THE LOXODROME ON AN ELLIPSOID

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ABSTRACT

These notes provide a detailed explanation of the geometry of the loxodrome on the ellipsoid. Equations are derived for azimuth and distance of a loxodrome between two points on an ellipsoid and these equations enable the development of algorithms for the solution of the direct and inverse problems of the loxodrome. A MATLAB function is provided that demonstrates an algorithm for the inverse problem.

INTRODUCTION

The loxodrome between P_1 and P_2 on the ellipsoid is a curved line such that every element of the curve ds intersects a meridian at a constant azimuth α. Unless $\alpha = 0^\circ, 90^\circ, 180^\circ$ or 270° the loxodrome will spiral around the ellipsoid and terminate at one of the poles. In other cases the loxodrome will lie along a meridian of longitude $\left(\alpha = 0^\circ, 180^\circ\right)$ or a parallel of latitude $\left(\alpha = 90^\circ, 270^\circ\right)$.

Figure 1: Loxodrome on the earth's surface

In marine and air navigation, aircraft and ships sailing or flying on fixed compass headings are moving along loxodromes, hence knowledge of loxodromes is important in navigation. Mercator's projection – a normal aspect cylindrical conformal projection – has the unique property that loxodromes on the earth's surface are projected as straight lines on the map.

In geodesy the direct problem (computing position given azimuth and distance from a known location) and the inverse problem (computing azimuth and distance between known positions) are fundamental operations and can be likened to the equivalent operations of plane surveying; radiations (computing coordinates of points given bearings and distances radiating from a point of known coordinates) and joins; (computing bearings and distances between points having known coordinates). The direct and inverse problems in geodesy are usually associated with the geodesic which is the unique curve defining the shortest path on the ellipsoid but they can also be associated with other curves. So;

> The direct problem of the loxodrome on the ellipsoid is: given latitude and longitude of P_1 and the azimuth α and distance s of a loxodrome between P_1 and P_2; compute the latitude and longitude of P_2.
>
> The inverse problem of the loxodrome on the ellipsoid is: given the latitude and longitude of P_1 and P_2; compute the azimuth α and distance s of the loxodrome between P_1 and P_2.

The equations necessary for the solution of the direct and inverse problems are derived from the differential geometry of the ellipsoid and in particular, relationships that can be obtained from the differential rectangle on the ellipsoid. Also, meridian distance (the distance along a meridian from the equator) is used in computing loxodrome distances. Discussions of differential geometry of the ellipsoid and meridian distance can be found in Deakin & Hunter (2008) or geodesy textbooks (e.g., Lauf 1983; Bomford 1980), and an excellent treatment of the loxodrome on the ellipsoid can be found in Bowring (1985).

THE ELLIPSOID

In geodesy, the ellipsoid is a surface of revolution created by rotating an ellipse about its minor axis. The size and shape of an ellipsoid is defined by one of three pairs of parameters: (i) a, b where a and b are the semi-major and semi-minor axes lengths of an ellipsoid respectively (and $a > b$), or (ii) a, f where f is the flattening of an ellipsoid, or (iii) a, e^2 where e^2 is the square of the first eccentricity of an ellipsoid.

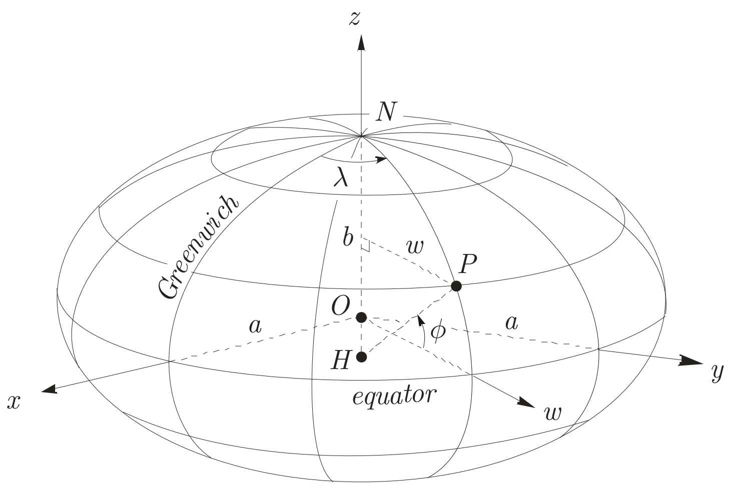


Figure 2: The reference ellipsoid

The ellipsoid parameters a, b, f, e^2 are related by the following equations

$$f = \frac{a-b}{a} = 1 - \frac{b}{a} \tag{1}$$

$$b = a(1-f) \tag{2}$$

$$e^2 = \frac{a^2 - b^2}{a^2} = 1 - \frac{b^2}{a^2} = f(2-f) \tag{3}$$

$$1 - e^2 = \frac{b^2}{a^2} = 1 - f(2-f) = (1-f)^2 \tag{4}$$

The <u>second eccentricity</u> e' of an ellipsoid is also of use and

$$e'^2 = \frac{a^2 - b^2}{b^2} = \frac{a^2}{b^2} - 1 = \frac{e^2}{1-e^2} = \frac{f(2-f)}{(1-f)^2} \tag{5}$$

$$e^2 = \frac{e'^2}{1+e'^2} \tag{6}$$

In Figure 2 the normal to the surface at P intersects the rotational axis of the ellipsoid (the z-axis) at H making an angle ϕ with the equatorial plane of the ellipsoid – this is the latitude of P. The longitude λ is the angle between the Greenwich meridian plane (a reference plane) and the meridian plane (the z-w plane) containing the normal through P. ϕ and λ are <u>curvilinear coordinates</u> and meridians of longitude (curves of constant λ) and parallels of latitude (curves of constant ϕ) are parametric curves on the ellipsoidal surface. At P on the surface of the ellipsoid, planes containing the normal to the ellipsoid intersect the surface creating elliptical sections known as <u>normal sections</u>. Amongst the infinite number of possible normal sections at a *P;* each having a certain radius of curvature, two are of interest: (i) the meridian section, containing the axis of revolution of the ellipsoid and having the least radius of curvature, denoted by ρ, and (ii) the prime vertical section, perpendicular to the meridian plane and having the greatest radius of curvature, denoted by ν.

$$\rho = \frac{a\left(1-e^2\right)}{\left(1-e^2\sin^2\phi\right)^{\frac{3}{2}}} = \frac{a\left(1-e^2\right)}{W^3} \tag{7}$$

$$\nu = \frac{a}{\left(1-e^2\sin^2\phi\right)^{\frac{1}{2}}} = \frac{a}{W} \tag{8}$$

$$W^2 = 1-e^2\sin^2\phi \tag{9}$$

For P, the centre of the radius of curvature of the prime vertical section is at H and $\nu = PH$. The centre of the radius of curvature of the meridian section lies on the normal between P and H.

Alternative equations for the radii of curvature ρ and ν are given by

$$\rho = \frac{a^2}{b\left(1+e'^2\cos^2\phi\right)^{\frac{3}{2}}} = \frac{c}{V^3} \tag{10}$$

$$\nu = \frac{a^2}{b\left(1+e'^2\cos^2\phi\right)^{\frac{1}{2}}} = \frac{c}{V} \tag{11}$$

$$c = \frac{a^2}{b} = \frac{a}{1-f} \tag{12}$$

$$V^2 = 1+e'^2\cos^2\phi \tag{13}$$

and c is the polar radius of curvature of the ellipsoid.
The latitude functions W and V are related as follows

$$W^2 = \frac{V^2}{1+e'^2} \quad \text{and} \quad W = \frac{V}{\left(1+e'^2\right)^{\frac{1}{2}}} = \frac{b}{a}V \tag{14}$$

Points on the ellipsoid surface have curvilinear coordinates ϕ, λ and Cartesian coordinates x,y,z where the x-z plane is the Greenwich meridian plane, the x-y plane is the equatorial plane and the y-z plane is a meridian plane 90° east of the Greenwich meridian plane. Cartesian and curvilinear coordinates are related by

$$\begin{aligned} x &= \nu\cos\phi\cos\lambda \\ y &= \nu\cos\phi\cos\lambda \\ z &= \nu\left(1-e^2\right)\sin\phi \end{aligned} \tag{15}$$

Note that $\nu\left(1-e^2\right)$ is the distance along the normal from a point on the surface to the point where the normal cuts the equatorial plane.

DIFFERENTIAL RELATIONSHIPS FOR THE LOXODROME ON THE ELLIPSOID

The derivation of equations relating to the loxodrome requires an understanding of the connection between differentially small quantities on the surface of the ellipsoid.

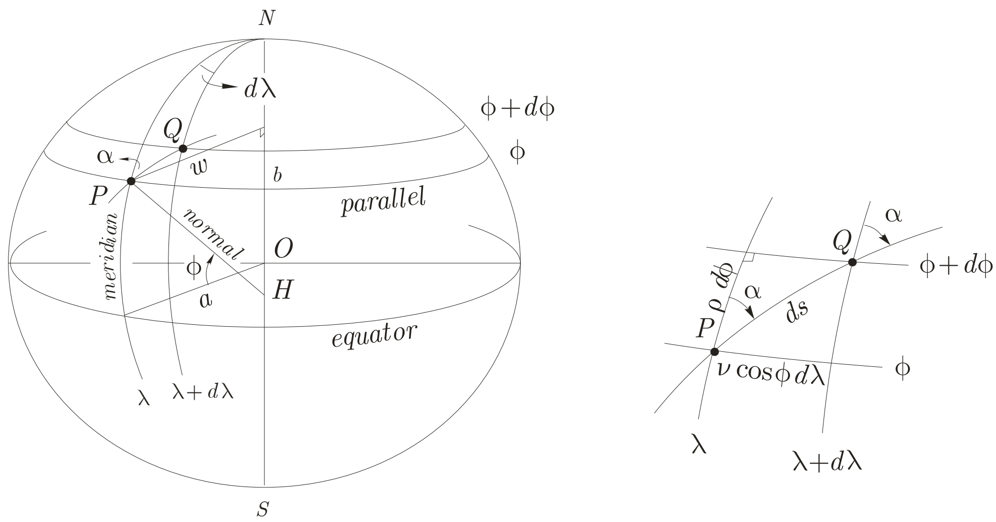


Figure 3: The differential rectangle on an ellipsoid (a,b)

These relationships can be derived from the differential rectangle, with diagonal PQ in Figure 3 which shows P and Q on an ellipsoid whose semi-axes are a and b $\left(a>b\right)$. P and Q are separated by differential changes in latitude $d\phi$ and longitude $d\lambda$ and are connected by a loxodrome of length ds making an angle α (the azimuth) with the meridian through P. The meridians λ and $\lambda+d\lambda$, and the parallels ϕ and $\phi+d\phi$ form a differential rectangle on the surface of the ellipsoid. The differential distances dp along the parallel ϕ and dm along the meridian λ are

$$dp = w\,d\lambda = \nu\cos\phi\,d\lambda \tag{16}$$

$$dm = \rho\,d\phi \tag{17}$$

where ρ and ν are radii of curvature in the meridian and prime vertical planes respectively and $w=\nu\cos\phi$ is the perpendicular distance from the rotational axis *NOS*.

From Figure 3, the differential distance ds is given by

$$\begin{aligned} ds &= \sqrt{dm^2 + dp^2} \\ &= \sqrt{\rho^2 d\phi^2 + \nu^2 \cos^2\phi\, d\lambda^2} \\ &= \nu\cos\phi\sqrt{\left(\frac{\rho\, d\phi}{\nu\cos\phi}\right)^2 + d\lambda^2} \\ &= \nu\cos\phi\sqrt{dq^2 + d\lambda^2} \end{aligned} \tag{18}$$

q is known as the isometric latitude defined by the differential relationship

$$dq = \frac{\rho}{\nu\cos\phi}\, d\phi \tag{19}$$

(q, λ) is a curvilinear coordinate system on the ellipsoid with isometric parameters where isometric means of equal measure (*iso* = equal; *metric* = able to be measured). We can see this from equation (18) where the differential distances along the parametric curves q and λ are $dm = \nu\cos\phi\, dq$ and $dp = \nu\cos\phi\, d\lambda$, i.e., the differential distances are equal for equal angular differentials dq and $d\lambda$.

Also from Figure 3 the azimuth α of the loxodrome is obtained from

$$\tan\alpha = \frac{\nu\cos\phi\, d\lambda}{\rho\, d\phi} = \frac{d\lambda}{dq} \tag{20}$$

and azimuth α and distance s are linked by the differential relationship

$$ds = \frac{dm}{\cos\alpha} = \frac{1}{\cos\alpha}\rho\, d\phi \tag{21}$$

ISOMETRIC LATITUDE

The isometric latitude is defined by the differential equation (19) from which we obtain

$$q = \int \frac{\rho}{\nu\cos\phi}\, d\phi + C_1 \tag{22}$$

where C_1 is a constant of integration.

Substituting into equation (22) expressions for ρ and ν given by equations (7) and (8), and simplifying gives

$$q = \int \frac{\left(1 - e^2\right)}{\left(1 - e^2\sin^2\phi\right)\cos\phi}\, d\phi + C_1 \tag{23}$$

The integrand of equation (23) can be separated into partial fractions

$$\frac{\left(1-e^2\right)}{\left(1-e^2\sin^2\phi\right)\cos\phi} = \frac{A}{\left(1-e^2\sin^2\phi\right)} + \frac{B}{\cos\phi} \tag{24}$$

Expanding and simplifying equation (24) gives

$$\begin{aligned} 1-e^2 &= A\cos\phi + B\left(1-e^2\sin^2\phi\right) \\ &= A\cos\phi + B - Be^2\left(1-\cos^2\phi\right) \\ &= B\left(1-e^2\right) + \left(A + Be^2\cos\phi\right)\cos\phi \end{aligned} \tag{25}$$

A and B are obtained by comparing the coefficients of $1-e^2$ and $\cos\phi$ in equation (25) giving

$$B = 1; \quad A = -e^2\cos\phi$$

Substituting these results into equation (24) gives the isometric latitude as

$$q = \int \frac{1}{\cos\phi}\,d\phi \; - \int \frac{e^2\cos\phi}{1-e^2\sin^2\phi}\,d\phi + C_1 \tag{26}$$

Put $e\sin\phi = \sin u$ then $e\cos\phi\,d\phi = \cos u\,du$ and

$$\begin{aligned} q &= \int \frac{1}{\cos\phi}\,d\phi \; - e\int \frac{\cos u}{1-\sin^2 u}\,du + C_1 \\ &= \int \frac{1}{\cos\phi}\,d\phi \; - e\int \frac{\cos u}{\cos^2 u}\,du + C_1 \\ &= \int \frac{1}{\cos\phi}\,d\phi \; - e\int \frac{1}{\cos u}\,du + C_1 \end{aligned} \tag{27}$$

From standard integrals $\int \frac{1}{\cos x}\,dx = \ln\left\{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}$ and from half-angle trigonometric formula $\tan\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$ giving $\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) = \sqrt{\frac{1-\cos\left(x+\pi/2\right)}{1+\cos\left(x+\pi/2\right)}} = \sqrt{\frac{1+\sin x}{1-\sin x}}$.

Substituting these results into equation (27) gives the isometric latitude as

$$q = \ln\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right) + C_2 - e\ln\left(\frac{1+e\sin\phi}{1-e\sin\phi}\right)^{\frac{1}{2}} - C_3 + C_1$$

where C_1, C_2 and C_3 are constants of integration. Using the laws of logarithms:

$\log_a MN = \log_a M + \log_a N$, $\log_a \frac{M}{N} = \log_a M - \log_a N$ and $\log_a M^p = p\log_a M$, and defining a new constant of integration $C = C_2 - C_3 + C_1$ gives

$$
\begin{aligned}
q &= \ln\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)+\ln\left(\frac{1-e\sin\phi}{1+e\sin\phi}\right)^{\frac{e}{2}}+C \\
&= \ln\left\{\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\left(\frac{1-e\sin\phi}{1+e\sin\phi}\right)^{\frac{e}{2}}\right\}+C
\end{aligned}
\tag{28}
$$

The constant C in equation (28) equals zero since if $\phi = 0$ then $q = 0$ and the isometric latitude q is obtained from

$$
q = \ln\left\{\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\left(\frac{1-e\sin\phi}{1+e\sin\phi}\right)^{\frac{e}{2}}\right\} \tag{29}
$$

This derivation follows Lauf (1983) where an integral identical to equation (22) is evaluated as part of the derivation of the equations for the ellipsoidal Mercator projection – a conformal projection of the ellipsoid. Thomas (1952) derives a similar equation in his development of conformal representation of the ellipsoid upon a plane.

THE EQUATION OF THE LOXODROME

By re-arranging equation (20) we have

$$
d\lambda = \tan\alpha\, dq
$$

and integrating both sides, noting that $\tan\alpha$ is a constant, gives

$$
\begin{aligned}
\int_{\lambda_1}^{\lambda_2} d\lambda &= \tan\alpha \int_{q_1}^{q_2} dq \\
\lambda_2 - \lambda_1 &= \tan\alpha\left(q_2 - q_1\right)
\end{aligned}
$$

And the equation of the loxodrome between P_1 and P_2 on the ellipsoid is

$$
\Delta\lambda = \Delta q \tan\alpha \tag{30}
$$

where $\Delta\lambda = \lambda_2 - \lambda_1$ and $\Delta q = q_2 - q_1$ are differences in longitude and isometric latitude respectively and α is the (constant) azimuth of the loxodrome.

THE AZIMUTH OF A LOXODROME

The azimuth α of a loxodrome between P_1 and P_2 on an ellipsoid can be obtained from equation (30) as

$$\alpha = \arctan\left(\frac{\Delta\lambda}{\Delta q}\right) = \arctan\left(\frac{\lambda_2 - \lambda_1}{q_2 - q_1}\right) \tag{31}$$

where q_1, q_2 are isometric latitudes of P_1 and P_2 respectively and q is given by equation (29). λ_1, λ_2 are the longitudes of P_1 and P_2.

DISTANCE ALONG A LOXODROME

Consider a loxodrome of constant azimuth α that crosses the equator and passes through P_1 and P_2. The distance s between P_1 and P_2 can be defined as $s = s_2 - s_1$ where s_1 and s_2 are distances from the equator to P_1 and P_2 respectively and from equations (21) and (7) we may write

$$s_1 = \frac{1}{\cos\alpha}\int_0^{\phi_1} \rho \; d\phi = \frac{a\left(1-e^2\right)}{\cos\alpha}\int_0^{\phi_1} \frac{1}{W^3} \; d\phi = \frac{m_1}{\cos\alpha} \tag{32}$$

and similarly

$$s_2 = \frac{m_2}{\cos\alpha} \tag{33}$$

m_1 and m_2 are meridian distances and meridian distance m is defined as the length of the arc of the meridian to a point in latitude ϕ. m is obtained from the differential relationship given by equation (17) and

$$m = \int_0^{\phi} \rho \; d\phi = a\left(1-e^2\right)\int_0^{\phi}\left(1-e^2\sin^2\phi\right)^{-\frac{3}{2}} \; d\phi = a\left(1-e^2\right)\int_0^{\phi}\frac{1}{W^3} \; d\phi \tag{34}$$

This is an elliptic integral of the second kind and cannot be evaluated directly; instead, the integrand $\dfrac{1}{W^3} = \left(1-e^2\sin^2\phi\right)^{-\frac{3}{2}}$ is expanded by using the binomial series and then evaluated by term-by-term integration. Following Deakin & Hunter (2008) we obtain an expression for the meridian distance as

$$m = a\left\{A_0\phi - A_2\sin 2\phi + A_4\sin 4\phi - A_6\sin 6\phi + A_8\sin 8\phi - A_{10}\sin 10\phi + \cdots\right\} \tag{35}$$

where

$$
\begin{aligned}
A_0 &= 1 - \frac{1}{4}e^2 - \frac{3}{64}e^4 - \frac{5}{256}e^6 - \frac{175}{16384}e^8 - \frac{441}{65536}e^{10} + \cdots \\
A_2 &= \frac{3}{8}\left(e^2 + \frac{1}{4}e^4 + \frac{15}{128}e^6 + \frac{35}{512}e^8 + \frac{735}{16384}e^{10} + \cdots\right) \\
A_4 &= \frac{15}{256}\left(e^4 + \frac{3}{4}e^6 + \frac{35}{64}e^8 + \frac{105}{256}e^{10} + \cdots\right) \\
A_6 &= \frac{35}{3072}\left(e^6 + \frac{5}{4}e^8 + \frac{315}{256}e^{10} + \cdots\right) \\
A_8 &= \frac{315}{131072}\left(e^8 + \frac{7}{4}e^{10} + \cdots\right) \\
A_{10} &= \frac{693}{131072}\left(e^{10} + \cdots\right)
\end{aligned}
\tag{36}
$$

Combining equations (32) and (33) gives the length of the loxodrome between P_1 and P_2 as

$$
s = \frac{m_2 - m_1}{\cos\alpha} \tag{37}
$$

where α is the (constant) azimuth and m_1 and m_2 are meridian distances for ϕ_1 and ϕ_2 obtained from equation (35).

THE DIRECT PROBLEM OF THE LOXODROME ON THE ELLIPSOID

The direct problem is: Given latitude and longitude of P_1, azimuth α_{12} of the loxodrome P_1P_2 and the arc length s along the loxodrome curve; compute the latitude and longitude of P_2 and the reverse azimuth α_{21}.

With the ellipsoid constants a, f, and e^2 and given $\phi_1, \lambda_1, \alpha_{12}$ and s the problem may be solved by the following sequence.

1. Compute m_1 the meridian distance of P_1 using equation (35).
2. Compute meridian distance m_2 from equation (37) where

$$
m_2 = s\cos\alpha_{12} + m_1
$$

3. Use Newton-Raphson iteration to compute latitude ϕ_2 using equation (35) re-arranged as

$$
f(\phi) = a\left\{A_0\phi - A_2\sin 2\phi + A_4\sin 4\phi - A_6\sin 6\phi + A_8\sin 8\phi - A_{10}\sin 10\phi\right\} - m = 0
$$

and the iterative equation $\phi_{(n+1)} = \phi_{(n)} - \dfrac{f\left(\phi_{(n)}\right)}{f'\left(\phi_{(n)}\right)}$ where $f'\left(\phi\right) = \dfrac{d}{d\phi}\left\{f\left(\phi\right)\right\}$ and

$$f'\left(\phi\right) = a\left\{A_0 - 2A_2\cos 2\phi + 4A_4\cos 4\phi - 6A_6\cos 6\phi + 8A_8\cos 8\phi - 10A_{10}\cos 10\phi\right\}$$

An initial value of $\phi_{(1)}$ (ϕ for $n = 1$) can be taken as the latitude of P_1 and the functions $f\left(\phi_{(1)}\right)$ and $f'\left(\phi_{(1)}\right)$ evaluated using ϕ_1. $\phi_{(2)}$ (ϕ for $n = 2$) can now be computed from the iterative equation and this process repeated to obtain values $\phi_{(3)}, \phi_{(4)}, \ldots$. This iterative process can be concluded when the difference between $\phi_{(n+1)}$ and $\phi_{(n)}$ reaches an acceptably small value.

4. Compute isometric latitudes q_1 and q_2 using equation (29) and then the difference in isometric latitudes $\Delta q = q_2 - q_1$
5. Compute the difference in longitude $\Delta\lambda = \lambda_2 - \lambda_1$ from equation (30)
6. Compute longitude λ_2 from $\lambda_2 = \lambda_1 + \Delta\lambda$
7. Compute reverse azimuth from $\alpha_{21} = \alpha_{12} \pm 180^\circ$

THE INVERSE PROBLEM OF THE LOXODROME ON THE ELLIPSOID

The inverse problem is: Given latitudes and longitudes of P_1 and P_2 on the ellipsoid, compute the azimuth α_{12} of the loxodrome P_1P_2, the arc length s along the loxodrome curve and the reverse azimuth α_{21}.

With the ellipsoid constants a, f, and e^2 and given ϕ_1, λ_1 and ϕ_2, λ_2 the problem may be solved by the following sequence.

1. Compute isometric latitudes q_1 and q_2 using equation (29) and then the difference in isometric latitudes $\Delta q = q_2 - q_1$
2. Compute the longitude difference $\Delta\lambda = \lambda_2 - \lambda_1$ and then the azimuth α_{12} using equation (31).
3. Compute meridian distances m_1 and m_2 using equation (35).
4. Compute the arc length s from equation (37).
5. Compute reverse azimuth from $\alpha_{21} = \alpha_{12} \pm 180^\circ$

MATLAB FUNCTIONS

A MATLAB function *loxodrome_ inverse.m* is shown below. This function computes the inverse problem of the loxodrome on the ellipsoid.

Output from the function is shown below for points on a great elliptic arc between the terminal points of the straight-line section of the Victorian–New South Wales border. This straight-line section of the border, between Murray Spring and Wauka 1978, is known as the Black-Allan Line in honour of the surveyors Black and Allan who set out the border line in 1870-71. Wauka 1978 (Gabo PM 4) is a geodetic concrete border pillar on the coast at Cape Howe and Murray Spring (Enamo PM 15) is a steel pipe driven into a spring of the Murray River that is closest to Cape Howe. The straight line is a normal section curve on the reference ellipsoid of the Geocentric Datum of Australia (GDA94) that contains the normal to the ellipsoid at Murray Spring. The GDA94 coordinates of Murray Spring and Wauka 1978 are:

$$\begin{array}{lll} \text{Murray Spring:} & \phi\ -37^\circ\,47'\,49.2232'' & \lambda\ 148^\circ\,11'\,48.3333'' \\ \text{Wauka 1978:} & \phi\ -37^\circ\,30'\,18.0674'' & \lambda\ 149^\circ\,58'\,32.9932'' \end{array}$$

The normal section azimuth and distance are:

$$116^\circ\,58'\,14.173757'' \qquad 176495.243760 \text{ m}$$

The geodesic azimuth and distance are:

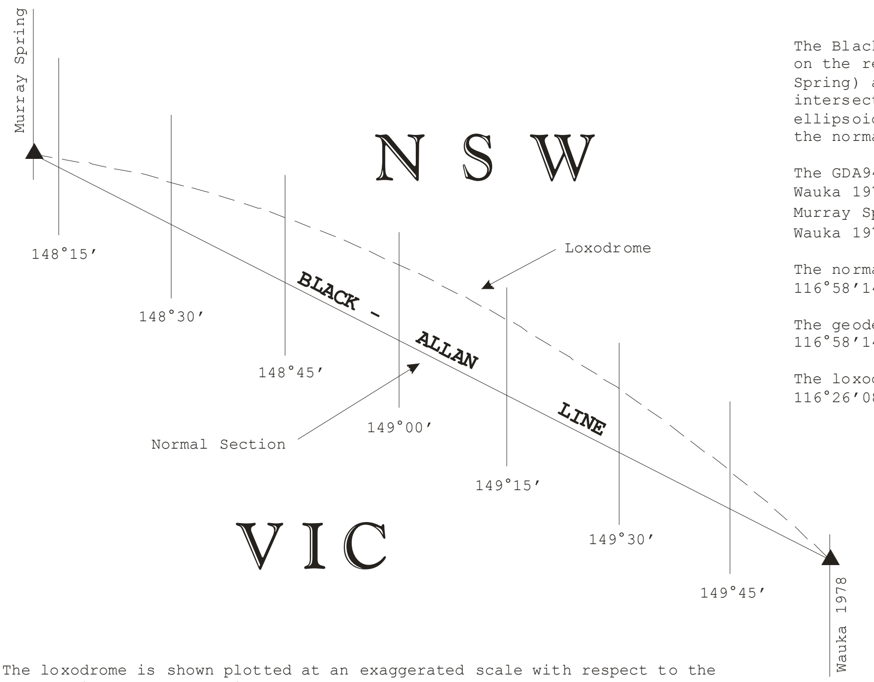
$$116^\circ\,58'\,14.219146'' \qquad 176495.243758 \text{ m}$$

The loxodrome azimuth and distance are:

$$116^\circ\,26'\,08.400701'' \qquad 176497.829952 \text{ m}$$

Figure 4 shows a schematic view of the Black-Allan line (normal section) and the great elliptic arc. The relationships between the great elliptic arc and the normal section have been computed at seven locations along the line (A, B, C, etc.) where meridians of longitude at $0^\circ\,15'$ intervals cut the line. These relationships are shown in Table 1.

BLACK-ALLAN LINE: VICTORIA/NSW BORDER



The Black-Allan Line is a normal section curve on the reference ellipsoid between P1 (Murray Spring) and P2 (Wauka 1978). This curve is the intersection of the normal section plane and the ellipsoid, and the normal section contains P1, the normal to the ellipsoid at P1, and P2.

The GDA94 coordinates of Murray Spring and Wauka 1978 are:
Murray Spring: ϕ -37°47'49.2232" λ 148°11'48.3333"
Wauka 1978: ϕ -37°30'18.0674" λ 149°58'32.9932"

The normal section azimuth and distance are:
116°58'14.173757" 176495.243760 m.

The geodesic azimuth and distance are:
116°58'14.219146" 176495.243758 m.

The loxodrome azimuth and distance are:
116°26'08.400701" 176497.829952 m.

The loxodrome is shown plotted at an exaggerated scale with respect to the Border Line (normal section).
At longitude 149°00'E. the loxodrome is 457.918 m north of the Border Line.
At longitude 149°30'E. the loxodrome is 361.250 m north of the Border Line.

Figure 4

BLACK-ALLAN LINE: VICTORIA/NSW BORDER

NAME	**GDA94**		**Ellipsoid values**		
	LATITUDE	**LONGITUDE**	**dφ**	**ρ**	**dm = ρ×dφ**
Murray Spring	-36°47´49.223200˝	148°11´48.333300˝			
A	-36°49´07.598047˝ N -36°49´05.849245˝ Lox	148°15´00.000000˝	+00´01.748802˝	6358356.102	+53.9089
B	-36°55´13.876510˝ N -36°55´05.371035˝ Lox	148°30´00.000000˝	+00´08.505475˝	6358465.209	+262.1958
C	-37°01´17.289080˝ N -37°01´04.418599˝ Lox	148°45´00.000000˝	+00´12.870481˝	6358573.577	+396.7613
D	-37°07´17.845554˝ N -37°07´02.991484˝ Lox	149°00´00.000000˝	+00´14.854070˝	6358681.204	+457.9177
E	-37°13´15.555723˝ N -37°13´01.089240˝ Lox	149°15´00.000000˝	+00´14.466483˝	6358788.089	+459.9767
F	-37°19´10.429372˝ N -37°18´58.711427˝ Lox	149°30´00.000000˝	+00´11.717945˝	6358894.232	+361.2501
G	-37°25´02.476276˝ N -37°24´55.857608˝ Lox	149°45´00.000000˝	+00´06.618668˝	6358999.632	+204.0489
Wauka 1978	-37°30´18.067400˝	149°58´32.993200˝			

TABLE 1: Points where the Great Elliptic Arc cuts meridians of A, B, C, etc at 0°15´ intervals of longitude along Border Line. N = Normal Section, Lox = Loxodrome

```
>> help loxodrome_inverse

  loxodrome_inverse:  This function computes the inverse case for a
    loxodrome on the reference ellipsoid.  That is, given the latitudes and
    longitudes of two points on the ellipsoid, compute the azimuth and the
    arc length of the loxodrome on the surface.

>> loxodrome_inverse

=======================
Loxodrome: Inverse Case
=======================
Ellipsoid parameters
a  = 6378137.0000
f  = 1/298.257222101

Terminal points of curve
Latitude  P1 =  -36 47 49.223200 (D M S)
Longitude P1 =  148 11 48.333300 (D M S)

Latitude  P2 =  -37 30 18.067400 (D M S)
Longitude P2 =  149 58 32.993200 (D M S)

isometric lat  P1 =  -39 23 36.268670 (D M S)
isometric lat  P2 =  -40 16 40.540366 (D M S)

diff isometric lat  P2-P1 =   -0 53  4.271697 (D M S)
diff in longitude P2-P1   =    1 46 44.659900 (D M S)

meridian distance P1 =  -4073983.614420
meridian distance P2 =  -4152559.155874

diff in mdist P2-P1  =    -78575.541454

Azimuth of loxodrome P1-P2
Az12  = 116 26  8.400701  (D M S)

loxodrome distance P1-P2
s =    176497.829952

>>
```

MATLAB function *loxodrome_ inverse.m*

```
function loxodrome_inverse
%
% loxodrome_inverse:  This function computes the inverse case for a
%   loxodrome on the reference ellipsoid.  That is, given the latitudes and
%   longitudes of two points on the ellipsoid, compute the azimuth and the
%   arc length of the loxodrome on the surface.

%------------------------------------------------------------------------
% Function:  loxodrome_inverse()
%
% Usage:     loxodrome_inverse
%
% Author:    R.E.Deakin,
%            School of Mathematical & Geospatial Sciences, RMIT University
%            GPO Box 2476V, MELBOURNE, VIC 3001, AUSTRALIA.
%            email: rod.deakin@rmit.edu.au
%            Version  1.0  5 October 2009
%            Version  1.1 11 January 2010
%
% Purpose:   This function computes the inverse case for a loxodrome on the
%   reference ellipsoid.  That is, given the latitudes and longitudes of
%   two points on the ellipsoid, compute the azimuth and the arc length of
%   the loxodrome on the surface.
%
% Functions required:
%  [D,M,S] = DMS(DecDeg)
%   isolat = isometric(flat,lat)
%    mdist = meridian_dist(a,flat,lat)
%
% Variables:
%  Az12     - azimuth of loxodrome P1-P2 (radians)
%  a        - semi-major axis of spheroid
%  d2r      - degree to radian conversion factor 57.29577951...
%  disolat  - difference in isometric latitudes (isolat2-isolat1)
%  dlon     - difference in longitudes (radian)
%  dm       - difference in meridian distances (dm = m2-m1)
%  e        - eccentricity of ellipsoid
%  e2       - eccentricity of ellipsoid squared
%  f        - f = 1/flat is the flattening of ellipsoid
%  flat     - denominator of flattening of ellipsoid
%  isolat1  - isometric latitude of P1 (radians)
%  isolat2  - isometric latitude of P2 (radians)
%  lat1     - latitude of P1 (radians)
%  lat2     - latitude of P2 (radians)
%  lon1     - longitude of P1 (radians)
%  lon2     - longitude of P2 (radians)
%  lox_s    - distance along loxodrome
%  m1,m2    - meridian distances of P1 and P2 (metres)
%  pion2    - pi/2
%
% Remarks:
%
% References:
%  [1] Deakin, R.E., 2010, 'The Loxodrome on an Ellipsoid', Lecture Notes,
%         School of Mathematical and Geospatial Sciences, RMIT University,
%         January 2010
%  [2] Bowring, B.R., 1985, 'The geometry of the loxodrome on the
%         ellipsoid', The Canadian Surveyor, Vol. 39, No. 3, Autumn 1985,
%         pp.223-230.
%  [3] Snyder, J.P., 1987, Map Projections-A Working Manual.  U.S.
%         Geological Survey Professional Paper 1395.  Washington, DC: U.S.
%         Government Printing Office, pp.15-16 and pp. 44-45.
%  [4] Thomas, P.D., 1952, Conformal Projections in Geodesy and
%         Cartography, Special Publication No. 251, Coast and Geodetic
%         Survey, U.S. Department of Commerce, Washington, DC: U.S.
%         Government Printing Office, p. 66.
```

```
%
%------------------------------------------------------------------------------

% Degree to radian conversion factor
d2r   = 180/pi;

% Set ellipsoid parameters
a    = 6378137;    % GRS80
flat = 298.257222101;

% Set lat and long of P1 and P2 on ellipsoid
lat1 = -(36  + 47/60 + 49.2232/3600)/d2r;   % Spring
lon1 =  (148 + 11/60 + 48.3333/3600)/d2r;
lat2 = -(37  + 30/60 + 18.0674/3600)/d2r;    % Wauka 1978
lon2 =  (149 + 58/60 + 32.9932/3600)/d2r;

% Compute isometric latitude of P1 and P2
isolat1 = isometric(flat,lat1);
isolat2 = isometric(flat,lat2);

% Compute changes in isometric latitude and longitude between P1 and P2
disolat = isolat2-isolat1;
dlon = lon2-lon1;

% Compute azimuth
Az12 = atan2(dlon,disolat);

% Compute distance along loxodromic curve
m1 = meridian_dist(a,flat,lat1);
m2 = meridian_dist(a,flat,lat2);
dm = m2-m1;
lox_s  = dm/cos(Az12);

%-----------------------
% Print result to screen
%-----------------------
fprintf('\n======================');
fprintf('\nLoxodrome: Inverse Case');
fprintf('\n======================');
fprintf('\nEllipsoid parameters');
fprintf('\na  = %12.4f',a);
fprintf('\nf  = 1/%13.9f',flat);

fprintf('\n\nTerminal points of curve');
% Print lat and lon of Point 1
[D,M,S] = DMS(lat1*d2r);
if D == 0 && lat1 < 0
    fprintf('\nLatitude  P1 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLatitude  P1 = %4d %2d %9.6f (D M S)',D,M,S);
end
[D,M,S] = DMS(lon1*d2r);
if D == 0 && lon1 < 0
    fprintf('\nLongitude P1 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLongitude P1 = %4d %2d %9.6f (D M S)',D,M,S);
end

% Print lat and lon of point 2
[D,M,S] = DMS(lat2*d2r);
if D == 0 && lat1 < 0
    fprintf('\n\nLatitude  P2 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\n\nLatitude  P2 = %4d %2d %9.6f (D M S)',D,M,S);
end
[D,M,S] = DMS(lon2*d2r);
if D == 0 && lon2 < 0
    fprintf('\nLongitude P2 =   -0 %2d %9.6f (D M S)',M,S);
else
```

```
    fprintf('\nLongitude P2 = %4d %2d %9.6f (D M S)',D,M,S);
end

% Print isometric latitudes of P1 and P2
[D,M,S] = DMS(isolat1*d2r);
if D == 0 && isolat1 < 0
    fprintf('\n\nisometric lat  P1 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\n\nisometric lat  P1 = %4d %2d %9.6f (D M S)',D,M,S);
end
[D,M,S] = DMS(isolat2*d2r);
if D == 0 && isolat2 < 0
    fprintf('\nisometric lat  P2 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nisometric lat  P2 = %4d %2d %9.6f (D M S)',D,M,S);
end

% Print differences in isometric latitudes and longitudes
[D,M,S] = DMS(disolat*d2r);
if D == 0 && disolat < 0
    fprintf('\n\ndiff isometric lat  P2-P1 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\ndiff isometric lat  P2-P1 = %4d %2d %9.6f (D M S)',D,M,S);
end
[D,M,S] = DMS(dlon*d2r);
if D == 0 && dlon < 0
    fprintf('\ndiff in longitude P2-P1   =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\ndiff in longitude P2-P1   = %4d %2d %9.6f (D M S)',D,M,S);
end

% Print meridian distances of P1 and P2
fprintf('\n\nmeridian distance P1 =  %15.6f',m1);
fprintf('\nmeridian distance P2 =  %15.6f',m2);
fprintf('\n\ndiff in mdist P2-P1  =  %15.6f',dm);

% Print azimuth of loxodrome
fprintf('\n\nAzimuth of loxodrome P1-P2');
[D,M,S] = DMS(Az12*d2r);
fprintf('\nAz12  = %3d %2d %9.6f  (D M S)',D,M,S);

% Print loxodrome distance P1-P2
fprintf('\n\nloxodrome distance P1-P2');
fprintf('\ns =  %15.6f',lox_s);

fprintf('\n\n');
```

MATLAB function *isometric.m*

```
function isolat = isometric(flat,lat)
%
% isolat=isometric(flat,lat)  Function computes the isometric latitude
%   (isolat) of a point whose latitude (lat) is given on an ellipsoid whose
%   denominator of flattening is flat.
%   Latitude (lat) must be in radians and the returned value of isometric
%   latitude (isolat) will also be in radians.
%   Example: isolat = isometric(298.257222101,-0.659895044028705);
%            should return isolat = -0.709660227088983 radians,
%            equal to -40 39 37.9292417795658 (DMS) for latitude equal to
%            -0.659895044028705 radians (-37 48 33.1234 (DMS)) on the GRS80
%            ellipsoid.

%--------------------------------------------------------------------------
% Function:  isometric(flat,lat)
%
% Syntax:    isolat = isometric(flat,lat);
%
% Author:    R.E.Deakin,
%            School of Mathematical & Geospatial Sciences, RMIT University
%            GPO Box 2476V, MELBOURNE, VIC 3001, AUSTRALIA.
%            email: rod.deakin@rmit.edu.au
%            Version  1.0  5 October 2009
%
% Purpose:   Function computes the isometric latitude of a point whose
%  latitude is given on an ellipsoid defined by semi-major axis (a) and
%  denominator of flattening (flat).
%
% Return value: Function isometric() returns isolat (isometric latitude in
%  radians)
%
% Variables:
%  e      - eccentricity of ellipsoid
%  e2     - eccentricity-squared
%  f      - flattening of ellipsoid
%  flat   - denominator of flattening
%
% Remarks:
%  Isometric latitude is an auxiliary latitude proportional to the spacing
%  of parallels of latitude on an ellipsoidal Mercator projection.
%
% References:
%  [1] Snyder, J.P., 1987, Map Projections-A Working Manual.  U.S.
%         Geological SurveyProfessional Paper 1395.  Washington, DC: U.S.
%         Government Printing Office, pp.15-16.
%
%
%--------------------------------------------------------------------------

% compute flattening f eccentricity squared e2
f   = 1/flat;
e2 = f*(2-f);
e  = sqrt(e2);

x = e*sin(lat);
y = (1-x)/(1+x);
z = pi/4 + lat/2;

% calculate the isometric latitude
isolat = log(tan(z)*(y^(e/2)));
```

MATLAB function *meridian_ dist.m*

```
function mdist = meridian_dist(a,flat,lat)
%
% mdist = meridian_dist(a,flat,lat) Function computes the meridian distance
%   on an ellipsoid defined by semi-major axis (a) and denominator of
%   flattening (flat) from the equator to a point having latitude (lat) in
%   radians.
%   e.g. mdist = (6378137, 298.257222101, -0.659895044028705) will compute
%   the meridian distance for a point having latitude -37 deg 48 min
%   33.1234 sec on the GRS80 ellipsoid (a = 6378137, f = 1/298.257222101).

%--------------------------------------------------------------------------
% Function:  meridian_dist()
%
% Usage:     mdist = meridian_dist(a,flat,lat)
%
% Author:    R.E.Deakin,
%            School of Mathematical & Geospatial Sciences, RMIT University
%            GPO Box 2476V, MELBOURNE, VIC 3001, AUSTRALIA.
%            email: rod.deakin@rmit.edu.au
%            Version  1.0  5 October 2009
%
% Purpose:   Function computes the meridian distance
%   on an ellipsoid defined by semi-major axis (a) and denominator of
%   flattening (flat) from the equator to a point having latitude (lat) in
%   radians.
%
% Functions required:
%
% Variables: a         - semi-major axis of spheroid
%            A,B,C...  - coefficients
%            e2        - eccentricity squared
%            e4,e6,... - powers of e2
%            f         - f = 1/flat is the flattening of ellipsoid
%            flat      - denominator of flattening of ellipsoid
%            mdist     - meridian distance
%
% Remarks:   The formulae used are given in Baeschlin, C.F., 1948,
%            "Lehrbuch Der Geodasie", Orell Fussli Verlag, Zurich, pp.47-50.
%            See also Deakin, R. E. and Hunter M. N., 2008, "Geometric
%            Geodesy - Part A", Lecture Notes, School of Mathematical and
%            geospatial Sciences, RMIT University, March 2008, pp. 60-65.
%
%--------------------------------------------------------------------------

% compute eccentricity squared
f  = 1/flat;
e2 = f*(2-f);

% powers of eccentricity
e4  = e2*e2;
e6  = e4*e2;
e8  = e6*e2;
e10 = e8*e2;

% coefficients of series expansion for meridian distance
A = 1+(3/4)*e2+(45/64)*e4+(175/256)*e6+(11025/16384)*e8+(43659/65536)*e10;
B = (3/4)*e2+(15/16)*e4+(525/512)*e6+(2205/2048)*e8+(72765/65536)*e10;
C = (15/64)*e4+(105/256)*e6+(2205/4096)*e8+(10395/16384)*e10;
D = (35/512)*e6+(315/2048)*e8+(31185/131072)*e10;
E = (315/16384)*e8+(3465/65536)*e10;
F = (693/131072)*e10;

term1 = A*lat;
term2 = (B/2)*sin(2*lat);
term3 = (C/4)*sin(4*lat);
term4 = (D/6)*sin(6*lat);
```

```
term5 = (E/8)*sin(8*lat);
term6 = (F/10)*sin(10*lat);

mdist = a*(1-e2)*(term1-term2+term3-term4+term5-term6);
```

MATLAB function *DMS.m*

```
function [D,M,S] = DMS(DecDeg)
% [D,M,S] = DMS(DecDeg)  This function takes an angle in decimal degrees and returns
%   Degrees, Minutes and Seconds

val = abs(DecDeg);
D = fix(val);
M = fix((val-D)*60);
S = (val-D-M/60)*3600;
if(DecDeg<0)
  D = -D;
end
return
```

REFERENCES

Bowring, B. R., (1985), 'The Geometry of the Loxodrome on the Ellipsoid', *The Canadian Surveyor,* Vol. 39, No. 3, Autumn 1985, pp. 223-230.

Bomford, G., (1980), *Geodesy,* 4th edition, Clarendon Press, Oxford.

Deakin, R. E. and Hunter, M. N., (2008), 'Geometric Geodesy - Part A', Lecture Notes, School of Mathematical & Geospatial Sciences, RMIT University, Melbourne, Australia, March 2008, 140 pages.

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