute the latitude and longitude of P_2.
>
> The inverse problem of the loxodrome on the ellipsoid is: given the latitude and longitude of P_1 and P_2; compute the azimuth α and distance s of the loxodrome between P_1 and P_2.

The equations necessary for the solution of the direct and inverse problems are derived from the differential geometry of the ellipsoid and in particular, relationships that can be obtained from the differential rectangle on the ellipsoid. Also, meridian distance (the distance along a meridian from the equator) is used in computing loxodrome distances. Discussions of differential geometry of the ellipsoid and meridian distance can be found in Deakin & Hunter (2008) or geodesy textbooks (e.g., Lauf 1983; Bomford 1980), and an excellent treatment of the loxodrome on the ellipsoid can be found in Bowring (1985).

THE ELLIPSOID

In geodesy, the ellipsoid is a surface of revolution created by rotating an ellipse about its minor axis. The size and shape of an ellipsoid is defined by one of three pairs of parameters: (i) a, b where a and b are the semi-major and semi-minor axes lengths of an ellipsoid respectively (and $a > b$), or (ii) a, f where f is the flattening of an ellipsoid, or (iii) a, e^2 where e^2 is the square of the first eccentricity of an ellipsoid.

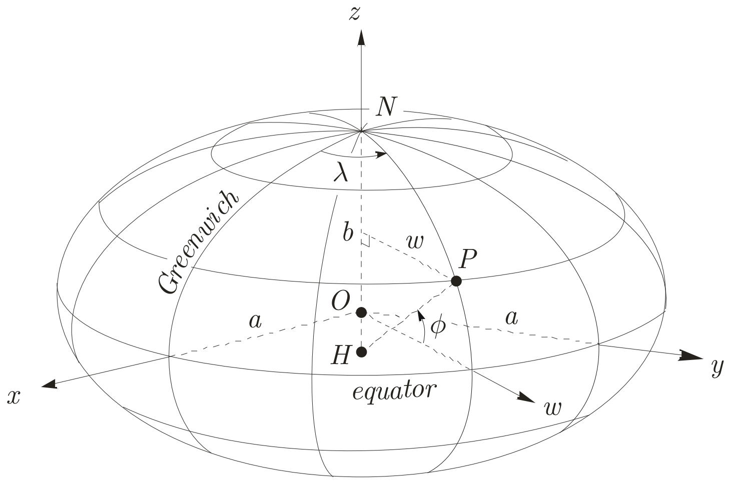


Figure 2: The reference ellipsoid

The ellipsoid parameters a, b, f, e^2 are related by the following equations

$$f = \frac{a-b}{a} = 1 - \frac{b}{a} \tag{1}$$

$$b = a(1-f) \tag{2}$$

$$e^2 = \frac{a^2 - b^2}{a^2} = 1 - \frac{b^2}{a^2} = f(2-f) \tag{3}$$

$$1 - e^2 = \frac{b^2}{a^2} = 1 - f(2-f) = (1-f)^2 \tag{4}$$

The <u>second eccentricity</u> e' of an ellipsoid is also of use and

$$e'^2 = \frac{a^2 - b^2}{b^2} = \frac{a^2}{b^2} - 1 = \frac{e^2}{1-e^2} = \frac{f(2-f)}{(1-f)^2} \tag{5}$$

$$e^2 = \frac{e'^2}{1+e'^2} \tag{6}$$

In Figure 2 the normal to the surface at P intersects the rotational axis of the ellipsoid (the z-axis) at H making an angle ϕ with the equatorial plane of the ellipsoid – this is the latitude of P. The longitude λ is the angle between the Greenwich meridian plane (a reference plane) and the meridian plane (the z-w plane) containing the normal through P. ϕ and λ are <u>curvilinear coordinates</u> and meridians of longitude (curves of constant λ) and parallels of latitude (curves of constant ϕ) are parametric curves on the ellipsoidal surface. At P on the surface of the ellipsoid, planes containing the normal to the ellipsoid intersect the surface creating elliptical sections known as <u>normal sections</u>. Amongst the infinite number of possible normal sections at a *P;* each having a certain radius of curvature, two are of interest: (i) the meridian section, containing the axis of revolution of the ellipsoid and having the least radius of curvature, denoted by ρ, and (ii) the prime vertical section, perpendicular to the meridian plane and having the greatest radius of curvature, denoted by ν.

$$\rho = \frac{a\left(1-e^2\right)}{\left(1-e^2\sin^2\phi\right)^{\frac{3}{2}}} = \frac{a\left(1-e^2\right)}{W^3} \tag{7}$$

$$\nu = \frac{a}{\left(1-e^2\sin^2\phi\right)^{\frac{1}{2}}} = \frac{a}{W} \tag{8}$$

$$W^2 = 1-e^2\sin^2\phi \tag{9}$$

For P, the centre of the radius of curvature of the prime vertical section is at H and $\nu = PH$. The centre of the radius of curvature of the meridian section lies on the normal between P and H.

Alternative equations for the radii of curvature ρ and ν are given by

$$\rho = \frac{a^2}{b\left(1+e'^2\cos^2\phi\right)^{\frac{3}{2}}} = \frac{c}{V^3} \tag{10}$$

$$\nu = \frac{a^2}{b\left(1+e'^2\cos^2\phi\right)^{\frac{1}{2}}} = \frac{c}{V} \tag{11}$$

$$c = \frac{a^2}{b} = \frac{a}{1-f} \tag{12}$$

$$V^2 = 1+e'^2\cos^2\phi \tag{13}$$

and c is the polar radius of curvature of the ellipsoid.
The latitude functions W and V are related as follows

$$W^2 = \frac{V^2}{1+e'^2} \quad \text{and} \quad W = \frac{V}{\left(1+e'^2\right)^{\frac{1}{2}}} = \frac{b}{a}V \tag{14}$$

Points on the ellipsoid surface have curvilinear coordinates ϕ,λ and Cartesian coordinates x,y,z where the x-z plane is the Greenwich meridian plane, the x-y plane is the equatorial plane and the y-z plane is a meridian plane 90° east of the Greenwich meridian plane. Cartesian and curvilinear coordinates are related by

$$\begin{aligned} x &= \nu\cos\phi\cos\lambda \\ y &= \nu\cos\phi\cos\lambda \\ z &= \nu\left(1-e^2\right)\sin\phi \end{aligned} \tag{15}$$

Note that $\nu\left(1-e^2\right)$ is the distance along the normal from a point on the surface to the point where the normal cuts the equatorial plane.

DIFFERENTIAL RELATIONSHIPS FOR THE LOXODROME ON THE ELLIPSOID

The derivation of equations relating to the loxodrome requires an understanding of the connection between differentially small quantities on the surface of the ellipsoid.

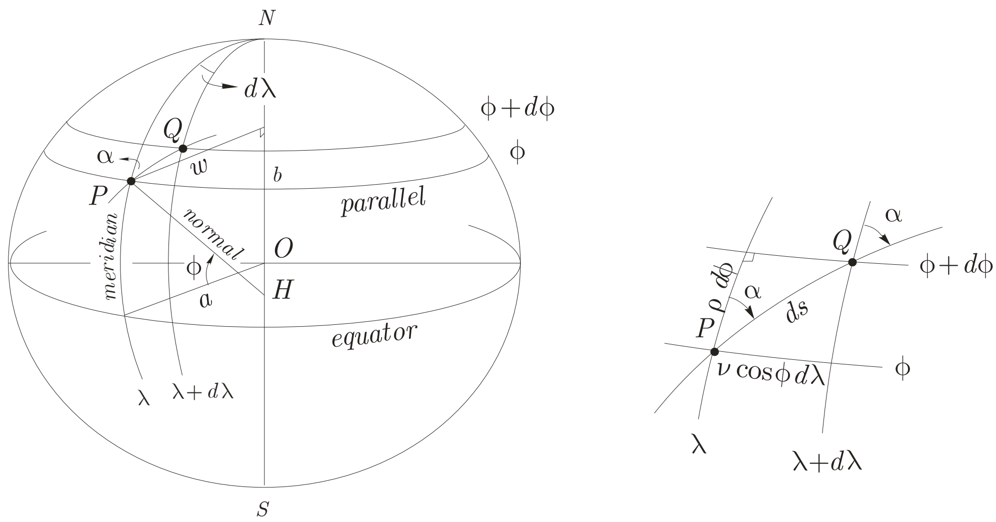


Figure 3: The differential rectangle on an ellipsoid (a,b)

These relationships can be derived from the differential rectangle, with diagonal PQ in Figure 3 which shows P and Q on an ellipsoid whose semi-axes are a and b $\left(a > b\right)$. P and Q are separated by differential changes in latitude $d\phi$ and longitude $d\lambda$ and are connected by a loxodrome of length ds making an angle α (the azimuth) with the meridian through P. The meridians λ and $\lambda + d\lambda$, and the parallels ϕ and $\phi + d\phi$ form a differential rectangle on the surface of the ellipsoid. The differential distances dp along the parallel ϕ and dm along the meridian λ are

$$dp = w\,d\lambda = \nu\cos\phi\,d\lambda \tag{16}$$

$$dm = \rho\,d\phi \tag{17}$$

where ρ and ν are radii of curvature in the meridian and prime vertical planes respectively and $w = \nu\cos\phi$ is the perpendicular distance from the rotational axis *NOS*.

From Figure 3, the differential distance ds is given by

$$\begin{aligned} ds &= \sqrt{dm^2 + dp^2} \\ &= \sqrt{\rho^2 d\phi^2 + \nu^2 \cos^2\phi\, d\lambda^2} \\ &= \nu\cos\phi \sqrt{\left(\frac{\rho\, d\phi}{\nu\cos\phi}\right)^2 + d\lambda^2} \\ &= \nu\cos\phi\sqrt{dq^2 + d\lambda^2} \end{aligned} \tag{18}$$

q is known as the isometric latitude defined by the differential relationship

$$dq = \frac{\rho}{\nu\cos\phi}\, d\phi \tag{19}$$

$\left(q,\lambda\right)$ is a curvilinear coordinate system on the ellipsoid with isometric parameters where isometric means of equal measure (*iso* = equal; *metric* = able to be measured). We can see this from equation (18) where the differential distances along the parametric curves q and λ are $dm = \nu\cos\phi\, dq$ and $dp = \nu\cos\phi\, d\lambda$, i.e., the differential distances are equal for equal angular differentials dq and $d\lambda$.

Also from Figure 3 the azimuth α of the loxodrome is obtained from

$$\tan\alpha = \frac{\nu\cos\phi\, d\lambda}{\rho\, d\phi} = \frac{d\lambda}{dq} \tag{20}$$

and azimuth α and distance s are linked by the differential relationship

$$ds = \frac{dm}{\cos\alpha} = \frac{1}{\cos\alpha}\rho\, d\phi \tag{21}$$

ISOMETRIC LATITUDE

The isometric latitude is defined by the differential equation (19) from which we obtain

$$q = \int \frac{\rho}{\nu\cos\phi}\, d\phi + C_1 \tag{22}$$

where C_1 is a constant of integration.

Substituting into equation (22) expressions for ρ and ν given by equations (7) and (8), and simplifying gives

$$q = \int \frac{\left(1-e^2\right)}{\left(1-e^2\sin^2\phi\right)\cos\phi}\, d\phi + C_1 \tag{23}$$

The integrand of equation (23) can be separated into partial fractions

$$\frac{\left(1-e^2\right)}{\left(1-e^2\sin^2\phi\right)\cos\phi}=\frac{A}{\left(1-e^2\sin^2\phi\right)}+\frac{B}{\cos\phi} \tag{24}$$

Expanding and simplifying equation (24) gives

$$\begin{aligned} 1-e^2 &= A\cos\phi+B\left(1-e^2\sin^2\phi\right) \\ &= A\cos\phi+B-Be^2\left(1-\cos^2\phi\right) \\ &= B\left(1-e^2\right)+\left(A+Be^2\cos\phi\right)\cos\phi \end{aligned} \tag{25}$$

A and B are obtained by comparing the coefficients of $1-e^2$ and $\cos\phi$ in equation (25) giving

$$B=1; \quad A=-e^2\cos\phi$$

Substituting these results into equation (24) gives the isometric latitude as

$$q=\int\frac{1}{\cos\phi}d\phi-\int\frac{e^2\cos\phi}{1-e^2\sin^2\phi}d\phi+C_1 \tag{26}$$

Put $e\sin\phi=\sin u$ then $e\cos\phi\,d\phi=\cos u\,du$ and

$$\begin{aligned} q &= \int\frac{1}{\cos\phi}d\phi-e\int\frac{\cos u}{1-\sin^2 u}du+C_1 \\ &= \int\frac{1}{\cos\phi}d\phi-e\int\frac{\cos u}{\cos^2 u}du+C_1 \\ &= \int\frac{1}{\cos\phi}d\phi-e\int\frac{1}{\cos u}du+C_1 \end{aligned} \tag{27}$$

From standard integrals $\int\frac{1}{\cos x}dx=\ln\left\{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}$ and from half-angle trigonometric formula $\tan\left(\frac{A}{2}\right)=\pm\sqrt{\frac{1-\cos A}{1+\cos A}}$ giving $\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)=\sqrt{\frac{1-\cos\left(x+\pi/2\right)}{1+\cos\left(x+\pi/2\right)}}=\sqrt{\frac{1+\sin x}{1-\sin x}}$.

Substituting these results into equation (27) gives the isometric latitude as

$$q=\ln\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)+C_2-e\ln\left(\frac{1+e\sin\phi}{1-e\sin\phi}\right)^{\frac{1}{2}}-C_3+C_1$$

where C_1, C_2 and C_3 are constants of integration. Using the laws of logarithms:

$\log_a MN=\log_a M+\log_a N$, $\log_a\frac{M}{N}=\log_a M-\log_a N$ and $\log_a M^p=p\log_a M$, and defining a new constant of integration $C=C_2-C_3+C_1$ gives

$$
\begin{aligned}
q &= \ln\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)+\ln\left(\frac{1-e\sin\phi}{1+e\sin\phi}\right)^{\frac{e}{2}}+C \\
&= \ln\left\{\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\left(\frac{1-e\sin\phi}{1+e\sin\phi}\right)^{\frac{e}{2}}\right\}+C
\end{aligned}
\tag{28}
$$

The constant C in equation (28) equals zero since if $\phi = 0$ then $q = 0$ and the isometric latitude q is obtained from

$$
q = \ln\left\{\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\left(\frac{1-e\sin\phi}{1+e\sin\phi}\right)^{\frac{e}{2}}\right\}
\tag{29}
$$

This derivation follows Lauf (1983) where an integral identical to equation (22) is evaluated as part of the derivation of the equations for the ellipsoidal Mercator projection – a conformal projection of the ellipsoid. Thomas (1952) derives a similar equation in his development of conformal representation of the ellipsoid upon a plane.

THE EQUATION OF THE LOXODROME

By re-arranging equation (20) we have

$$
d\lambda = \tan\alpha\, dq
$$

and integrating both sides, noting that $\tan\alpha$ is a constant, gives

$$
\begin{aligned}
\int_{\lambda_1}^{\lambda_2} d\lambda &= \tan\alpha \int_{q_1}^{q_2} dq \\
\lambda_2 - \lambda_1 &= \tan\alpha\left(q_2 - q_1\right)
\end{aligned}
$$

And the equation of the loxodrome between P_1 and P_2 on the ellipsoid is

$$
\Delta\lambda = \Delta q \tan\alpha
\tag{30}
$$

where $\Delta\lambda = \lambda_2 - \lambda_1$ and $\Delta q = q_2 - q_1$ are differences in longitude and isometric latitude respectively and α is the (constant) azimuth of the loxodrome.

THE AZIMUTH OF A LOXODROME

The azimuth α of a loxodrome between P_1 and P_2 on an ellipsoid can be obtained from equation (30) as

$$\alpha = \arctan\left(\frac{\Delta\lambda}{\Delta q}\right) = \arctan\left(\frac{\lambda_2 - \lambda_1}{q_2 - q_1}\right) \tag{31}$$

where q_1, q_2 are isometric latitudes of P_1 and P_2 respectively and q is given by equation (29). λ_1, λ_2 are the longitudes of P_1 and P_2.

DISTANCE ALONG A LOXODROME

Consider a loxodrome of constant azimuth α that crosses the equator and passes through P_1 and P_2. The distance s between P_1 and P_2 can be defined as $s = s_2 - s_1$ where s_1 and s_2 are distances from the equator to P_1 and P_2 respectively and from equations (21) and (7) we may write

$$s_1 = \frac{1}{\cos\alpha}\int_0^{\phi_1} \rho\ d\phi = \frac{a\left(1-e^2\right)}{\cos\alpha}\int_0^{\phi_1} \frac{1}{W^3}\ d\phi = \frac{m_1}{\cos\alpha} \tag{32}$$

and similarly

$$s_2 = \frac{m_2}{\cos\alpha} \tag{33}$$

m_1 and m_2 are meridian distances and meridian distance m is defined as the length of the arc of the meridian to a point in latitude ϕ. m is obtained from the differential relationship given by equation (17) and

$$m = \int_0^{\phi} \rho\ d\phi = a\left(1-e^2\right)\int_0^{\phi}\left(1-e^2\sin^2\phi\right)^{-\frac{3}{2}}\ d\phi = a\left(1-e^2\right)\int_0^{\phi}\frac{1}{W^3}\ d\phi \tag{34}$$

This is an elliptic integral of the second kind and cannot be evaluated directly; instead, the integrand $\dfrac{1}{W^3} = \left(1-e^2\sin^2\phi\right)^{-\frac{3}{2}}$ is expanded by using the binomial series and then evaluated by term-by-term integration. Following Deakin & Hunter (2008) we obtain an expression for the meridian distance as

$$m = a\left\{A_0\phi - A_2\sin 2\phi + A_4\sin 4\phi - A_6\sin 6\phi + A_8\sin 8\phi - A_{10}\sin 10\phi + \cdots\right\} \tag{35}$$

where

$$
\begin{aligned}
A_0 &= 1 - \frac{1}{4}e^2 - \frac{3}{64}e^4 - \frac{5}{256}e^6 - \frac{175}{16384}e^8 - \frac{441}{65536}e^{10} + \cdots \\
A_2 &= \frac{3}{8}\left(e^2 + \frac{1}{4}e^4 + \frac{15}{128}e^6 + \frac{35}{512}e^8 + \frac{735}{16384}e^{10} + \cdots\right) \\
A_4 &= \frac{15}{256}\left(e^4 + \frac{3}{4}e^6 + \frac{35}{64}e^8 + \frac{105}{256}e^{10} + \cdots\right) \\
A_6 &= \frac{35}{3072}\left(e^6 + \frac{5}{4}e^8 + \frac{315}{256}e^{10} + \cdots\right) \\
A_8 &= \frac{315}{131072}\left(e^8 + \frac{7}{4}e^{10} + \cdots\right) \\
A_{10} &= \frac{693}{131072}\left(e^{10} + \cdots\right)
\end{aligned}
\tag{36}
$$

Combining equations (32) and (33) gives the length of the loxodrome between P_1 and P_2 as

$$
s = \frac{m_2 - m_1}{\cos\alpha} \tag{37}
$$

where α is the (constant) azimuth and m_1 and m_2 are meridian distances for ϕ_1 and ϕ_2 obtained from equation (35).

THE DIRECT PROBLEM OF THE LOXODROME ON THE ELLIPSOID

The direct problem is: Given latitude and longitude of P_1, azimuth α_{12} of the loxodrome P_1P_2 and the arc length s along the loxodrome curve; compute the latitude and longitude of P_2 and the reverse azimuth α_{21}.

With the ellipsoid constants a, f, and e^2 and given $\phi_1, \lambda_1, \alpha_{12}$ and s the problem may be solved by the following sequence.

1. Compute m_1 the meridian distance of P_1 using equation (35).
2. Compute meridian distance m_2 from equation (37) where

$$
m_2 = s\cos\alpha_{12} + m_1
$$

3. Use Newton-Raphson iteration to compute latitude ϕ_2 using equation (35) re-arranged as

$$
f(\phi) = a\left\{A_0\phi - A_2\sin 2\phi + A_4\sin 4\phi - A_6\sin 6\phi + A_8\sin 8\phi - A_{10}\sin 10\phi\right\} - m = 0
$$

and the iterative equation $\phi_{(n+1)} = \phi_{(n)} - \dfrac{f\left(\phi_{(n)}\right)}{f'\left(\phi_{(n)}\right)}$ where $f'\left(\phi\right) = \dfrac{d}{d\phi}\left\{f\left(\phi\right)\right\}$ and

$$f'\left(\phi\right) = a\left\{A_0 - 2A_2\cos 2\phi + 4A_4\cos 4\phi - 6A_6\cos 6\phi + 8A_8\cos 8\phi - 10A_{10}\cos 10\phi\right\}$$

An initial value of $\phi_{(1)}$ (ϕ for $n = 1$) can be taken as the latitude of P_1 and the functions $f\left(\phi_{(1)}\right)$ and $f'\left(\phi_{(1)}\right)$ evaluated using ϕ_1. $\phi_{(2)}$ (ϕ for $n = 2$) can now be computed from the iterative equation and this process repeated to obtain values $\phi_{(3)}, \phi_{(4)}, \ldots$. This iterative process can be concluded when the difference between $\phi_{(n+1)}$ and $\phi_{(n)}$ reaches an acceptably small value.

4. Compute isometric latitudes q_1 and q_2 using equation (29) and then the difference in isometric latitudes $\Delta q = q_2 - q_1$
5. Compute the difference in longitude $\Delta\lambda = \lambda_2 - \lambda_1$ from equation (30)
6. Compute longitude λ_2 from $\lambda_2 = \lambda_1 + \Delta\lambda$
7. Compute reverse azimuth from $\alpha_{21} = \alpha_{12} \pm 180^\circ$

THE INVERSE PROBLEM OF THE LOXODROME ON THE ELLIPSOID

The inverse problem is: Given latitudes and longitudes of P_1 and P_2 on the ellipsoid, compute the azimuth α_{12} of the loxodrome P_1P_2, the arc length s along the loxodrome curve and the reverse azimuth α_{21}.

With the ellipsoid constants a, f, and e^2 and given ϕ_1, λ_1 and ϕ_2, λ_2 the problem may be solved by the following sequence.

1. Compute isometric latitudes q_1 and q_2 using equation (29) and then the difference in isometric latitudes $\Delta q = q_2 - q_1$
2. Compute the longitude difference $\Delta\lambda = \lambda_2 - \lambda_1$ and then the azimuth α_{12} using equation (31).
3. Compute meridian distances m_1 and m_2 using equation (35).
4. Compute the arc length s from equation (37).
5. Compute reverse azimuth from $\alpha_{21} = \alpha_{12} \pm 180^\circ$

MATLAB FUNCTIONS

A MATLAB function *loxodrome_ inverse.m* is shown below. This function computes the inverse problem of the loxodrome on the ellipsoid.

Output from the function is shown below for points on a great elliptic arc between the terminal points of the straight-line section of the Victorian–New South Wales border. This straight-line section of the border, between Murray Spring and Wauka 1978, is known as the Black-Allan Line in honour of the surveyors Black and Allan who set out the border line in 1870-71. Wauka 1978 (Gabo PM 4) is a geodetic concrete border pillar on the coast at Cape Howe and Murray Spring (Enamo PM 15) is a steel pipe driven into a spring of the Murray River that is closest to Cape Howe. The straight line is a normal section curve on the reference ellipsoid of the Geocentric Datum of Australia (GDA94) that contains the normal to the ellipsoid at Murray Spring. The GDA94 coordinates of Murray Spring and Wauka 1978 are:

$$\begin{array}{lll} \text{Murray Spring:} & \phi\ -37^\circ\,47'\,49.2232'' & \lambda\ 148^\circ\,11'\,48.3333'' \\ \text{Wauka 1978:} & \phi\ -37^\circ\,30'\,18.0674'' & \lambda\ 149^\circ\,58'\,32.9932'' \end{array}$$

The normal section azimuth and distance are:

$$116^\circ\,58'\,14.173757'' \qquad 176495.243760 \text{ m}$$

The geodesic azimuth and distance are:

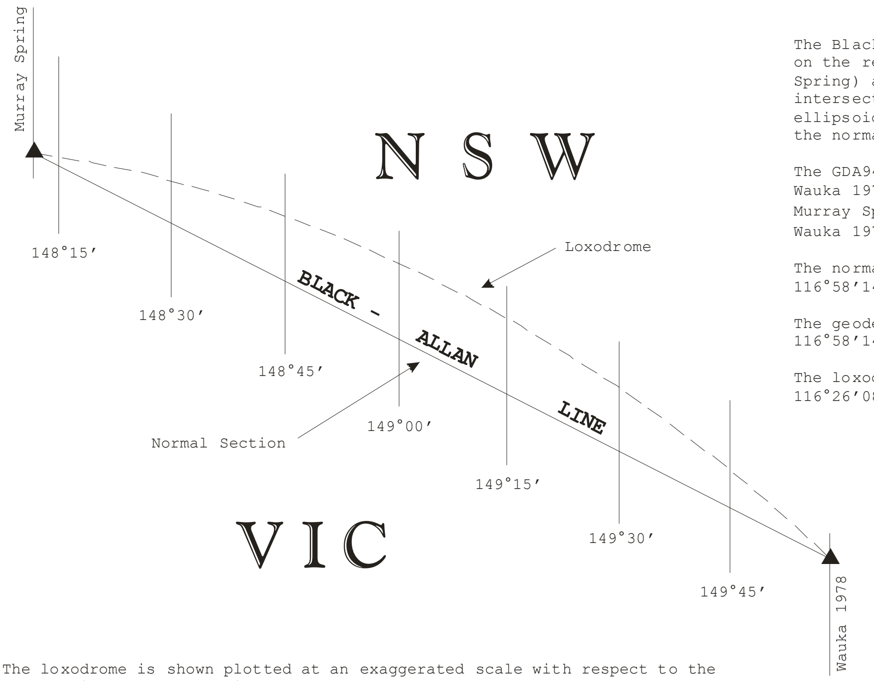
$$116^\circ\,58'\,14.219146'' \qquad 176495.243758 \text{ m}$$

The loxodrome azimuth and distance are:

$$116^\circ\,26'\,08.400701'' \qquad 176497.829952 \text{ m}$$

Figure 4 shows a schematic view of the Black-Allan line (normal section) and the great elliptic arc. The relationships between the great elliptic arc and the normal section have been computed at seven locations along the line (A, B, C, etc.) where meridians of longitude at $0^\circ\,15'$ intervals cut the line. These relationships are shown in Table 1.

BLACK-ALLAN LINE: VICTORIA/NSW BORDER



The Black-Allan Line is a normal section curve on the reference ellipsoid between P1 (Murray Spring) and P2 (Wauka 1978). This curve is the intersection of the normal section plane and the ellipsoid, and the normal section contains P1, the normal to the ellipsoid at P1, and P2.

The GDA94 coordinates of Murray Spring and Wauka 1978 are:
Murray Spring: ϕ -37°47′49.2232″ λ 148°11′48.3333″
Wauka 1978: ϕ -37°30′18.0674″ λ 149°58′32.9932″

The normal section azimuth and distance are:
116°58′14.173757″ 176495.243760 m.

The geodesic azimuth and distance are:
116°58′14.219146″ 176495.243758 m.

The loxodrome azimuth and distance are:
116°26′08.400701″ 176497.829952 m.

The loxodrome is shown plotted at an exaggerated scale with respect to the Border Line (normal section).
At longitude 149°00′E. the loxodrome is 457.918 m north of the Border Line.
At longitude 149°30′E. the loxodrome is 361.250 m north of the Border Line.

Figure 4

BLACK-ALLAN LINE: VICTORIA/NSW BORDER

NAME	**GDA94**		**Ellipsoid values**		
	LATITUDE	**LONGITUDE**	**dφ**	**ρ**	**dm = ρ×dφ**
Murray Spring	-36°47′49.223200″	148°11′48.333300″			
A	-36°49′07.598047″ N -36°49′05.849245″ Lox	148°15′00.000000″	+00′01.748802″	6358356.102	+53.9089
B	-36°55′13.876510″ N -36°55′05.371035″ Lox	148°30′00.000000″	+00′08.505475″	6358465.209	+262.1958
C	-37°01′17.289080″ N -37°01′04.418599″ Lox	148°45′00.000000″	+00′12.870481″	6358573.577	+396.7613
D	-37°07′17.845554″ N -37°07′02.991484″ Lox	149°00′00.000000″	+00′14.854070″	6358681.204	+457.9177
E	-37°13′15.555723″ N -37°13′01.089240″ Lox	149°15′00.000000″	+00′14.466483″	6358788.089	+459.9767
F	-37°19′10.429372″ N -37°18′58.711427″ Lox	149°30′00.000000″	+00′11.717945″	6358894.232	+361.2501
G	-37°25′02.476276″ N -37°24′55.857608″ Lox	149°45′00.000000″	+00′06.618668″	6358999.632	+204.0489
Wauka 1978	-37°30′18.067400″	149°58′32.993200″			

TABLE 1: Points where the Great Elliptic Arc cuts meridians of A, B, C, etc at 0°15′ intervals of longitude along Border Line. N = Normal Section, Lox = Loxodrome

```
>> help loxodrome_inverse

  loxodrome_inverse:  This function computes the inverse case for a
    loxodrome on the reference ellipsoid.  That is, given the latitudes and
    longitudes of two points on the ellipsoid, compute the azimuth and the
    arc length of the loxodrome on the surface.

>> loxodrome_inverse

=======================
Loxodrome: Inverse Case
=======================
Ellipsoid parameters
a  = 6378137.0000
f  = 1/298.257222101

Terminal points of curve
Latitude  P1 =  -36 47 49.223200 (D M S)
Longitude P1 =  148 11 48.333300 (D M S)

Latitude  P2 =  -37 30 18.067400 (D M S)
Longitude P2 =  149 58 32.993200 (D M S)

isometric lat  P1 =  -39 23 36.268670 (D M S)
isometric lat  P2 =  -40 16 40.540366 (D M S)

diff isometric lat  P2-P1 =   -0 53  4.271697 (D M S)
diff in longitude P2-P1   =    1 46 44.659900 (D M S)

meridian distance P1 =  -4073983.614420
meridian distance P2 =  -4152559.155874

diff in mdist P2-P1  =    -78575.541454

Azimuth of loxodrome P1-P2
Az12  = 116 26  8.400701  (D M S)

loxodrome distance P1-P2
s =    176497.829952

>>
```

MATLAB function *loxodrome_inverse.m*

```
function loxodrome_inverse
%
% loxodrome_inverse:  This function computes the inverse case for a
%   loxodrome on the reference ellipsoid.  That is, given the latitudes and
%   longitudes of two points on the ellipsoid, compute the azimuth and the
%   arc length of the loxodrome on the surface.

%--------------------------------------------------------------------------
% Function:  loxodrome_inverse()
%
% Usage:     loxodrome_inverse
%
% Author:    R.E.Deakin,
%            School of Mathematical & Geospatial Sciences, RMIT University
%            GPO Box 2476V, MELBOURNE, VIC 3001, AUSTRALIA.
%            email: rod.deakin@rmit.edu.au
%            Version  1.0  5 October 2009
%            Version  1.1 11 January 2010
%
% Purpose:   This function computes the inverse case for a loxodrome on the
%   reference ellipsoid.  That is, given the latitudes and longitudes of
%   two points on the ellipsoid, compute the azimuth and the arc length of
%   the loxodrome on the surface.
%
% Functions required:
%  [D,M,S] = DMS(DecDeg)
%   isolat = isometric(flat,lat)
%    mdist = meridian_dist(a,flat,lat)
%
% Variables:
%  Az12     - azimuth of loxodrome P1-P2 (radians)
%  a        - semi-major axis of spheroid
%  d2r      - degree to radian conversion factor 57.29577951...
%  disolat  - difference in isometric latitudes (isolat2-isolat1)
%  dlon     - difference in longitudes (radian)
%  dm       - difference in meridian distances (dm = m2-m1)
%  e        - eccentricity of ellipsoid
%  e2       - eccentricity of ellipsoid squared
%  f        - f = 1/flat is the flattening of ellipsoid
%  flat     - denominator of flattening of ellipsoid
%  isolat1  - isometric latitude of P1 (radians)
%  isolat2  - isometric latitude of P2 (radians)
%  lat1     - latitude of P1 (radians)
%  lat2     - latitude of P2 (radians)
%  lon1     - longitude of P1 (radians)
%  lon2     - longitude of P2 (radians)
%  lox_s    - distance along loxodrome
%  m1,m2    - meridian distances of P1 and P2 (metres)
%  pion2    - pi/2
%
% Remarks:
%
% References:
%  [1] Deakin, R.E., 2010, 'The Loxodrome on an Ellipsoid', Lecture Notes,
%         School of Mathematical and Geospatial Sciences, RMIT University,
%         January 2010
%  [2] Bowring, B.R., 1985, 'The geometry of the loxodrome on the
%         ellipsoid', The Canadian Surveyor, Vol. 39, No. 3, Autumn 1985,
%         pp.223-230.
%  [3] Snyder, J.P., 1987, Map Projections-A Working Manual.  U.S.
%         Geological Survey Professional Paper 1395.  Washington, DC: U.S.
%         Government Printing Office, pp.15-16 and pp. 44-45.
%  [4] Thomas, P.D., 1952, Conformal Projections in Geodesy and
%         Cartography, Special Publication No. 251, Coast and Geodetic
%         Survey, U.S. Department of Commerce, Washington, DC: U.S.
%         Government Printing Office, p. 66.
```

```
%
%------------------------------------------------------------------------------

% Degree to radian conversion factor
d2r   = 180/pi;

% Set ellipsoid parameters
a    = 6378137;    % GRS80
flat = 298.257222101;

% Set lat and long of P1 and P2 on ellipsoid
lat1 = -(36  + 47/60 + 49.2232/3600)/d2r;   % Spring
lon1 =  (148 + 11/60 + 48.3333/3600)/d2r;
lat2 = -(37  + 30/60 + 18.0674/3600)/d2r;    % Wauka 1978
lon2 =  (149 + 58/60 + 32.9932/3600)/d2r;

% Compute isometric latitude of P1 and P2
isolat1 = isometric(flat,lat1);
isolat2 = isometric(flat,lat2);

% Compute changes in isometric latitude and longitude between P1 and P2
disolat = isolat2-isolat1;
dlon = lon2-lon1;

% Compute azimuth
Az12 = atan2(dlon,disolat);

% Compute distance along loxodromic curve
m1 = meridian_dist(a,flat,lat1);
m2 = meridian_dist(a,flat,lat2);
dm = m2-m1;
lox_s  = dm/cos(Az12);

%-----------------------
% Print result to screen
%-----------------------
fprintf('\n======================');
fprintf('\nLoxodrome: Inverse Case');
fprintf('\n======================');
fprintf('\nEllipsoid parameters');
fprintf('\na  = %12.4f',a);
fprintf('\nf  = 1/%13.9f',flat);

fprintf('\n\nTerminal points of curve');
% Print lat and lon of Point 1
[D,M,S] = DMS(lat1*d2r);
if D == 0 && lat1 < 0
    fprintf('\nLatitude  P1 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLatitude  P1 = %4d %2d %9.6f (D M S)',D,M,S);
end
[D,M,S] = DMS(lon1*d2r);
if D == 0 && lon1 < 0
    fprintf('\nLongitude P1 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLongitude P1 = %4d %2d %9.6f (D M S)',D,M,S);
end

% Print lat and lon of point 2
[D,M,S] = DMS(lat2*d2r);
if D == 0 && lat1 < 0
    fprintf('\n\nLatitude  P2 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\n\nLatitude  P2 = %4d %2d %9.6f (D M S)',D,M,S);
end
[D,M,S] = DMS(lon2*d2r);
if D == 0 && lon2 < 0
    fprintf('\nLongitude P2 =   -0 %2d %9.6f (D M S)',M,S);
else
```

```
    fprintf('\nLongitude P2 = %4d %2d %9.6f (D M S)',D,M,S);
end

% Print isometric latitudes of P1 and P2
[D,M,S] = DMS(isolat1*d2r);
if D == 0 && isolat1 < 0
    fprintf('\n\nisometric lat  P1 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\n\nisometric lat  P1 = %4d %2d %9.6f (D M S)',D,M,S);
end
[D,M,S] = DMS(isolat2*d2r);
if D == 0 && isolat2 < 0
    fprintf('\nisometric lat  P2 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nisometric lat  P2 = %4d %2d %9.6f (D M S)',D,M,S);
end

% Print differences in isometric latitudes and longitudes
[D,M,S] = DMS(disolat*d2r);
if D == 0 && disolat < 0
    fprintf('\n\ndiff isometric lat  P2-P1 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\ndiff isometric lat  P2-P1 = %4d %2d %9.6f (D M S)',D,M,S);
end
[D,M,S] = DMS(dlon*d2r);
if D == 0 && dlon < 0
    fprintf('\ndiff in longitude P2-P1   =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\ndiff in longitude P2-P1   = %4d %2d %9.6f (D M S)',D,M,S);
end

% Print meridian distances of P1 and P2
fprintf('\n\nmeridian distance P1 =  %15.6f',m1);
fprintf('\nmeridian distance P2 =  %15.6f',m2);
fprintf('\n\ndiff in mdist P2-P1  =  %15.6f',dm);

% Print azimuth of loxodrome
fprintf('\n\nAzimuth of loxodrome P1-P2');
[D,M,S] = DMS(Az12*d2r);
fprintf('\nAz12  = %3d %2d %9.6f  (D M S)',D,M,S);

% Print loxodrome distance P1-P2
fprintf('\n\nloxodrome distance P1-P2');
fprintf('\ns =  %15.6f',lox_s);

fprintf('\n\n');
```

MATLAB function *isometric.m*

```
function isolat = isometric(flat,lat)
%
% isolat=isometric(flat,lat)  Function computes the isometric latitude
%   (isolat) of a point whose latitude (lat) is given on an ellipsoid whose
%   denominator of flattening is flat.
%   Latitude (lat) must be in radians and the returned value of isometric
%   latitude (isolat) will also be in radians.
%   Example: isolat = isometric(298.257222101,-0.659895044028705);
%            should return isolat = -0.709660227088983 radians,
%            equal to -40 39 37.9292417795658 (DMS) for latitude equal to
%            -0.659895044028705 radians (-37 48 33.1234 (DMS)) on the GRS80
%            ellipsoid.

%--------------------------------------------------------------------------
% Function:  isometric(flat,lat)
%
% Syntax:    isolat = isometric(flat,lat);
%
% Author:    R.E.Deakin,
%            School of Mathematical & Geospatial Sciences, RMIT University
%            GPO Box 2476V, MELBOURNE, VIC 3001, AUSTRALIA.
%            email: rod.deakin@rmit.edu.au
%            Version  1.0  5 October 2009
%
% Purpose:   Function computes the isometric latitude of a point whose
%  latitude is given on an ellipsoid defined by semi-major axis (a) and
%  denominator of flattening (flat).
%
% Return value: Function isometric() returns isolat (isometric latitude in
%  radians)
%
% Variables:
%  e      - eccentricity of ellipsoid
%  e2     - eccentricity-squared
%  f      - flattening of ellipsoid
%  flat   - denominator of flattening
%
% Remarks:
%  Isometric latitude is an auxiliary latitude proportional to the spacing
%  of parallels of latitude on an ellipsoidal Mercator projection.
%
% References:
%  [1] Snyder, J.P., 1987, Map Projections-A Working Manual.  U.S.
%         Geological SurveyProfessional Paper 1395.  Washington, DC: U.S.
%         Government Printing Office, pp.15-16.
%
%
%--------------------------------------------------------------------------

% compute flattening f eccentricity squared e2
f   = 1/flat;
e2 = f*(2-f);
e  = sqrt(e2);

x = e*sin(lat);
y = (1-x)/(1+x);
z = pi/4 + lat/2;

% calculate the isometric latitude
isolat = log(tan(z)*(y^(e/2)));
```

MATLAB function *meridian_ dist.m*

```
function mdist = meridian_dist(a,flat,lat)
%
% mdist = meridian_dist(a,flat,lat) Function computes the meridian distance
%   on an ellipsoid defined by semi-major axis (a) and denominator of
%   flattening (flat) from the equator to a point having latitude (lat) in
%   radians.
%   e.g. mdist = (6378137, 298.257222101, -0.659895044028705) will compute
%   the meridian distance for a point having latitude -37 deg 48 min
%   33.1234 sec on the GRS80 ellipsoid (a = 6378137, f = 1/298.257222101).

%--------------------------------------------------------------------------
% Function:  meridian_dist()
%
% Usage:     mdist = meridian_dist(a,flat,lat)
%
% Author:    R.E.Deakin,
%            School of Mathematical & Geospatial Sciences, RMIT University
%            GPO Box 2476V, MELBOURNE, VIC 3001, AUSTRALIA.
%            email: rod.deakin@rmit.edu.au
%            Version  1.0  5 October 2009
%
% Purpose:   Function computes the meridian distance
%   on an ellipsoid defined by semi-major axis (a) and denominator of
%   flattening (flat) from the equator to a point having latitude (lat) in
%   radians.
%
% Functions required:
%
% Variables: a         - semi-major axis of spheroid
%            A,B,C...  - coefficients
%            e2        - eccentricity squared
%            e4,e6,... - powers of e2
%            f         - f = 1/flat is the flattening of ellipsoid
%            flat      - denominator of flattening of ellipsoid
%            mdist     - meridian distance
%
% Remarks:   The formulae used are given in Baeschlin, C.F., 1948,
%            "Lehrbuch Der Geodasie", Orell Fussli Verlag, Zurich, pp.47-50.
%            See also Deakin, R. E. and Hunter M. N., 2008, "Geometric
%            Geodesy - Part A", Lecture Notes, School of Mathematical and
%            geospatial Sciences, RMIT University, March 2008, pp. 60-65.
%
%--------------------------------------------------------------------------

% compute eccentricity squared
f  = 1/flat;
e2 = f*(2-f);

% powers of eccentricity
e4  = e2*e2;
e6  = e4*e2;
e8  = e6*e2;
e10 = e8*e2;

% coefficients of series expansion for meridian distance
A = 1+(3/4)*e2+(45/64)*e4+(175/256)*e6+(11025/16384)*e8+(43659/65536)*e10;
B = (3/4)*e2+(15/16)*e4+(525/512)*e6+(2205/2048)*e8+(72765/65536)*e10;
C = (15/64)*e4+(105/256)*e6+(2205/4096)*e8+(10395/16384)*e10;
D = (35/512)*e6+(315/2048)*e8+(31185/131072)*e10;
E = (315/16384)*e8+(3465/65536)*e10;
F = (693/131072)*e10;

term1 = A*lat;
term2 = (B/2)*sin(2*lat);
term3 = (C/4)*sin(4*lat);
term4 = (D/6)*sin(6*lat);
```

```
term5 = (E/8)*sin(8*lat);
term6 = (F/10)*sin(10*lat);

mdist = a*(1-e2)*(term1-term2+term3-term4+term5-term6);
```

MATLAB function *DMS.m*

```
function [D,M,S] = DMS(DecDeg)
% [D,M,S] = DMS(DecDeg)  This function takes an angle in decimal degrees and returns
%   Degrees, Minutes and Seconds

val = abs(DecDeg);
D = fix(val);
M = fix((val-D)*60);
S = (val-D-M/60)*3600;
if(DecDeg<0)
  D = -D;
end
return
```

REFERENCES

Bowring, B. R., (1985), 'The Geometry of the Loxodrome on the Ellipsoid', *The Canadian Surveyor,* Vol. 39, No. 3, Autumn 1985, pp. 223-230.

Bomford, G., (1980), *Geodesy,* 4th edition, Clarendon Press, Oxford.

Deakin, R. E. and Hunter, M. N., (2008), 'Geometric Geodesy - Part A', Lecture Notes, School of Mathematical & Geospatial Sciences, RMIT University, Melbourne, Australia, March 2008, 140 pages.

Lauf, G. B., (1983), *Geodesy and Map Projections,* TAFE Publications Unit, Collingwood, Vic, Australia.

Thomas, P. D., (1952), *Conformal Projections in Geodesy and Cartography,* Special Publication No. 251, Coast and Geodetic Survey, United States Department of Commerce, Washington, D.C.