RELATIONSHIP BETWEEN ASTRONOMIC COORDINATES ϕ_A, λ_A, H AND GEODETIC COORDINATES ϕ_G, λ_G, h

In geodesy it is important to know the relationships between observed quantities such as horizontal directions (or azimuths) and zenith distances - both of which are related to the astronomic meridian and the *vertical* (the normal to the equipotential surface at the place of observation) - and the complementary values related to the geodetic meridian and the *normal* to the ellipsoid. These relationships can be used to reduce observations made between points on the terrestrial surface to quasi-observations between corresponding points on the ellipsoid. These relationships are often given in a form known as *Laplace's* equation[1].

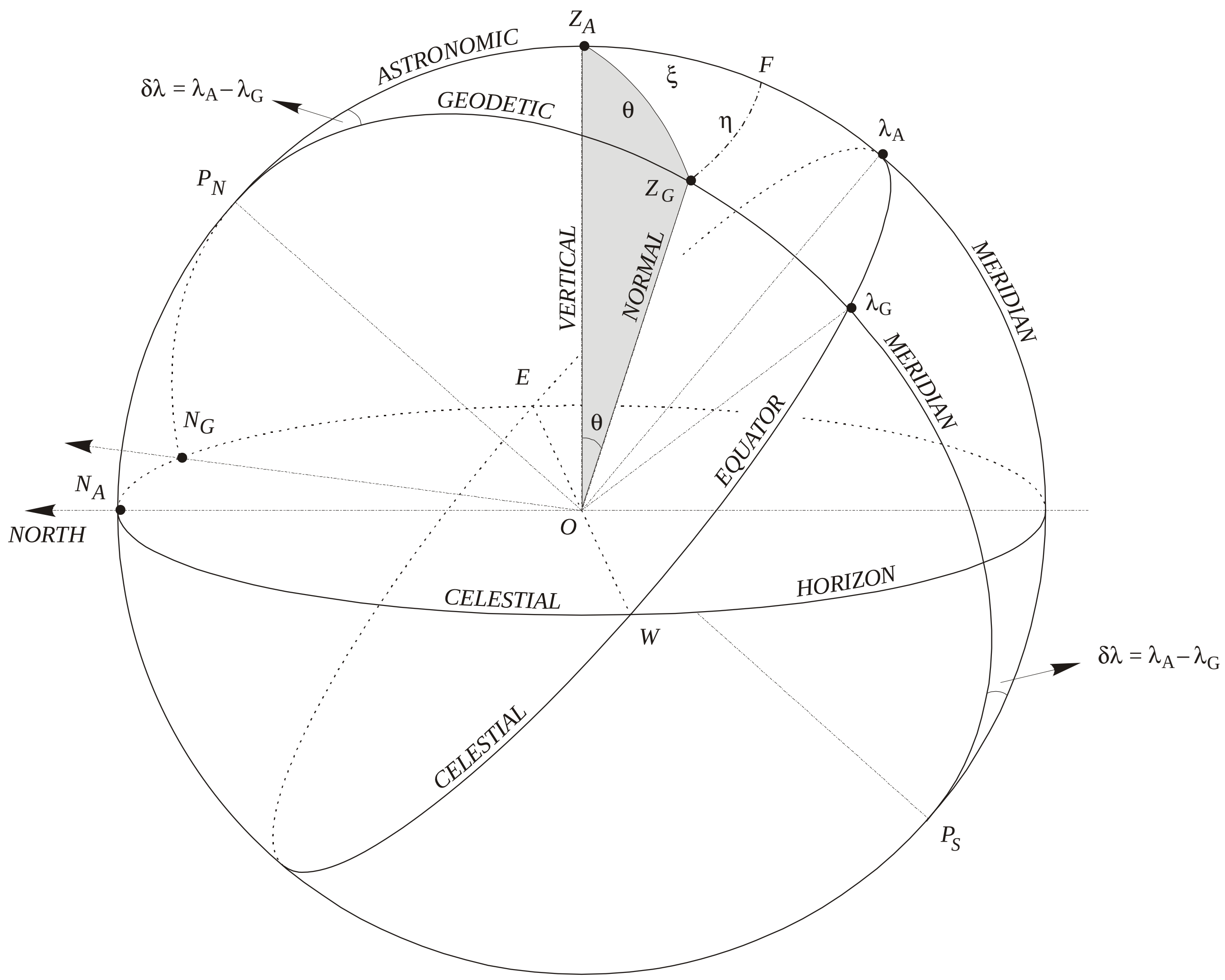


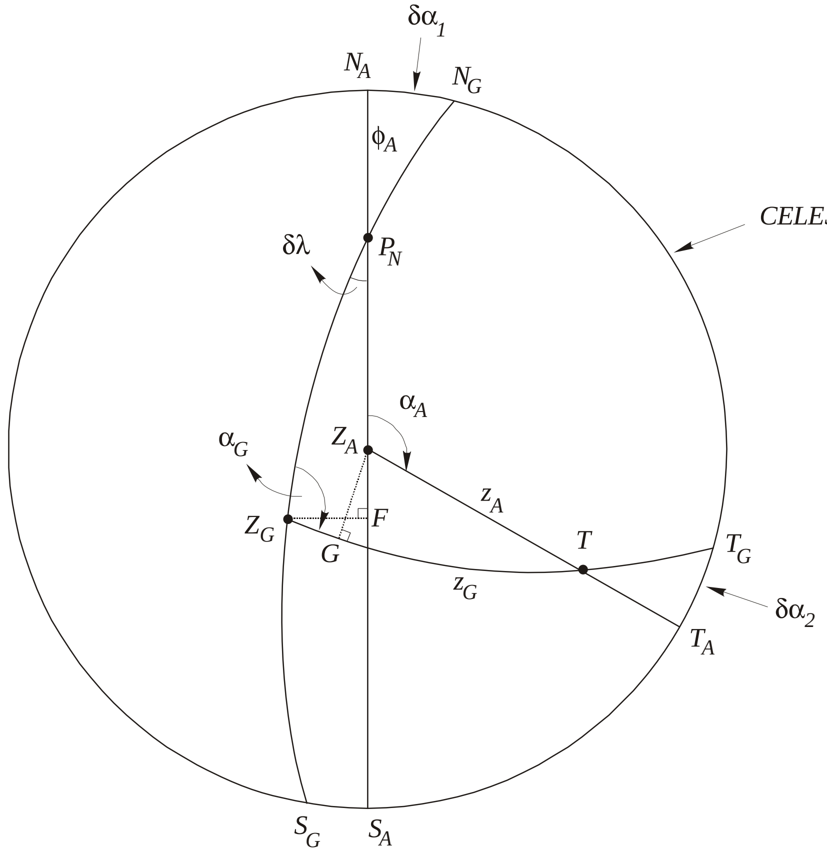
Figure 1 The celestial sphere for an observer at O on the Earth's surface

[1]Pierre Simon, Marquis de Laplace (1749-1827), distinguished French mathematician. Best known for his equation $\nabla^2 V = 0$ (where V is the gravitational potential) published in his great work *Mécanique Céleste*. Other equations in geodesy also bear his name.

Figure 1 shows the celestial sphere for an observer on Earth at *O*. P_N and P_S are the projections of north and south poles of the Earth on the celestial sphere. The polar axis (rotational axis of the Earth) is also the axis of the ellipsoid (an ellipse of revolution). The plane tangential to the equipotential surface at *O* intersects the celestial sphere in a line known as the celestial horizon.

The tangent to the *vertical* at *O* is perpendicular to the horizon plane and intersects the celestial sphere at the astronomic zenith Z_A (subscript $_A$ referring to astronomic), and the astronomic meridian plane passing through Z_A intersects the horizon at N_A - the reference point for astronomic azimuth α_A. The *normal* to the ellipsoid which passes through the observer at *O* intersects the celestial sphere at Z_G (subscript $_G$ referring to geodetic), and the geodetic meridian plane passing through Z_G intersects the horizon at N_G - the reference point for geodetic azimuth α_G.

In general, the vertical and the normal at *O* do not coincide but diverge by a small angle θ known as the deflection of the vertical whose components are ξ (Xi) in the plane of the astronomic meridian, and η (Eta) along a great circle perpendicular to the astronomic meridian and passing through Z_G. Expressions for ξ and η can be obtained from spherical trigonometry and Figure 2.

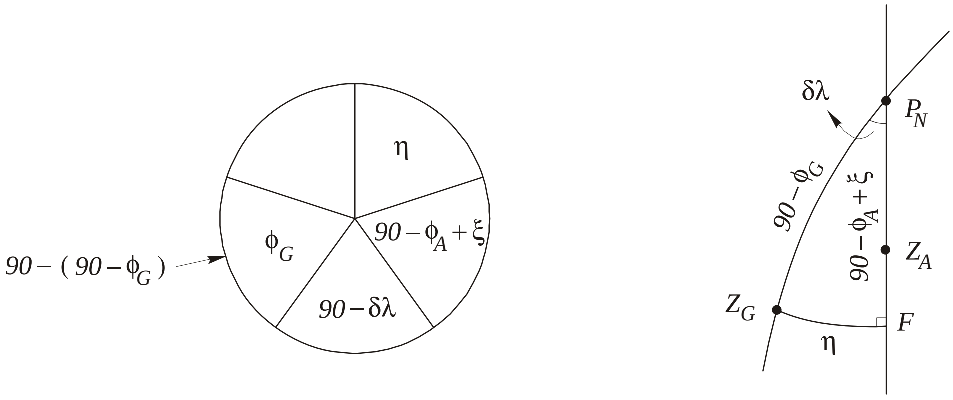


$$\eta = Z_G F$$
$$\xi = Z_A F$$
$$\beta = Z_A G$$
$$\delta\lambda = \lambda_A - \lambda_G$$

Figure 2 The celestial sphere as viewed from above.

In Figure 2 the astronomic and geodetic meridians are the great circles $N_A P_N Z_A S_A$ and $N_G P_N Z_G S_G$ respectively. T is an elevated target whose astronomic azimuth α_A is the angle between the astronomic meridian plane and the great circle plane $Z_A T \; T_A$. The geodetic azimuth α_G is the angle between the geodetic meridian plane and the great circle plane $Z_G T \; T_G$. The astronomic and geodetic zenith distances of *T* are z_A and z_G respectively.

Using Napier's Rules of Circular Parts in the right-angled spherical triangle $P_N\ F\ Z_G$



gives

$$\sin\eta \;=\; \cos(90-\delta\lambda)\,\cos\phi_G \tag{i}$$

and

$$\sin\phi_G \;=\; \cos\eta\,\cos(90-\phi_A+\xi) \tag{ii}$$

Now, $\cos(90-x)=\sin x$ and $\sin\eta\approx\eta$ and $\sin\delta\lambda\approx\delta\lambda=\lambda_A-\lambda_G$ since η and $\delta\lambda$ are small, hence equation (i) becomes

$$\sin\eta \;=\; \sin\delta\lambda\,\cos\phi_G$$

$$\eta \;\approx\; (\lambda_A-\lambda_G)\cos\phi_G \tag{1}$$

and equation (ii) becomes

$$\sin\phi_G \;=\; \cos\eta\,\sin(\phi_A-\xi)$$

$$\sin\phi_G \;\approx\; \sin(\phi_A-\xi)$$

and taking the sine of both sides gives

$$\xi \;=\; \phi_A-\phi_G \tag{2}$$

The difference in azimuth is

$$\delta\alpha \;=\; \alpha_A-\alpha_G \tag{3}$$

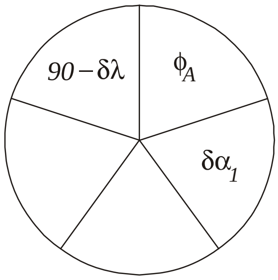
and from Figure 2

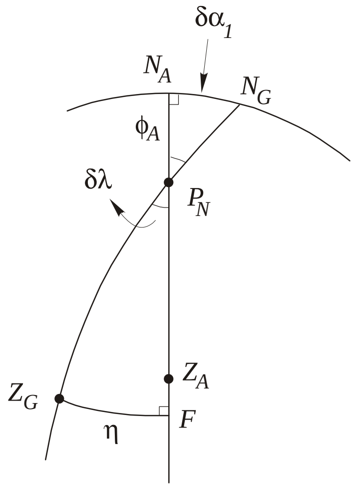
$$\alpha_A-(\delta\alpha_1+\delta\alpha_2) \;=\; \alpha_G$$

hence

$$\delta\alpha \;=\; \delta\alpha_1+\delta\alpha_2 \tag{4}$$

Now $\delta\alpha_1$ can be calculated from the right-angled spherical triangle $P_N\, N_A\, N_G$ and using Napier's Rules for Circular Parts





gives

$$\sin\phi_A = \tan\delta\alpha_1 \tan(90-\delta\lambda)$$

giving

$$\tan\delta\alpha_1 = \tan\delta\lambda \sin\phi_A$$

and since $\delta\alpha$ and $\delta\lambda$ are small, then $\tan\delta\alpha_1 \approx \delta\alpha_1$ and $\tan\delta\lambda \approx \delta\lambda = (\lambda_A - \lambda_G)$

$$\delta\alpha_1 \approx (\lambda_A - \lambda_G)\sin\phi_A \tag{5}$$

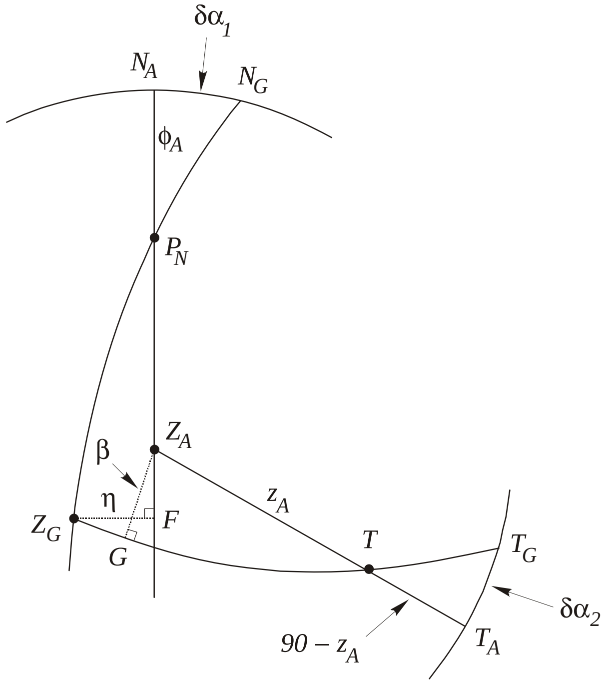
Re-arranging equation (1) and substituting into equation (5) we have

$$\delta\alpha_1 = \frac{\eta}{\cos\phi_G}\sin\phi_A$$

and since $\xi = \phi_A - \phi_G$ is a small quantity then $\cos\phi_G \approx \cos\phi_A$ and we may write

$$\delta\alpha_1 \approx \eta \tan\phi_A \tag{6}$$

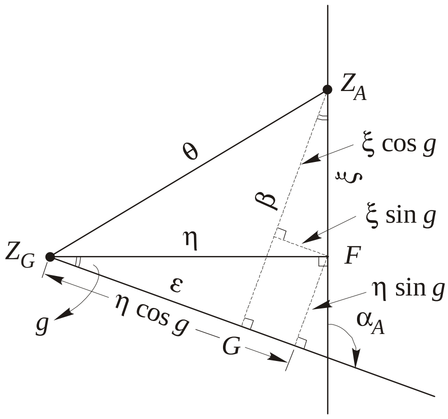
Now the figure $P_N\,N_A\,N_G - P_N\,F\,Z_G$ is similar to $T\,T_G\,T_A - T\,G\,Z_A$



and using similar reasoning to above we may write

$$\delta\alpha_2 \approx \beta \tan(90 - z_A) = \beta \cot z_A \qquad (7)$$

and since ξ and η are very small, the figure below (extracted from Figure 2) can be considered plane



$$g = \alpha_A - 90°$$
$$\sin g = -\cos\alpha_A$$
$$\cos g = \sin\alpha_A$$
$$Z_G F = \eta$$
$$Z_A F = \xi$$
$$Z_A G = \beta$$
$$Z_G G = \varepsilon$$

$$\beta = \xi \cos g + \eta \sin g$$
$$\beta = \xi \sin\alpha_A - \eta\cos\alpha_A \qquad (8)$$
$$\varepsilon = -\xi \sin g + \eta \cos g$$
$$\varepsilon = \xi \cos\alpha_A + \eta \sin\alpha_A \qquad (9)$$

Substituting equations (6) and (7) into equation (4) gives

$$\delta\alpha = \delta\alpha_1 + \delta\alpha_2 = \eta \tan\phi_A + \beta \cot z_A$$

and substituting equation (8) gives

$$\delta\alpha = \delta\alpha_1 + \delta\alpha_2 = \eta \tan\phi_A + (\xi \sin\alpha_A - \eta\cos\alpha_A)\cot z_A \qquad (10)$$

Equation (10) is often expressed as

$$\delta\alpha = +\xi \sin \alpha_A \cot z_A + \eta \left(\tan \phi_A - \cos \alpha_A \cot z_A\right) \tag{11}$$

Equations (10) and (11) give an expression for the difference between *astronomic* and *geodetic* azimuth

Using similar reasoning and the diagram above we have $z_A + \varepsilon = z_G$ and hence an expression for the difference between *astronomic* and *geodetic* zenith distance is given by

$$\delta z = z_A - z_G = -\varepsilon = -\xi \cos \alpha_A - \eta \sin \alpha_A \tag{12}$$

Now from Figure 2, when z_A is approximately 90°, as it often is in geodetic survey work, then $\delta\alpha_2 \approx 0$ and the correction for the elevation of the target *T* can be ignored. This is the second term in equation (10), and as a consequence, the correction in azimuth $\delta\alpha$ is entirely due to the elevation of the pole P_N . Hence a common form of the azimuth correction is written as

$$\delta\alpha = \alpha_A - \alpha_G = \eta \tan \phi_A \tag{13}$$

Now from equation (1) $\eta = (\lambda_A - \lambda_G) \cos \phi_G$ which when substituted into equation (13) gives

$$\delta\alpha = \alpha_A - \alpha_G = \left(\lambda_A - \lambda_G\right) \cos \phi_G \frac{\sin \phi_A}{\cos \phi_A}$$

and since ξ is a small quantity then $\cos \phi_G \approx \cos \phi_A$ and we may write

$$\delta\alpha = \alpha_A - \alpha_G \approx \left(\lambda_A - \lambda_G\right) \sin \phi_A \tag{14}$$

Equation (14) is known as *Laplace's equation* for azimuths.

CONNECTION BETWEEN ASTRONOMIC AND GEODETIC COORDINATES

From equations (1) and (2) the following relationships between astronomic and geodetic latitude ϕ_A and ϕ_G respectively and astronomic and geodetic longitude λ_A and λ_G respectively are

$$\phi_G = \phi_A - \xi \tag{15}$$

$$\lambda_G = \lambda_A - \eta \sec \phi_G \tag{16}$$

CONNECTION BETWEEN ASTRONOMIC AND GEODETIC AZIMUTH

From equation (13) the connection between geodetic azimuth α_G and astronomic azimuth α_A is given by

$$\alpha_G = \alpha_A - \eta \tan \phi_A \tag{17}$$

CONNECTION BETWEEN ELLIPSOIDAL AND ORTHOMETRIC HEIGHTS

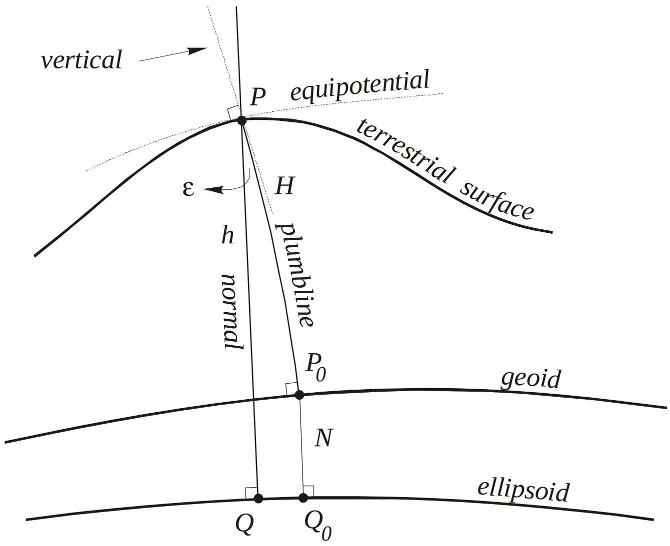


Figure 3 A sectional view of the ellipsoid, geoid and the terrestrial surface.

In Figure 3 the point *P* on the Earth's terrestrial surface is projected onto the ellipsoid at *Q* via the normal *PQ* and onto the geoid at P_0 via the curved plumbline PP_0. P_0 on the geoid is projected onto the ellipsoid at Q_0 via the normal P_0Q_0. It should be noted that the plumbline is a space curve, i.e., it has tortion and twist and in general P_0 (and Q_0) will not lie in the plane containing *P* and *Q*.

The ellipsoidal height *h* is the distance *PQ* measured along the normal. The orthometric height *H* is the distanced measured along the curved plumbline PP_0.

The geoid-ellipsoid separation *N* is the distance between the geoid and ellipsoid measured along the normal P_0Q_0 but for all practical purposes the geoid and the ellipsoid can be considered as parallel surfaces in the vicinity of P_0 and *Q* and the connection between ellipsoidal and orthometric heights is given by

$$h = H + N \tag{18}$$

DEFLECTION COMPONENT ε IN THE DIRECTION OF THE AZIMUTH α

In Figure 3, the vertical at *P* is tangential to the plumbline at *P* and perpendicular to the equipotential surface through *P*. The vertical pierces the celestial sphere at Z_A, the astronomic zenith and the normal pierces the celestial sphere at Z_G, the geodetic zenith. The angle between the two zeniths is the deflection of the vertical θ (see Figure 1) but Figure 3 is a section in a particular azimuth α and the component of the deflection, denoted by ε, is given by equation (9) as

$$\varepsilon = \xi \cos \alpha_A + \eta \sin \alpha_A \tag{19}$$

CORRECTIONS TO OBSERVED DIRECTIONS DUE TO ξ AND η

It should be noted that the correction for azimuth $\delta\alpha$ is made up of two terms $\delta\alpha_1$ and $\delta\alpha_2$ (see equation 10)). The first term, $\delta\alpha_1 = \eta \tan \phi_A$, is the same for every target, independent of its azimuth and zenith distance and results from the astronomical azimuth α_A being reckoned from astronomical north N_A rather than geodetic north N_G. It thus represents a shift of the zero point or the Reference Object in a set of observed directions.
The second term $\delta\alpha_2 = (\xi \sin \alpha_A - \eta \cos \alpha_A) \cot z_A$ depends on the azimuth and zenith distance of the particular target and arises because the target T is projected from Z_A and Z_G onto different points T_A and T_G of the horizon. It thus represents a correction to an observed direction and is analogous to the correction for an inaccurately levelled theodolite. The corrected direction is given by

$$\text{corrected direction} = \text{observed direction} + \left\{ \frac{-\xi \sin \alpha_A + \eta \cos \alpha_A}{\tan z_A} \right\} \tag{20}$$

where the corrected direction relates to a theodolite whose rotational axis is coincident with the normal to the ellipsoid. The observed direction relates to a theodolite whose rotational axis is coincident with the vertical.

CORRECTIONS TO OBSERVED ZENITH DISTANCES DUE TO ξ AND η

The correction to an observed zenith distance z_A due to deflection components ξ and η is given by equation (12) as

$$\delta z = z_A - z_G = -\varepsilon = -\xi \cos \alpha_A - \eta \sin \alpha_A$$

and the corrected zenith distance z_G is given by

$$\text{corrected zenith distance} = \text{observed zenith distance} + (\xi \cos \alpha_A + \eta \sin \alpha_A) \tag{21}$$

where the corrected zenith distance relates to a theodolite whose rotational axis is coincident with the normal to the ellipsoid. The observed zenith distance relates to a theodolite whose rotational axis is coincident with the vertical.

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