MAP PROJECTION THEORY

1. INTRODUCTION

A map projection is the mathematical transformation of coordinates on a datum surface to coordinates on a projection surface. In all the map projections we will be dealing with, the datum surface is a 3-Dimensional surface, either a sphere or ellipsoid, representing the Earth and on this surface, there are imaginary sets of reference curves, or parametric curves, that we use to coordinate points. We know these parametric curves as parallels of latitude ϕ and meridians of longitude λ and along these curves one of the parameters, ϕ or λ is constant. Points on the datum surface having particular values of ϕ and λ are said to have curvilinear coordinates. These curvilinear coordinates are more commonly called geographical or geodetic coordinates. Points on the datum surface can also have *x,y,z* Cartesian coordinates and there are mathematical connections between the curvilinear and Cartesian that we call functional relationships and write as

$$x = f_1(\phi,\lambda)$$
$$y = f_2(\phi,\lambda)$$
$$z = f_3(\phi,\lambda)$$

Figure 1.1(a) below shows a spherical datum surface representing the Earth with meridians and parallels (the ϕ,λ parametric curves) and the continental outlines. The *x,y,z* Cartesian axes are shown with the *z*-axis passing through the North pole. The *x*-*y* plane is the Earth's equatorial plane and the *x*-*z* plane is the Greenwich meridian plane. The *x*-axis passes through the intersection of the Greenwich meridian and the equator and the *y*-axis is advanced 90º eastwards along the equator. The longitude of *P* is the angular measure between the Greenwich meridian plane and the meridian plane passing through *P* and the latitude is the angular measure between the equatorial plane and the normal to the datum surface passing through *P*. Longitude is measured 0º to 180º positive east and negative west of the Greenwich meridian and latitude is measured 0º to 90º positive north, and negative south of the equator.

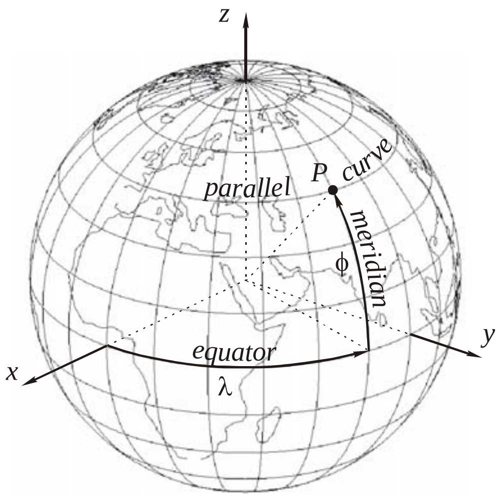


Figure 1.1(a)

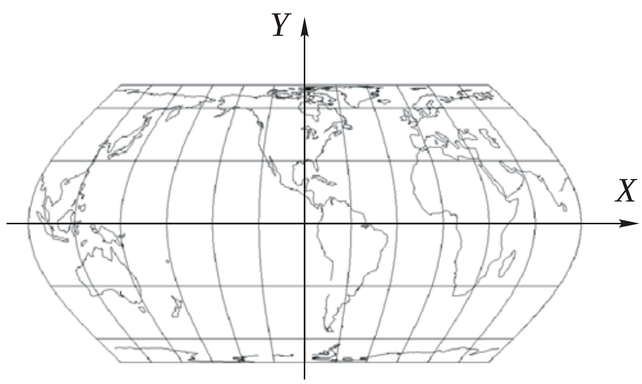


Figure 1.1(b)

The functional relationships between the *x,y,z* Cartesian coordinates and the ϕ,λ curvilinear coordinates written in the general form above can be expressed in the more familiar form

$$\begin{aligned} x &= f_1(\phi,\lambda) = R\cos\phi\cos\lambda \\ y &= f_2(\phi,\lambda) = R\cos\phi\sin\lambda \\ z &= f_3(\phi,\lambda) = R\sin\phi \end{aligned}$$

where *R* is the radius of the spherical Earth.

Figure 1.1(b) shows the projection surface, which we commonly refer to as the map projection. In this case, as in all cases we will deal with in this study, the projection surface is a plane. In general, the projection surface may be another curved 3D surface and we use this general concept in the theoretical development that follows. On the projection surface, there are sets of parametric curves, say *U,V* curves and points on the projection surface have *U,V* curvilinear coordinates. These coordinates are related to another 3D Cartesian coordinate system *X,Y,Z* and the two systems are related by another set of functional relationships

$$\begin{aligned} X &= F_1(U,V) \\ Y &= F_2(U,V) \\ Z &= F_3(U,V) \end{aligned}$$

In the case of a plane projection surface $Z = 0$, and what we would like to establish are the connections between the curvilinear coordinates ϕ,λ on the datum surface and *X,Y* Cartesian coordinates of the projection plane, i.e., we wish to find the functional relationships

$$\begin{aligned} X &= \bar{f}_1(\phi,\lambda) \\ Y &= \bar{f}_2(\phi,\lambda) \end{aligned}$$

In Figure 1.1(b) the map projection is a *modified Sinusoidal projection* and the projection equations (the functional relationships) are

$$\begin{aligned} X &= R\frac{(\lambda-\lambda_0)}{Mn}(m+\cos\alpha) \\ Y &= RM\alpha \end{aligned}$$

R is the radius of the spherical Earth, λ_0 is the longitude of the central meridian (the *Y*-axis) and α is a function of the latitude. *M, m* and *n* are constants related to the *axes ratio* (the ratio between the lengths of the *X* and *Y* axes) and the *pole-equator ratio* (the ratio between the lengths of the pole line and the equator).

Inspection of the map projection reveals distortions that we see as misshapen continental outlines (Antarctica), points projected as lines (the north and south poles) and straight lines projected as curves (the meridians). Every map projection has distortions of one sort or another and we would like to quantify these distortions. It turns out, as we shall see later, that distortions can be related to scale factors where scale is the ratio of elemental distances on the datum surface and the projection surface, and a knowledge of scale factors allow us to "uncover" the projection equations by enforcing scale conditions and particular geometric constraints. The following theoretical development gives us the tools to understand the limitations and properties of several projections that are used in practice.

2. SURFACES AND PROJECTION FORMULAE

Consider the diagrams shown in Figures 2(a) and 2(b) below. Both diagrams show surfaces; one is the datum surface and the other is the projection surface. Both of these surfaces are connected to Cartesian coordinate axes, x,y,z for the datum surface and X,Y,Z for the projection surface. We shall use lower case letters for the datum surface and capital letters for the projection surface.

On the datum surface, there are a system of reference curves or parametric curves of constant u and v. The reference curves are used to coordinate points on the datum surface, i.e., a point P on the datum surface has u,v curvilinear coordinates and x,y,z Cartesian coordinates and both systems are connected by the functional relationships

$$\begin{aligned} x &= f_1(u,v) \\ y &= f_2(u,v) \\ z &= f_3(u,v) \end{aligned} \tag{2.1}$$

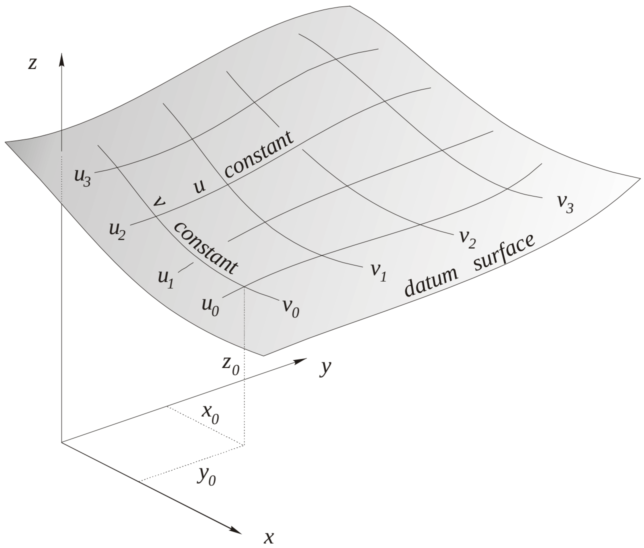
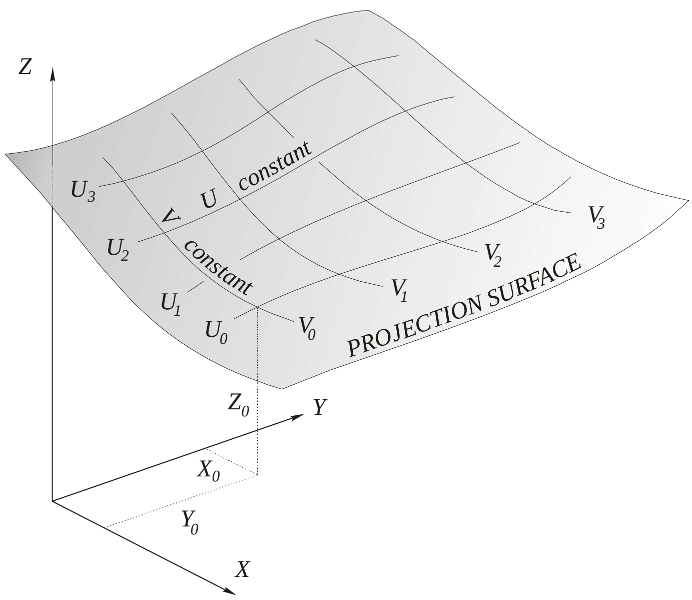


Figure 2.1(a) Datum Surface



On the projection surface there is also a system of reference curves or parametric curves U,V that are related to the X,Y,Z coordinates by the functional relationships

$$\begin{aligned} X &= F_1(U,V) \\ Y &= F_2(U,V) \\ Z &= F_3(U,V) \end{aligned} \tag{2.2}$$

Figure 2.1(b) Projection Surface

The parametric curves on the datum surface and the parametric curves on the projection surface are related by two sets of functions

$$\begin{aligned} U &= g_1(u,v) \\ V &= g_2(u,v) \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} u &= G_1(U,V) \\ v &= G_2(U,V) \end{aligned} \tag{2.4}$$

These functions satisfy the basic requirements of any map projection or transformation, which are:

(i) the transformation must be unique (i.e., a 1 to 1 relationship) and
(ii) the transformation must be reversible

These basic requirements mean that a point on the datum surface should correspond to one, and only one point on the projection – and that the reverse holds true. These requirements are satisfied by equations (2.3) and (2.4) which mean that *u* and *v* are solvable as explicit functions of *U* and *V*.

Substituting (2.3) into (2.2) gives <u>the general transformation or projection equations</u>

$$\begin{aligned} x &= f_1(u,v) \\ y &= f_2(u,v) \\ z &= f_3(u,v) \end{aligned} \quad \text{and} \quad \begin{aligned} X &= F_1(U,V) = F_1\left(g_1(u,v), g_2(u,v)\right) = \bar{f}_1(u,v) \\ Y &= F_2(U,V) = F_2\left(g_1(u,v), g_2(u,v)\right) = \bar{f}_2(u,v) \\ Z &= F_3(U,V) = F_3\left(g_1(u,v), g_2(u,v)\right) = \bar{f}_3(u,v) \end{aligned} \tag{2.5}$$

3. THE GAUSSIAN FUNDAMENTAL QUANTITIES ON THE DATUM SURFACE

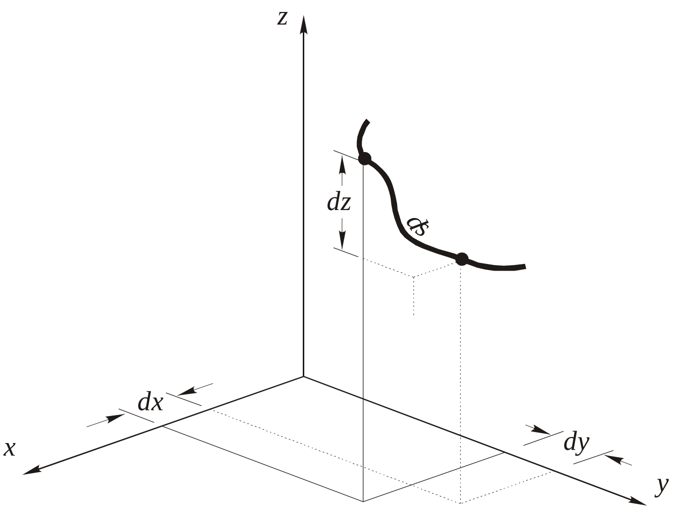


Figure 3.1 The elemental distance *ds*

From differential geometry, the square of the length of a <u>differentially small</u> part of a curve on the datum surface is

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{3.1}$$

From the functional relationships of (2.1) and (2.5) the <u>total differentials</u> are

$$\begin{aligned} dx &= \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv \\ dy &= \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv \\ dz &= \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv \end{aligned} \tag{3.2}$$

Substituting the total differentials into (3.1) gives

$$\begin{aligned} ds^2 = &\left(\frac{\partial x}{\partial u}\right)^2 du^2 + \left(\frac{\partial x}{\partial v}\right)^2 dv^2 + 2\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}du\,dv \\ &+\left(\frac{\partial y}{\partial u}\right)^2 du^2 + \left(\frac{\partial y}{\partial v}\right)^2 dv^2 + 2\frac{\partial y}{\partial u}\frac{\partial y}{\partial v}du\,dv \\ &+\left(\frac{\partial z}{\partial u}\right)^2 du^2 + \left(\frac{\partial z}{\partial v}\right)^2 dv^2 + 2\frac{\partial z}{\partial u}\frac{\partial z}{\partial v}du\,dv \end{aligned} \tag{3.3}$$

Gathering terms gives

$$\begin{aligned} ds^2 = &\left\{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2\right\}du^2 \\ &+\left\{\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v}\right\}2du\,dv \\ &+\left\{\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right\}dv^2 \end{aligned} \tag{3.4}$$

The coefficients of du^2, $du\,dv$ and dv^2 are called the <u>Gaussian Fundamental Quantities</u>. They are invariably indicated in the map projection literature by E, F and G or e, f and g.

The equation for the <u>elemental distance</u> ds is usually written with the Gaussian Fundamental Quantities e, f and g as

$$ds^2 = e\,du^2 + 2\,f\,du\,dv + g\,dv^2 \tag{3.5}$$

where

$$\begin{aligned} e &= \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 \\ f &= \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v} \\ g &= \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 \end{aligned} \tag{3.6}$$

3.1. Elemental distance on the surface of the spherical Earth

For a sphere of radius *R* representing the Earth the *u* and *v* curves are the latitude curves ϕ and longitude curves λ respectively and the functional relationships connecting *x*,*y*,*z* Cartesian coordinates and ϕ,λ curvilinear coordinates are

$$\begin{aligned} x &= f_1(\phi,\lambda) = R\cos\phi\cos\lambda \\ y &= f_2(\phi,\lambda) = R\cos\phi\sin\lambda \\ z &= f_3(\phi,\lambda) = R\sin\phi \end{aligned} \tag{3.7}$$

The Gaussian Fundamental Quantities *e*, *f* and *g* are

$$\begin{aligned} e &= \left(\frac{\partial x}{\partial\phi}\right)^2 + \left(\frac{\partial y}{\partial\phi}\right)^2 + \left(\frac{\partial z}{\partial\phi}\right)^2 \\ f &= \frac{\partial x}{\partial\phi}\frac{\partial x}{\partial\lambda} + \frac{\partial y}{\partial\phi}\frac{\partial y}{\partial\lambda} + \frac{\partial z}{\partial\phi}\frac{\partial z}{\partial\lambda} \\ g &= \left(\frac{\partial x}{\partial\lambda}\right)^2 + \left(\frac{\partial y}{\partial\lambda}\right)^2 + \left(\frac{\partial z}{\partial\lambda}\right)^2 \end{aligned} \tag{3.8}$$

Differentiating equations (3.7) gives

$$\begin{aligned} \frac{\partial x}{\partial\phi} &= -R\sin\phi\cos\lambda & \qquad \frac{\partial x}{\partial\lambda} &= -R\cos\phi\sin\lambda \\ \frac{\partial y}{\partial\phi} &= -R\sin\phi\sin\lambda & \qquad \frac{\partial x}{\partial\lambda} &= R\cos\phi\cos\lambda \\ \frac{\partial x}{\partial\phi} &= R\cos\phi & \qquad \frac{\partial x}{\partial\lambda} &= 0 \end{aligned}$$

Substituting these derivatives into (3.8) gives

$$\begin{aligned} e &= (-R\sin\phi\cos\lambda)^2 + (-R\sin\phi\sin\lambda)^2 + (R\cos\phi)^2 \\ &= R^2\left(\sin^2\phi\cos^2\lambda + \sin^2\phi\sin^2\lambda + \cos^2\phi\right) \\ &= R^2\left(\sin^2\phi\left(\cos^2\lambda + \sin^2\lambda\right) + \cos^2\phi\right) \\ &= R^2\left(\sin^2\phi + \cos^2\phi\right) \\ &= R^2 \end{aligned} \tag{3.9}$$

$$\begin{aligned} f &= (-R\sin\phi\cos\lambda)(-R\cos\phi\sin\lambda) + (-R\sin\phi\sin\lambda)(R\cos\phi\cos\lambda) + (R\cos\phi)(0) \\ &= \left(R^2\sin\phi\cos\lambda\cos\phi\sin\lambda\right) - \left(R^2\sin\phi\sin\lambda\cos\phi\cos\lambda\right) + (0) \\ &= 0 \end{aligned} \tag{3.10}$$

$$\begin{aligned} g &= \left(-R\cos\phi\sin\lambda\right)^2 + \left(R\cos\phi\cos\lambda\right)^2 + \left(0\right)^2 \\ &= R^2\left(\cos^2\phi\sin^2\lambda + \cos^2\phi\cos^2\lambda\right) \\ &= R^2\left(\cos^2\phi\left(\cos^2\lambda + \sin^2\lambda\right)\right) \\ &= R^2\cos^2\phi \end{aligned} \tag{3.11}$$

The elemental distance on the surface of a sphere of radius R with parametric curves ϕ (latitude) and λ (longitude) is

$$ds^2 = e\,d\phi^2 + 2f\,d\phi\,d\lambda + g\,d\lambda^2 \tag{3.12}$$

and the Gaussian Fundamental Quantities for this surface are

$$\begin{aligned} e &= R^2 \\ f &= 0 \\ g &= R^2\cos^2\phi \end{aligned} \tag{3.13}$$

Example

Consider two points P and Q on the spherical Earth of radius 6371 km. P has latitude 37º 48' 00" South and Q is 0º 00' 01" North and 0º 00' 01" East of P. What is the distance on the surface of the sphere between P and Q?

$$\begin{aligned} ds^2 &= R^2\left(4.8481\times10^{-6}\right)^2 + R^2\cos^2\phi\left(4.8481\times10^{-6}\right)^2 \\ &= 1.5497\times10^{-3}\ \text{km}^2 \end{aligned}$$

giving

$$ds = 39.366\ \text{m}$$

4. THE ELEMENTAL PARALLELOGRAM ON THE DATUM SURFACE

The elemental distance *ds* on the datum surface may be shown as the diagonal of a differentially small parallelogram

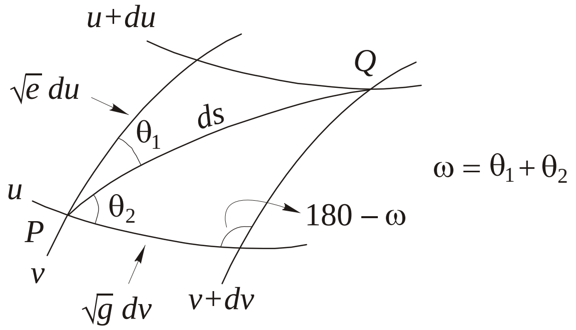


Figure 4.1 The elemental parallelogram

Figure 4.1 shows two differentially close points *P* and *Q* on the datum surface. The parametric curves *u* and *v* pass through *P* and the curves $u+du$ and $v+dv$ pass through *Q*. The distance between *P* and *Q* is the elemental distance *ds*. The differential parallelogram formed by the curves may be regarded as a plane figure, whose opposite sides are parallel straight lines enclosing a differentially small area *da*. The angle between the parametric curves *u* and *v* is equal to ω and $\omega = \theta_1 + \theta_2$

4.1. Elemental distance along parametric curves on the datum surface

The elemental distances along the *u* and *v* curves forming the sides of the parallelogram can be obtained from (3.5) considering the fact that along the *u*-curve, *u* is a constant value, hence $du = 0$ and along the *v*-curve, *v* is a constant value, hence $dv = 0$.

Elemental distance along the *v*-curve:

$$\begin{aligned} ds_v^2 &= e\,du^2 + 2\,f\,du\,dv + g\,dv^2 \\ &= e\,du^2 + 2\,f\,du(0) + g\,(0)^2 \\ &= e\,du^2 \end{aligned}$$

and
$$ds_v = \sqrt{e}\,du \tag{4.1}$$

Elemental distance along the *u*-curve:

$$\begin{aligned} ds_u^2 &= e\,du^2 + 2\,f\,du\,dv + g\,dv^2 \\ &= e(0)^2 + 2\,f\,(0)dv + g\,dv^2 \\ &= g\,dv^2 \end{aligned}$$

and
$$ds_u = \sqrt{g}\,dv \tag{4.2}$$

<u>The quantities $\sqrt{e}$ and $\sqrt{g}$ are units of measure along the *u* and *v* curves on the datum surface.</u>

4.2. The angle between parametric curves on the datum surface

The elemental parallelogram can be regarded as a plane parallelogram within its infinitely small area, hence from the cosine rule for plane trigonometry and bearing in mind that $\cos(180 - x) = -\cos x$

$$\begin{aligned} ds^2 &= e\,du^2 + g\,dv^2 - 2\left(\sqrt{e}\,du\right)\left(\sqrt{g}\,dv\right)\cos(180 - \omega) \\ &= e\,du^2 + g\,dv^2 + 2\sqrt{eg}\,du\,dv\cos\omega \end{aligned} \tag{4.3}$$

Equating (3.5) and (4.3) gives

$$\cos\omega = \frac{f}{\sqrt{eg}} \tag{4.4}$$

Thus, we may say: <u>If the parametric curves on the datum surface intersect at right angles (i.e., an orthogonal system) then $\omega = 90^\circ$ and $\cos\omega = 0$. This implies that $f = 0$</u>

Also $\sin\omega = \sqrt{1-\cos^2\omega} = \sqrt{1-\frac{f^2}{eg}} = \sqrt{\frac{eg-f^2}{eg}}$ and we may define a useful quantity *j* as

$$j^2 = eg - f^2 \tag{4.5}$$

Hence
$$\sin\omega = \frac{j}{\sqrt{eg}} \tag{4.6}$$

4.3. The elemental area on the datum surface

Referring again to Figure 4.1 and treating the elemental parallelogram as a plane parallelogram, the <u>elemental area</u> *da* is

$$\begin{aligned} da &= \left(\sqrt{e}\,du\right)\left(\sqrt{g}\,dv\right)\sin\omega \\ &= \sqrt{eg}\,\sin\omega\,du\,dv \end{aligned}$$

Using (4.6) gives an expression for the elemental area

$$da = j\,du\,dv \tag{4.7}$$

5. ELEMENTAL QUANTITIES ON THE PROJECTION SURFACE

Referring to Figure 2.1(b) which shows the projection surface with parametric curves *U* and *V* on the surface and the *X,Y,Z* Cartesian coordinate system connected to the *U,V* curvilinear coordinates by the functional relationships (2.2). Using similar developments as we used for the datum surface, the following relationships for the projection surface may be derived.

5.1. Elemental distance on the projection surface

The elemental distance *dS* on the projection surface is

$$dS^2 = dX^2 + dY^2 + dZ^2 \tag{5.1}$$

5.2. Gaussian Fundamental Quantities for the projection surface

Using equations (2.2), the Gaussian Fundamental Quantities $\overline{E}$, $\overline{F}$ and $\overline{G}$ can be derived in a similar manner as those for the datum surface.

$$dS^2 = \overline{E}\,dU^2 + 2\overline{F}\,dU\,dV + \overline{G}\,dV^2 \tag{5.2}$$

where

$$\begin{aligned}
\overline{E} &= \left(\frac{\partial X}{\partial U}\right)^2 + \left(\frac{\partial Y}{\partial U}\right)^2 + \left(\frac{\partial Z}{\partial U}\right)^2 \\
\overline{F} &= \frac{\partial X}{\partial U}\frac{\partial X}{\partial V} + \frac{\partial Y}{\partial U}\frac{\partial Y}{\partial V} + \frac{\partial Z}{\partial U}\frac{\partial Z}{\partial V} \\
\overline{G} &= \left(\frac{\partial X}{\partial V}\right)^2 + \left(\frac{\partial Y}{\partial V}\right)^2 + \left(\frac{\partial Z}{\partial V}\right)^2
\end{aligned} \tag{5.3}$$

5.3. The angle between parametric curves on the projection surface

$$\cos\Omega = \frac{\overline{F}}{\sqrt{\overline{E}\,\overline{G}}} \tag{5.4}$$

5.4. The elemental area on the projection surface

$$dA = \overline{J}\,dU\,dV \tag{5.5}$$

where

$$\overline{J}^2 = \overline{E}\,\overline{G} - \overline{F}^2 \tag{5.6}$$

6. THE FUNDAMENTAL TRANSFORMATION MATRIX

Referring again to the general transformation or projection equations given by (2.5) and re-stated again

$$\begin{aligned} x &= f_1(u,v) \\ y &= f_2(u,v) \\ z &= f_3(u,v) \end{aligned} \qquad \text{and} \qquad \begin{aligned} X &= \overline{f}_1(u,v) \\ Y &= \overline{f}_2(u,v) \\ Z &= \overline{f}_3(u,v) \end{aligned} \tag{6.1}$$

We may write another equation for the elemental distance on the projection plane and a third set of Gaussian Fundamental Quantities *E, F* and *G* as

$$dS^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2 \tag{6.2}$$

where

$$E = \left(\frac{\partial X}{\partial u}\right)^2 + \left(\frac{\partial Y}{\partial u}\right)^2 + \left(\frac{\partial Z}{\partial u}\right)^2$$
$$F = \frac{\partial X}{\partial u}\frac{\partial X}{\partial v} + \frac{\partial Y}{\partial u}\frac{\partial Y}{\partial v} + \frac{\partial Z}{\partial u}\frac{\partial Z}{\partial v} \tag{6.3}$$
$$G = \left(\frac{\partial X}{\partial v}\right)^2 + \left(\frac{\partial Y}{\partial v}\right)^2 + \left(\frac{\partial Z}{\partial v}\right)^2$$

The angle Ω between parametric curves on the projection surface is given by $\cos\Omega = \frac{F}{\sqrt{EG}}$ and the elemental area on the projection surface given by $dA = J\,du\,dv$ where $J^2 = EG - F^2$.

Note that these relationships link the *u*,*v* curvilinear coordinates of the datum surface with the *X*,*Y*,*Z* Cartesian coordinates of the projection surface. What we would like to find is a connection between $\bar{E}$, $\bar{F}$ and $\bar{G}$ and *E*, *F* and *G*. We can derive this connection using calculus and algebra and present the connection as <u>The Fundamental Transformation Matrix</u>.

Differentiating equations (6.1) gives

$$\frac{\partial X}{\partial u} = \frac{\partial X}{\partial U}\frac{\partial U}{\partial u} + \frac{\partial X}{\partial V}\frac{\partial V}{\partial u}$$
$$\frac{\partial X}{\partial v} = \frac{\partial X}{\partial U}\frac{\partial U}{\partial v} + \frac{\partial X}{\partial V}\frac{\partial V}{\partial v}$$

Similarly, we may derive relationships for $\frac{\partial Y}{\partial u}, \frac{\partial Y}{\partial v}, \frac{\partial Z}{\partial u}, \frac{\partial Z}{\partial v}$

Substituting these differential equations into (6.3) gives

$$\begin{aligned} E = {} & \left(\frac{\partial X}{\partial U}\right)^2\left(\frac{\partial U}{\partial u}\right)^2 + 2\frac{\partial X}{\partial U}\frac{\partial U}{\partial u}\frac{\partial X}{\partial V}\frac{\partial V}{\partial u} + \left(\frac{\partial X}{\partial V}\right)^2\left(\frac{\partial V}{\partial u}\right)^2 \\ & + \left(\frac{\partial Y}{\partial U}\right)^2\left(\frac{\partial U}{\partial u}\right)^2 + 2\frac{\partial Y}{\partial U}\frac{\partial U}{\partial u}\frac{\partial Y}{\partial V}\frac{\partial V}{\partial u} + \left(\frac{\partial Y}{\partial V}\right)^2\left(\frac{\partial V}{\partial u}\right)^2 \\ & + \left(\frac{\partial Z}{\partial U}\right)^2\left(\frac{\partial U}{\partial u}\right)^2 + 2\frac{\partial Z}{\partial U}\frac{\partial U}{\partial u}\frac{\partial Z}{\partial V}\frac{\partial V}{\partial u} + \left(\frac{\partial Z}{\partial V}\right)^2\left(\frac{\partial V}{\partial u}\right)^2 \end{aligned}$$

$$\begin{aligned} F = {} & \left(\frac{\partial X}{\partial U}\right)^2\frac{\partial U}{\partial u}\frac{\partial U}{\partial v} + 2\frac{\partial X}{\partial U}\frac{\partial X}{\partial V}\left(\frac{\partial U}{\partial v}\frac{\partial V}{\partial u} + \frac{\partial U}{\partial u}\frac{\partial V}{\partial u}\right) + \left(\frac{\partial X}{\partial V}\right)^2\frac{\partial V}{\partial u}\frac{\partial V}{\partial v} \\ & + \left(\frac{\partial Y}{\partial U}\right)^2\frac{\partial U}{\partial u}\frac{\partial U}{\partial v} + 2\frac{\partial Y}{\partial U}\frac{\partial Y}{\partial V}\left(\frac{\partial U}{\partial v}\frac{\partial V}{\partial u} + \frac{\partial U}{\partial u}\frac{\partial V}{\partial u}\right) + \left(\frac{\partial Y}{\partial V}\right)^2\frac{\partial V}{\partial u}\frac{\partial V}{\partial v} \\ & + \left(\frac{\partial Z}{\partial U}\right)^2\frac{\partial U}{\partial u}\frac{\partial U}{\partial v} + 2\frac{\partial Z}{\partial U}\frac{\partial Z}{\partial V}\left(\frac{\partial U}{\partial v}\frac{\partial V}{\partial u} + \frac{\partial U}{\partial u}\frac{\partial V}{\partial u}\right) + \left(\frac{\partial Z}{\partial V}\right)^2\frac{\partial V}{\partial u}\frac{\partial V}{\partial v} \end{aligned}$$

$$G = \left(\frac{\partial X}{\partial U}\right)^2 \left(\frac{\partial U}{\partial v}\right)^2 + 2\frac{\partial X}{\partial U}\frac{\partial U}{\partial v}\frac{\partial X}{\partial V}\frac{\partial V}{\partial v} + \left(\frac{\partial X}{\partial V}\right)^2 \left(\frac{\partial V}{\partial v}\right)^2$$
$$+ \left(\frac{\partial Y}{\partial U}\right)^2 \left(\frac{\partial U}{\partial v}\right)^2 + 2\frac{\partial Y}{\partial U}\frac{\partial U}{\partial v}\frac{\partial Y}{\partial V}\frac{\partial V}{\partial v} + \left(\frac{\partial Y}{\partial V}\right)^2 \left(\frac{\partial V}{\partial v}\right)^2$$
$$+ \left(\frac{\partial Z}{\partial U}\right)^2 \left(\frac{\partial U}{\partial v}\right)^2 + 2\frac{\partial Z}{\partial U}\frac{\partial U}{\partial v}\frac{\partial Z}{\partial V}\frac{\partial V}{\partial v} + \left(\frac{\partial Z}{\partial V}\right)^2 \left(\frac{\partial V}{\partial v}\right)^2$$

Using the relationships in (5.3) we may write

$$E = \left(\frac{\partial U}{\partial u}\right)^2 \bar{E} + 2\frac{\partial U}{\partial u}\frac{\partial V}{\partial u}\bar{F} + \left(\frac{\partial V}{\partial u}\right)^2 \bar{G} \tag{6.4}$$

$$F = \left(\frac{\partial U}{\partial u}\frac{\partial U}{\partial v}\right)\bar{E} + \left(\frac{\partial U}{\partial v}\frac{\partial V}{\partial u} + \frac{\partial U}{\partial u}\frac{\partial V}{\partial v}\right)\bar{F} + \left(\frac{\partial V}{\partial u}\frac{\partial V}{\partial v}\right)\bar{G} \tag{6.5}$$

$$G = \left(\frac{\partial U}{\partial v}\right)^2 \bar{E} + 2\frac{\partial U}{\partial v}\frac{\partial V}{\partial v}\bar{F} + \left(\frac{\partial V}{\partial v}\right)^2 \bar{G} \tag{6.6}$$

Equations (6.4), (6.5) and (6.6) may be conveniently expressed as a matrix equation

$$\begin{bmatrix} E \\ F \\ G \end{bmatrix} = \begin{bmatrix} \left(\frac{\partial U}{\partial u}\right)^2 & 2\frac{\partial U}{\partial u}\frac{\partial V}{\partial u} & \left(\frac{\partial V}{\partial u}\right)^2 \\ \frac{\partial U}{\partial u}\frac{\partial U}{\partial v} & \frac{\partial U}{\partial v}\frac{\partial V}{\partial u} + \frac{\partial U}{\partial u}\frac{\partial V}{\partial v} & \frac{\partial V}{\partial u}\frac{\partial V}{\partial v} \\ \left(\frac{\partial U}{\partial v}\right)^2 & 2\frac{\partial U}{\partial v}\frac{\partial V}{\partial v} & \left(\frac{\partial V}{\partial v}\right)^2 \end{bmatrix} \begin{bmatrix} \bar{E} \\ \bar{F} \\ \bar{G} \end{bmatrix} \tag{6.7}$$

This is <u>The Transformation Matrix</u>, the coefficient matrix of (6.7), and is fundamental in the design of computer programs for map projections. Using this transformation matrix, we may deduce the basic relationships between the curvilinear coordinates on the datum surface and the Cartesian coordinates on the projection.

The term $J^2 = EG - F^2$ can be derived by combining equations (6.4), (6.5) and (6.6) to give

$$J^2 = EG - F^2 = \left(\bar{E}\bar{G} - \bar{F}^2\right)\left(\frac{\partial U}{\partial u}\frac{\partial V}{\partial v} - \frac{\partial U}{\partial v}\frac{\partial V}{\partial u}\right)^2 \tag{6.8}$$

or as the product of two determinants

$$J^2 = EG - F^2 = \begin{vmatrix} \bar{E} & \bar{F} \\ \bar{F} & \bar{G} \end{vmatrix} \begin{vmatrix} \frac{\partial U}{\partial u} & \frac{\partial U}{\partial v} \\ \frac{\partial V}{\partial v} & \frac{\partial V}{\partial u} \end{vmatrix}^2 \tag{6.9}$$

The determinant $\begin{vmatrix} \dfrac{\partial U}{\partial u} & \dfrac{\partial U}{\partial v} \\ \dfrac{\partial V}{\partial v} & \dfrac{\partial V}{\partial u} \end{vmatrix}$ is the Jacobian determinant (U,V) with respect to (u,v)

Note: (i) The Gaussian Fundamental Quantities pertaining to the <u>datum surface</u> (*x,y,z* as functions of *u* and *v*) are denoted by *e, f* and *g*.

(ii) The Gaussian Fundamental Quantities pertaining to the <u>projection surface</u> (*X,Y,Z* as functions of *U* and *V*) are denoted by $\bar{E}$, $\bar{F}$ and $\bar{G}$.

(iii) The Gaussian Fundamental Quantities pertaining to the <u>projection surface</u> (*X,Y,Z* as functions of *u* and *v*) are denoted by *E, F* and *G*.

7. SCALE FACTOR

Knowledge of <u>scale factors</u> is fundamental in understanding map projections. Using certain scale factors, or scale relationships, we may create map projections with certain useful properties. For example, map projections that preserve angles at a point (i.e., an angle between two lines on the datum surface is transformed into the same angle between the complimentary lines on the projection) are known as <u>conformal</u> and conformal projections have the unique property that scale factor is the same in every direction at a point on the projection. Therefore, we may develop the equations for a conformal map projection by enforcing a particular scale relationship.

7.1. General equation for scale factor

A general equation for scale factor *m* can be developed in the following manner.

The <u>scale factor</u> *m* is defined as the ratio elemental distances *dS* on the projection surface and *ds* on the datum surface

$$\text{scale factor} = \frac{\text{elemental distance on PROJECTION SURFACE}}{\text{elemental distance on DATUM SURFACE}}$$

or

$$m^2 = \frac{dS^2}{ds^2} \tag{7.1}$$

Using the relationships (5.2) and (3.5) the scale factor is given by

$$m^2 = \frac{dS^2}{ds^2} = \frac{\bar{E}\,dU^2 + 2\bar{F}\,dU\,dV + \bar{G}\,dV^2}{e\,du^2 + 2\,f\;du\,dv + g\,dv^2}$$

Alternatively, we may use the relationships (6.2) and (3.5)

$$m^2 = \frac{dS^2}{ds^2} = \frac{E\,du^2 + 2F\,du\,dv + G\,dv^2}{e\,du^2 + 2f\,du\,dv + g\,dv^2} \tag{7.2}$$

Dividing numerator and denominator of (7.2) by dv^2 gives

$$m^2 = \frac{E\left(\dfrac{du}{dv}\right)^2 + 2F\dfrac{du}{dv} + G}{e\left(\dfrac{du}{dv}\right)^2 + 2f\dfrac{du}{dv} + g} \tag{7.3}$$

Inspection of equation (7.3) shows that in general the scale factor at a point depends directly on the term du/dv since for the datum and projection surfaces *e, f, g* and *e, F, G* are constant for a particular point. du/dv is the ratio between elemental changes *du* and *dv* and for any curve on the datum surface this ratio will vary according to the direction of the curve. From Figure 4.1 we may express the direction of a curve on the surface as

$$\tan\alpha = \frac{\sqrt{g}\,dv}{\sqrt{e}\,du}$$

where α is a positive clockwise angle from the *v*-curve. This equation may be re-arranged to give expressions for the ratio du/dv

$$\frac{du}{dv} = \frac{\sqrt{g}}{\sqrt{e}\tan\alpha}, \quad \text{and} \quad \left(\frac{du}{dv}\right)^2 = \frac{g}{e\tan^2\alpha}$$

Substituting these expressions into (7.3) gives

$$m^2 = \frac{E\left(\dfrac{g}{e\tan^2\alpha}\right) + 2F\left(\dfrac{\sqrt{g}}{\sqrt{e}\tan\alpha}\right) + G}{e\left(\dfrac{g}{e\tan^2\alpha}\right) + 2f\left(\dfrac{\sqrt{g}}{\sqrt{e}\tan\alpha}\right) + g}$$

Noting that $1 + \tan^2\alpha = \sec^2\alpha = \dfrac{1}{\cos^2\alpha}$ and $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha}$, the denominator of this equation can be simplified giving

$$m^2 = \frac{E\left(\dfrac{g}{e\tan^2\alpha}\right) + 2F\left(\dfrac{\sqrt{g}}{\sqrt{e}\tan\alpha}\right) + G}{\dfrac{g}{\sin^2\alpha}\left(1 + \dfrac{2f}{\sqrt{eg}}\sin\alpha\cos\alpha\right)}$$

Multiplying the numerator and denominator by $\dfrac{\sin^2\alpha}{g}$ and simplifying the trigonometric expressions gives the <u>general equation for scale factor</u> *m*

$$m^2 = \frac{\left(\frac{E}{e}\right)\cos^2\alpha + \left(\frac{F}{f}\right)\frac{f}{\sqrt{eg}}2\sin\alpha\cos\alpha + \left(\frac{G}{g}\right)\sin^2\alpha}{1+\frac{f}{\sqrt{eg}}2\sin\alpha\cos\alpha} \tag{7.4}$$

7.2. Important results from the general equation for scale factor

Several important results can be deduced from the general equation for scale factor (7.4).

1. In general, scale factor varies everywhere on the map projection.

This fact can be deduced from the general equation when it is realised that the Gaussian Fundamental Quantities are functions of the curvilinear coordinates u,v of the datum surface; see equations (3.6) and (6.3). Therefore, as points vary across the datum surface their complimentary points on the projection will have varying scale factor.

2. When $\frac{E}{e} = \frac{F}{f} = \frac{G}{g}$ the scale factor is independent of direction, i.e., m is the same value in every direction about a point on the projection. Such projections are known as CONFORMAL.

We can see this by substituting a constant $K = \frac{E}{e} = \frac{F}{f} = \frac{G}{g}$ into (7.4)

$$\begin{aligned} m^2 &= \frac{K\cos^2\alpha + K\frac{f}{\sqrt{eg}}2\sin\alpha\cos\alpha + K\sin^2\alpha}{1+\frac{f}{\sqrt{eg}}2\sin\alpha\cos\alpha} \\ &= \frac{K\left(\cos^2\alpha + \sin^2\alpha + \frac{f}{\sqrt{eg}}2\sin\alpha\cos\alpha\right)}{1+\frac{f}{\sqrt{eg}}2\sin\alpha\cos\alpha} \\ &= \frac{K\left(1 + \frac{f}{\sqrt{eg}}2\sin\alpha\cos\alpha\right)}{1+\frac{f}{\sqrt{eg}}2\sin\alpha\cos\alpha} \\ &= K \end{aligned}$$

Conformal projections, where the scale factor is the same in every direction around a point have the property that shape is preserved. By this, we mean that an object on the datum surface, say a square, is transformed into a square on the projection surface although it may be enlarged or reduced by a constant amount. Preservation of shape also means that angles at a point are preserved. By this, we mean that an angle between two lines radiating from a point on the datum surface will be identical to the angle between the two projected lines on the projection surface. There is one minor drawback: these properties only hold true for differentially small areas since the relationships have been established from the differential ratio $m^2 = dS^2 / ds^2$. Nevertheless, these properties make conformal projections the most appropriate map projections for topographic mapping; since measurements in the field, corrected to the appropriate datum surface, need little or no further correction and can be added directly to a conformal map. This fact becomes more obvious when we consider the size of the Earth (the datum surface) and any practicable mapping area we might be working on. Consider a 1:100,000 Topographic map sheet used in Australia. This map series is based on a conformal projection (Universal Transverse Mercator) of latitudes and longitudes of points related to the ellipsoid and cover 0º 30' of latitude and longitude. This equates roughly to 2,461,581,000 m^2 of the Earth's surface. The surface area of the Australian National Spheroid, a reasonable approximation to the Earth, is 5.1006927×10^{14} m^2, which means the map sheet is 0.000483% of the Earth's surface. Thus, the entire map sheet can be regarded as an extremely small portion of the Earth's surface.

3. In practical applications of map projection, the *u*,*v* curvilinear coordinates of the datum surface relate to a set of parametric *u* and *v* curves that intersect everywhere at right angles. Such networks of lines are known as orthogonal coordinate systems; and on the surface of the Earth (sphere or ellipsoid) we have such a system: parallels of latitude (ϕ curves) and meridians of longitude (λ curves). Therefore, in any of the general relationships we have developed we may replace *u* with ϕ and *v* with λ. Furthermore, since meridians and parallels intersect everywhere at right angles $\left(\omega = 90^\circ\right)$ implies that the Gaussian Fundamental Quantity $f = 0$; see equation (4.4). This fact was verified in Section 3.1 where the Gaussian Fundamental Quantities were computed for a spherical Earth as the datum surface, giving $e = R^2,\ f = 0,\ g = R^2 \cos^2 \phi$. Using this fact, the general equation for scale, where the datum surface is a sphere or ellipsoid with meridian and parallels as the parametric curves is

$$m^2 = \left(\frac{E}{e}\right)\cos^2\alpha + \frac{F}{\sqrt{eg}} 2\sin\alpha\cos\alpha + \left(\frac{G}{g}\right)\sin^2\alpha \tag{7.5}$$

4. Consider the case where the datum surface is a sphere or ellipsoid with meridian and parallels as the parametric curves and two points *P* and *Q* an elemental distance *ds* apart. When *Q* is on the meridian passing through *P* then α, the azimuth of the line *PQ* on the datum surface, is 0° or 180° and $\cos\alpha = 1$ and $\sin\alpha = 0$. We can see from (7.5) that when $\alpha = 0^\circ$ or 180° the second and third terms vanish and we have

The meridian scale factor *h*

$$h = \frac{\sqrt{E}}{\sqrt{e}} \tag{7.6}$$

Similarly, when *Q* is on the parallel passing through *P* then $\alpha = 90^\circ$ or 270° and $\cos\alpha = 0$ and $\sin\alpha = 1$. We can see from (7.5) that the first and second terms vanish and we have

The parallel scale factor k

$$k = \frac{\sqrt{G}}{\sqrt{g}} \tag{7.7}$$

5. In many map projections, the parametric curves U,V on the projection surface form an orthogonal network of lines (or curves); this implies that the Gaussian Fundamental Quantity $\bar{F} = 0$ and as we shall see in following sections if these lines or curves coincide with the projection surface coordinate system (X,Y for Cartesian coordinates or r,θ for polar coordinates) then $F = 0$. In such cases, where the datum surface is a sphere or ellipsoid with meridian and parallels as the parametric curves the general equation for scale becomes

$$m^2 = \left(\frac{E}{e}\right)\cos^2\alpha + \left(\frac{G}{g}\right)\sin^2\alpha \tag{7.8}$$

In these cases, if $\frac{E}{e} = \frac{G}{g}$ then the scale factor is a constant value and is independent of direction, i.e., the projection is conformal. We can see this by substituting $K = \frac{E}{e} = \frac{G}{g}$ into (7.8) giving

$$\begin{aligned} m^2 &= K\cos^2\alpha + K\sin^2\alpha \\ &= K\left(\cos^2\alpha + \sin^2\alpha\right) \\ &= K \end{aligned}$$

This leads us to another definition of conformal projections:

If $f = F = 0$ and $h = k$ then the projection is conformal

7.3. Tissot's Indicatrix Ellipse

By making the substitutions $E' = E/e$, $F' = F/\sqrt{eg}$, $G' = G/g$ equation (7.5) can be written as

$$m^2 = E'\cos^2\alpha + 2F'\sin\alpha\cos\alpha + G'\sin^2\alpha \tag{7.9}$$

Equation (7.9) defines the *pedal curve* of Tissot's Indicatrix Ellipse

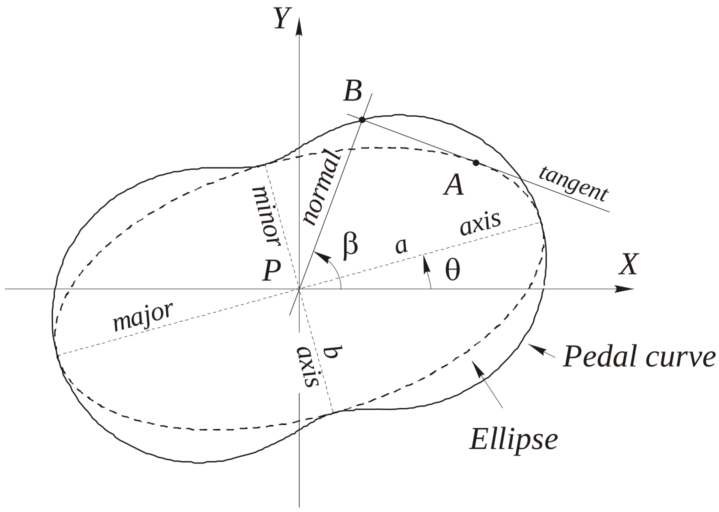


Figure 7.1. Tissot's Indicatrix Ellipse and pedal curve

In Figure 7.1, *A* is a point on an ellipse. The tangent to the ellipse at *A* intersects a normal to the tangent passing through *P* at *B*. As *A* moves around the ellipse, the locus of all points *B* is the pedal curve of the ellipse. The distance $PB = m^2$ for the angle β. The maximum and minimum values of m^2 define the directions and lengths of the axes of Tissot's Indicatrix Ellipse. It can be shown that m^2 has a maximum value when

$$\tan 2\theta = \frac{2F'}{E' - G'} \tag{7.10}$$

The semi-axes lengths of the ellipse are

$$\begin{aligned} a &= \sqrt{m^2_{\max}} = \sqrt{\tfrac{1}{2}\left(E' + G' + W\right)} \\ b &= \sqrt{m^2_{\min}} = \sqrt{\tfrac{1}{2}\left(E' + G' - W\right)} \end{aligned} \tag{7.11}$$

where $W = \sqrt{\left(E' - G'\right)^2 + 4\left(F'\right)^2}$. Tissot's Indicatrix Ellipse provides a visual display of the scale distortions on a map projection and the parameters *a* and *b* (the semi-axes lengths) can be used to give

Maximum angular distortion Ω

$$\Omega = \sin^{-1}\left(\frac{a-b}{a+b}\right) \tag{7.12}$$

Area scale factor $J' = \dfrac{J}{j}$

$$J' = a\,b \tag{7.13}$$

7.4. Area scale factor

The area scale factor is defined as the ratio of elemental areas on the datum surface and the projection surface

$$\text{area scale factor} = \frac{\text{elemental area on PROJECTION SURFACE}}{\text{elemental area on DATUM SURFACE}} = \frac{dA}{da}$$

Using equations (5.5) and (4.7)

$$\text{area scale factor} = \frac{dA}{da} = \frac{\bar{J}\, dU\, dV}{j\, du\, dv} \tag{7.14}$$

But, from (6.8) $J^2 = EG - F^2 = \left(\bar{E}\bar{G} - \bar{F}^2\right)\left(\frac{\partial U}{\partial u}\frac{\partial V}{\partial v} - \frac{\partial U}{\partial v}\frac{\partial V}{\partial u}\right)^2$ and since $\bar{J}^2 = \bar{E}\bar{G} - \bar{F}^2$ we may write

$$\bar{J} = \frac{J}{\frac{\partial U}{\partial u}\frac{\partial V}{\partial v} - \frac{\partial U}{\partial v}\frac{\partial V}{\partial u}}$$

Substituting this expression for $\bar{J}$ into (7.14) gives

$$\text{area scale factor} = \frac{1}{\left(\frac{\partial U}{\partial u}\frac{\partial V}{\partial v} - \frac{\partial U}{\partial v}\frac{\partial V}{\partial u}\right)}\left\{\frac{J\, dU\, dV}{j\, du\, dv}\right\} \tag{7.15}$$

Equation (7.15) is given in terms of differentials and partial derivatives of the parametric curves on the datum surface and the projection surface.

A more useful expression for area scale factor can be written in terms of differentials of the parametric curves on the datum surface only. Referring to Section 6, equation (6.2) where we expressed elemental distance on the projection surface in terms of elemental changes in the parametric curves on the datum surface, i.e., $dS^2 = E\, du^2 + 2F\, du\, dv + G\, dv^2$ where *E, F* and *G* are given by (6.3) we also have elemental area

$$dA = J\, du\, dv \tag{7.16}$$

where $J^2 EG - F^2$. Hence, area scale factor can be expressed as

$$\text{area scale factor} = \frac{dA}{da} = \frac{J\, du\, dv}{j\, du\, dv} = \frac{J}{j} \tag{7.17}$$

8. GAUSSIAN FUNDAMENTAL QUANTITIES FOR THE SPHERE

For most of our study of map projections, the datum surface is a sphere of radius R representing the Earth, with parallels of latitude (ϕ curves) and meridians of longitude (λ curves) as the parametric curves on the surface. The functional relationships connecting x,y,z Cartesian coordinates and ϕ,λ curvilinear coordinates are

$$\begin{aligned} x &= f_1(\phi,\lambda) = R\cos\phi\cos\lambda \\ y &= f_2(\phi,\lambda) = R\cos\phi\sin\lambda \\ z &= f_3(\phi,\lambda) = R\sin\phi \end{aligned} \tag{8.1}$$

The Gaussian Fundamental Quantities e, f and g are given in the general formula (3.6) and we may replace u with ϕ and v with λ and write them as

$$\begin{aligned} e &= \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2 \\ f &= \frac{\partial x}{\partial \phi}\frac{\partial x}{\partial \lambda} + \frac{\partial y}{\partial \phi}\frac{\partial y}{\partial \lambda} + \frac{\partial z}{\partial \phi}\frac{\partial z}{\partial \lambda} \\ g &= \left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2 + \left(\frac{\partial z}{\partial \lambda}\right)^2 \end{aligned} \tag{8.2}$$

Differentiating equations (8.1) gives

$$\begin{aligned} \frac{\partial x}{\partial \phi} &= -R\sin\phi\cos\lambda & \qquad \frac{\partial x}{\partial \lambda} &= -R\cos\phi\sin\lambda \\ \frac{\partial y}{\partial \phi} &= -R\sin\phi\sin\lambda & \qquad \frac{\partial x}{\partial \lambda} &= R\cos\phi\cos\lambda \\ \frac{\partial x}{\partial \phi} &= R\cos\phi & \qquad \frac{\partial x}{\partial \lambda} &= 0 \end{aligned}$$

Substituting these derivatives into (8.2) gives

$$\begin{aligned} e &= (-R\sin\phi\cos\lambda)^2 + (-R\sin\phi\sin\lambda)^2 + (R\cos\phi)^2 \\ &= R^2\left(\sin^2\phi\cos^2\lambda + \sin^2\phi\sin^2\lambda + \cos^2\phi\right) \\ &= R^2\left(\sin^2\phi\left(\cos^2\lambda + \sin^2\lambda\right) + \cos^2\phi\right) \\ &= R^2\left(\sin^2\phi + \cos^2\phi\right) \\ &= R^2 \end{aligned} \tag{8.3}$$

$$\begin{aligned} f &= (-R\sin\phi\cos\lambda)(-R\cos\phi\sin\lambda) + (-R\sin\phi\sin\lambda)(R\cos\phi\cos\lambda) + (R\cos\phi)(0) \\ &= \left(R^2\sin\phi\cos\lambda\cos\phi\sin\lambda\right) - \left(R^2\sin\phi\sin\lambda\cos\phi\cos\lambda\right) + (0) \\ &= 0 \end{aligned} \tag{8.4}$$

$$
\begin{aligned}
g &= \left(-R\cos\phi\sin\lambda\right)^2 + \left(R\cos\phi\cos\lambda\right)^2 + \left(0\right)^2 \\
&= R^2\left(\cos^2\phi\sin^2\lambda + \cos^2\phi\cos^2\lambda\right) \\
&= R^2\left(\cos^2\phi\left(\cos^2\lambda + \sin^2\lambda\right)\right) \\
&= R^2\cos^2\phi
\end{aligned}
\tag{8.5}
$$

The elemental distance on the sphere of radius R with parametric curves ϕ (latitude) and λ (longitude) is

$$
ds^2 = e\,d\phi^2 + 2f\,d\phi\,d\lambda + g\,d\lambda^2 \tag{8.6}
$$

and the Gaussian Fundamental Quantities for the sphere are

$$
\begin{aligned}
e &= R^2 \\
f &= 0 \\
g &= R^2\cos^2\phi
\end{aligned}
\tag{8.7}
$$

9. CYLINDRICAL PROJECTIONS

In elementary texts on map projections, the projection surfaces are often described as developable surfaces, such as the cylinder (cylindrical projections) and the cone (conical projections), or a plane (azimuthal projections). These surfaces are imagined as enveloping or touching the datum surface and by some means, usually geometric, the meridians, parallels and features are projected onto these surfaces. In the case of the cylinder, it is cut and laid flat (developed). If the axis of the cylinder coincides with the axis of the Earth, the projection is said to be normal aspect, if the axis lies in the plane of the equator the projection is known as transverse and in any other orientation it is known as oblique. [It is usual that the descriptor "normal" is implied in the name of a projection, but for different orientations, the words "transverse" or "oblique" are added to the name.] This simplified approach is not adequate for developing a general theory of projections (which as we can see is quite mathematical) but is useful for describing characteristics of certain projections. In the case of cylindrical projections, some characteristics are a common feature:

(i) Meridians of longitude and parallels of latitude form an orthogonal network of straight parallel lines.
(ii) Meridians are equally spaced straight parallel lines intersecting parallels at right angles.
(iii) Parallels, in general, are unequally spaced straight parallel lines but are symmetric about the equator.

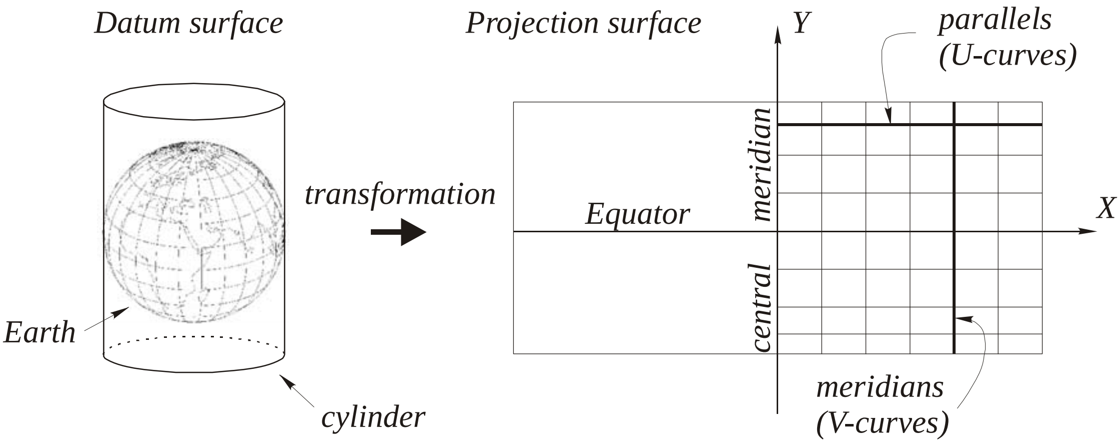


Figure 9.1 Cylindrical projection. *u*,*v* curves on the datum surface projected as *U*,*V* curves on the projection surface

Figure 9.1 shows a schematic diagram of a cylindrical projection demonstrating the basic characteristics common to all cylindrical projections (normal aspect). Cylindrical projections often have a square or rectangular shape. In the case of projections showing the whole of the earth (or nearly so), the *X*-axis coincides with the equator and the *Y*-axis coincides with the central meridian. As can be seen, the *U*-curves (parallels of latitude) are parallel to the *X*-axis and the *V*-curves (meridians of longitude) are parallel to the *Y*-axis.

9.1. The Gaussian Fundamental Quantities of Cylindrical projections

For cylindrical projections, the projection surface is a plane and the *U*,*V* curvilinear coordinate system is an orthogonal system of *U* and *V*-curves that are straight lines parallel with the *X* and *Y*-axes respectively. The functional relationships connecting the *U,V* coordinate system with the *X,Y,Z* Cartesian coordinate system were given previously by equations (2.2) and are restated here in more explicit form that recognises the fact that *X* is a function of *V* only, *Y* is a function of *U* only and $Z = 0$

$$\begin{aligned} X &= F_1(U,V) = V \\ Y &= F_2(U,V) = U \\ Z &= F_3(U,V) = 0 \end{aligned} \tag{9.1}$$

The Gaussian Fundamental Quantities of the projection surface are

$$\begin{aligned} \bar{E} &= \left(\frac{\partial X}{\partial U}\right)^2 + \left(\frac{\partial Y}{\partial U}\right)^2 &= 1 \\ \bar{F} &= \frac{\partial X}{\partial U}\frac{\partial X}{\partial V} + \frac{\partial Y}{\partial U}\frac{\partial Y}{\partial V} &= 0 \\ \bar{G} &= \left(\frac{\partial X}{\partial V}\right)^2 + \left(\frac{\partial Y}{\partial V}\right)^2 &= 1 \end{aligned} \tag{9.2}$$

Now, considering the datum surface to be a sphere of radius *R* and the *u*,*v* curves as parallels and meridians ϕ, λ, the Gaussian Fundamental Quantities *E*,*F*,*G* relating to the functional relationships

$$X = \bar{f}_1(\phi, \lambda) \qquad Y = \bar{f}_2(\phi, \lambda) \tag{9.3}$$

are

$$E = \left(\frac{\partial X}{\partial \phi}\right)^2 + \left(\frac{\partial Y}{\partial \phi}\right)^2 \qquad F = \frac{\partial X}{\partial \phi}\frac{\partial X}{\partial \lambda} + \frac{\partial Y}{\partial \phi}\frac{\partial Y}{\partial \lambda} \qquad G = \left(\frac{\partial X}{\partial \lambda}\right)^2 + \left(\frac{\partial Y}{\partial \lambda}\right)^2 \tag{9.4}$$

Equations (9.4) can be obtained from the Transformation Matrix (6.7) in the following way

$$\begin{bmatrix} E \\ F \\ G \end{bmatrix} = \begin{bmatrix} \left(\frac{\partial Y}{\partial \phi}\right)^2 & 2\frac{\partial Y}{\partial \phi}\frac{\partial X}{\partial \phi} & \left(\frac{\partial X}{\partial \phi}\right)^2 \\ \frac{\partial Y}{\partial \phi}\frac{\partial Y}{\partial \lambda} & \frac{\partial Y}{\partial \lambda}\frac{\partial X}{\partial \phi} + \frac{\partial Y}{\partial \phi}\frac{\partial X}{\partial \lambda} & \frac{\partial X}{\partial \phi}\frac{\partial X}{\partial \lambda} \\ \left(\frac{\partial Y}{\partial \lambda}\right)^2 & 2\frac{\partial Y}{\partial \lambda}\frac{\partial X}{\partial \lambda} & \left(\frac{\partial X}{\partial \lambda}\right)^2 \end{bmatrix} \begin{bmatrix} \bar{E} \\ \bar{F} \\ \bar{G} \end{bmatrix} \tag{9.5}$$

where *X* has replaced *V* and *Y* has replaced *U*, since they are parallel to the *X* and *Y* axes and where $\bar{E} = 1,\ \bar{F} = 0,\ \bar{G} = 1$.

Now in Cylindrical projections, *Y* is a function of ϕ only and *X* is a function of λ only, since our *U*-curves are parallels of latitude and *V*-curves are meridians of longitude. Hence, equations (2.3) can be written as

$$X = g_1(\lambda) \qquad Y = g_2(\phi) \tag{9.6}$$

noting that *X* has replaced *V*, *Y* has replaced *U*, ϕ has replaced *u* and λ has replaced *v*. From equations (9.6) we can see that

$$\frac{\partial X}{\partial \phi} = 0 \quad \text{and} \quad \frac{\partial Y}{\partial \lambda} = 0$$

Substituting these derivatives into (9.4) gives the <u>Gaussian Fundamental Quantities *E*,*F*,*G* for normal aspect Cylindrical projections</u>

$$E = \left(\frac{\partial Y}{\partial \phi}\right)^2, \quad F = 0, \quad G = \left(\frac{\partial X}{\partial \lambda}\right)^2 \tag{9.7}$$

Using these differential relationships and particular geometric and scale conditions we can derive *CONFORMAL*, *EQUAL AREA* and *EQUIDISTANT* Cylindrical projections.

The scale conditions are:

For CONFORMAL Cylindrical projections: $\frac{E}{e} = \frac{G}{g} = m^2$

EQUAL AREA Cylindrical projections: $\frac{J}{j} = 1$

EQUIDISTANT Cylindrical projections: $\frac{E}{e} = 1$

In addition, since the datum surface is a sphere of radius *R*, the Gaussian Fundamental Quantities of the datum surface are

$$\begin{aligned} e &= R^2 \\ f &= 0 \\ g &= R^2 \cos^2 \phi \end{aligned} \tag{9.8}$$

9.2. Conformal Cylindrical Projection (Mercator's projection)

For a Conformal Cylindrical projection the scale condition to be enforced is

$$\frac{E}{e} = \frac{G}{g} = m^2 \tag{9.9}$$

Alternatively, using the notation for meridian and parallel scale factors we may write the *scale condition* as

$$h = k \tag{9.10}$$

where

$$h = \frac{\sqrt{E}}{\sqrt{e}} = \frac{dY}{R\,d\phi} \tag{9.11}$$

and

$$k = \frac{\sqrt{G}}{\sqrt{g}} = \frac{dX}{R\cos\phi\,d\lambda} \tag{9.12}$$

Using (9.10), (9.11) and (9.12) gives this scale condition as

$$\frac{dY}{d\phi} = \frac{1}{\cos\phi}\frac{dX}{d\lambda} \tag{9.13}$$

To simplify this equation, we can enforce a particular scale condition: <u>that the scale factor along the equator be unity</u>. Using (9.12), this condition can be written as

$$k_0 = \frac{dX}{R\cos\phi_0\, d\lambda} = 1 \tag{9.14}$$

Now since $\phi_0 = 0^\circ$ and $\cos\phi_0 = 1$ this particular scale condition gives the differential equation

$$dX = R\, d\lambda \tag{9.15}$$

Integrating (9.15) gives

$$X = R\int d\lambda = R\lambda + C_1$$

C_1 is a constant of integration that can be evaluated by considering that when $X = 0,\ \lambda = \lambda_0$ and $C_1 = -R\lambda_0$ giving

$$\boxed{X = R\left(\lambda - \lambda_0\right)} \tag{9.16}$$

Substituting (9.15) into (9.13) gives the differential equation

$$dY = \frac{R\, d\phi}{\cos\phi} \tag{9.17}$$

Integration gives

$$\begin{aligned} Y &= R\int \frac{1}{\cos\phi}\, d\phi \\ &= R\ln\left\{\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)\right\} + C_2 \end{aligned}$$

C_2 is a constant of integration that can be evaluated by considering that when $Y = 0,\ \phi = 0^\circ,\ \tan\left(\frac{\pi}{4} + \frac{0}{2}\right) = \tan\frac{\pi}{4} = 1$ and $\ln(1) = 0$. Hence $C_2 = 0$ giving

$$\boxed{Y = R\ln\left\{\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)\right\}} \tag{9.18}$$

Equations (9.16) and (9.18) are the projection equations for a Conformal Cylindrical projection, known commonly as *Mercator's projection*.

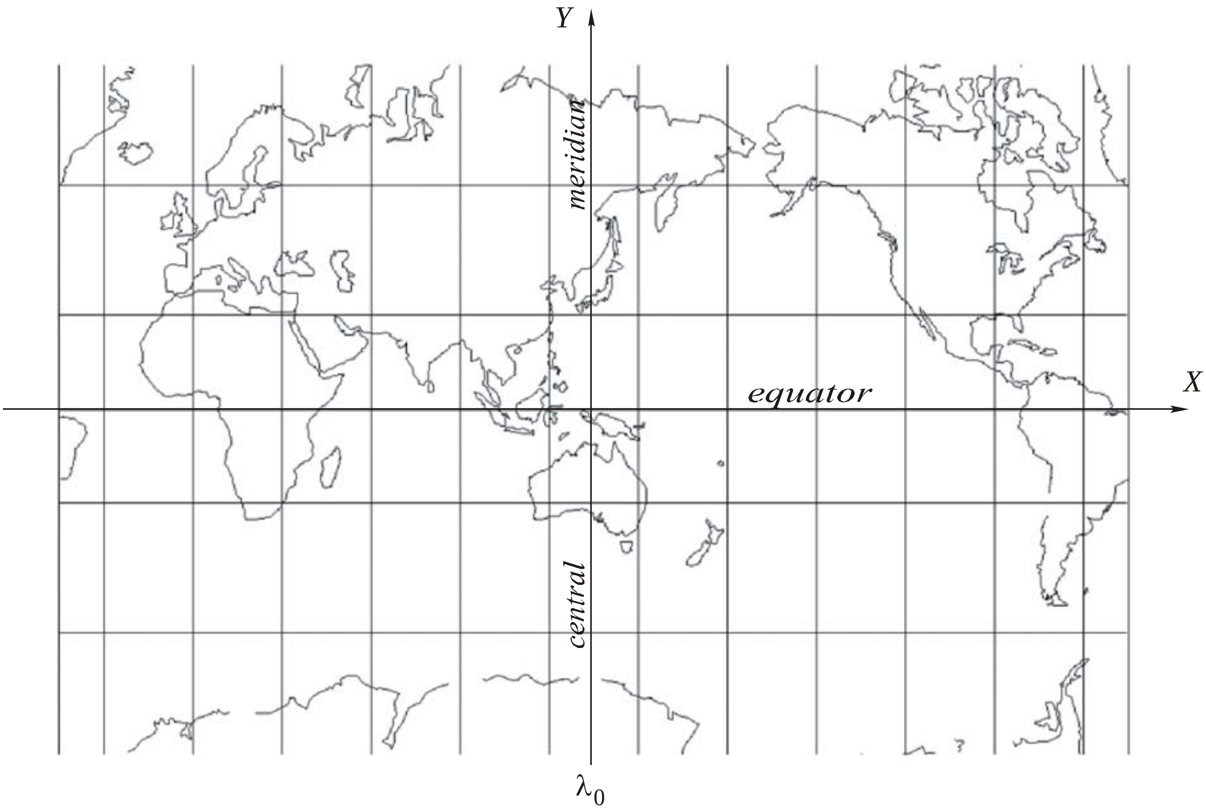


Figure 9.2 Mercator's projection (Cylindrical Conformal).
Scale 1:270 million, graticule interval 30º, central meridian $\lambda_0 = 135^\circ$

Alternative expressions for *Y* may be obtained by the following:

(i) From the *half-angle* trigonometric identities

$$\tan\frac{A}{2} = \sqrt{\frac{1-\cos A}{1+\cos A}}$$

and $$\tan\left(\frac{\pi}{4}+\frac{A}{2}\right) = \sqrt{\frac{1-\cos\left(\frac{\pi}{2}+A\right)}{1+\cos\left(\frac{\pi}{2}+A\right)}}$$

and since $\cos\left(\frac{\pi}{2}+A\right) = -\sin A$ then $\tan\left(\frac{\pi}{4}+\frac{A}{2}\right) = \sqrt{\frac{1+\sin A}{1-\sin A}}$

Substituting into (9.18) gives $$Y = R\ln\left(\frac{1+\sin\phi}{1-\sin\phi}\right)^{\frac{1}{2}} \tag{9.19}$$

(ii) Using the *law of logarithms*

$$\log_a M^p = p\log_a M$$

equation (9.19) becomes
$$Y=\frac{R}{2}\ln\left(\frac{1+\sin\phi}{1-\sin\phi}\right) \tag{9.20}$$

(iii) If colatitudes χ (chi) are used then $\chi=90-\phi$ or $\phi=90-\chi$. Therefore

$$\frac{\pi}{4}+\frac{\phi}{2}=\frac{\pi}{4}+\frac{1}{2}\left(\frac{\pi}{2}-\chi\right)=\frac{\pi}{4}+\frac{\pi}{4}-\frac{\chi}{2}=\frac{\pi}{2}-\frac{\chi}{2}$$

and
$$Y=R\ln\left\{\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\right\}=R\ln\left\{\tan\left(\frac{\pi}{2}-\frac{\chi}{2}\right)\right\}$$
$$=R\ln\left\{\cot\left(\frac{\chi}{2}\right)\right\}$$

and
$$Y=-R\ln\left\{\tan\left(\frac{\chi}{2}\right)\right\} \tag{9.21}$$

9.2.1. Properties of Mercator's Projection

(i) Projection is Conformal

(ii) Gaussian Fundamental Quantities *E*,*F*,*G*

The projection equations are:
$$\boxed{\begin{aligned}X&=R(\lambda-\lambda_0)\\ Y&=R\ln\left\{\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\right\}\end{aligned}} \tag{9.22}$$

Using these equations, we may determine expressions for the derivatives
$\dfrac{\partial X}{\partial\phi},\dfrac{\partial X}{\partial\lambda},\dfrac{\partial Y}{\partial\phi},\dfrac{\partial Y}{\partial\lambda}$

$$\frac{\partial X}{\partial\phi}=0 \quad\text{and}\quad \frac{\partial X}{\partial\lambda}=R \tag{9.23}$$

To find $\dfrac{\partial Y}{\partial\phi}$ Let $z=\dfrac{\pi}{4}+\dfrac{\phi}{2}$, then $\dfrac{\partial z}{\partial\phi}=\dfrac{1}{2}$ and $Y=R\ln(\tan z)$

Now, let $u=\tan z$, then $Y=R\ln u$, $\dfrac{\partial Y}{\partial u}=\dfrac{R}{u}$ and $\dfrac{\partial u}{\partial z}=\sec^2 z$

$\dfrac{\partial Y}{\partial\phi}$ can now be found using the chain rule for derivatives

$$\frac{\partial Y}{\partial \phi} = \frac{\partial Y}{\partial u}\frac{\partial u}{\partial z}\frac{\partial z}{\partial \phi} = \left(\frac{R}{u}\right)\left(\sec^2 z\right)\left(\frac{1}{2}\right)$$

$$= \frac{R}{2}\frac{\cos\left(\frac{\pi}{4}+\frac{\phi}{2}\right)}{\sin\left(\frac{\pi}{4}+\frac{\phi}{2}\right)}\frac{1}{\cos^2\left(\frac{\pi}{4}+\frac{\phi}{2}\right)}$$

$$= \frac{R}{2\sin A\cos A} \qquad \text{where } A = \frac{\pi}{4}+\frac{\phi}{2}$$

$$= \frac{R}{\sin 2A}$$

$$= \frac{R}{\sin\left(\frac{\pi}{2}+\phi\right)}$$

$$= \frac{R}{\cos\phi}$$

hence $\dfrac{\partial Y}{\partial \phi} = \dfrac{R}{\cos\phi}$ and $\dfrac{\partial Y}{\partial \lambda} = 0$ (9.24)

Substituting (9.23) and (9.24) into the general equations for the Gaussian Fundamental Quantities E,F,G gives

$$\begin{aligned} E &= \left(\frac{\partial X}{\partial \phi}\right)^2 + \left(\frac{\partial Y}{\partial \phi}\right)^2 &&= \frac{R^2}{\cos^2\phi} \\ F &= \frac{\partial X}{\partial \phi}\frac{\partial X}{\partial \lambda} + \frac{\partial Y}{\partial \phi}\frac{\partial Y}{\partial \lambda} &&= 0 \\ G &= \left(\frac{\partial X}{\partial \lambda}\right)^2 + \left(\frac{\partial Y}{\partial \lambda}\right)^2 &&= R^2 \\ J &= \sqrt{EG-F^2} &&= \frac{R^2}{\cos\phi} \end{aligned} \tag{9.25}$$

(iii) Scale Factors *h* (meridian) and *k* (parallel)

Using equations (9.11) and (9.12)

$$\begin{aligned} h &= \frac{\sqrt{E}}{\sqrt{e}} = \frac{R}{\cos\phi}\frac{1}{R} = \frac{1}{\cos\phi} \\ k &= \frac{\sqrt{G}}{\sqrt{g}} = R\frac{1}{\cos\phi} = \frac{1}{\cos\phi} \end{aligned} \tag{9.26}$$

Note that the scale factors *h* and *k* are equal and the projection is conformal since $f = F = 0$.

(iv) Scale factor along equator $k_0 = 1$

9.2.2. The Loxodrome and Mercator's projection

Mercator's projection has the unique property that a loxodrome on the datum surface of the Earth (sphere or ellipsoid) is shown as a straight line on the projection.

Figure 9.3 – Loxodrome on the Earth's surface.

A *loxodrome* or *rhumb line* is a curved line on the sphere (or ellipsoid) such that every element of the curve *ds* intersects a meridian at a fixed angle α. In marine and air navigation, aircraft and ships sailing or flying on fixed compass headings are moving along loxodromes, hence knowledge of loxodromes is important in navigation.

Formulae for computation of loxodromic distance and azimuth <u>on the sphere</u> can be derived by considering an *elemental rectangle* on the surface

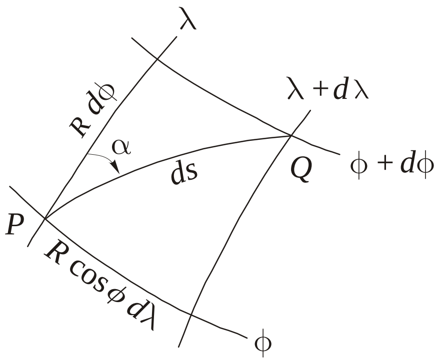


Figure 9.4 – The Elemental Rectangle on the spherical Earth

In Figure 9.4, *P* and *Q* are two points on the surface of the spherical Earth separated by $d\phi$ and $d\lambda$; elemental changes in latitude and longitude respectively (the parametric curves of meridians and parallels). *R* is the radius of the Earth, *ds* is the elemental distance between *P* and *Q* and α is the azimuth of the element of distance.

Two differential relationships can be determined from the diagram

$$\tan\alpha = \frac{R\cos\phi\, d\lambda}{R\, d\phi} \tag{9.27}$$

$$ds\cos\alpha = R\, d\phi \tag{9.28}$$

To determine the azimuth of a loxodrome between two points we may write (9.27) as

$$d\lambda = \tan\alpha \frac{d\phi}{\cos\phi}$$

and since the azimuth is constant then $\tan\alpha = \text{constant}$ and integration gives

$$\int_{\lambda_1}^{\lambda_2} d\lambda = \tan\alpha \int_{\phi_1}^{\phi_2} \frac{d\phi}{\cos\phi}$$

Knowing that $\int \frac{d\phi}{\cos\phi} = \ln\left\{\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)\right\}$ (a standard integral result) gives

$$\lambda_2 - \lambda_1 = \tan\alpha\left[\ln\left\{\tan\left(\frac{\pi}{4} + \frac{\phi_2}{2}\right)\right\} - \ln\left\{\tan\left(\frac{\pi}{4} + \frac{\phi_1}{2}\right)\right\}\right]$$

which can be rearranged to give

$$\tan\alpha = \frac{\lambda_2 - \lambda_1}{\ln\left\{\tan\left(\frac{\pi}{4} + \frac{\phi_2}{2}\right)\right\} - \ln\left\{\tan\left(\frac{\pi}{4} + \frac{\phi_1}{2}\right)\right\}} \tag{9.29}$$

The distance along the loxodrome between two points can be obtained by writing (9.28) as

$$ds = \frac{R\,d\phi}{\cos\alpha}$$

and since the azimuth and *R* are constants, the length of the loxodrome is given by integration as

$$s = \int ds = \frac{R}{\cos\alpha}\int_{\phi_1}^{\phi_2} d\phi$$

which is evaluated as

$$s = \frac{R}{\cos\alpha}\left(\phi_2 - \phi_1\right) \tag{9.30}$$

Hence given two points on the surface of the spherical Earth, the azimuth of the loxodrome between them is computed from (9.29) and then the distance is computed from (9.30).

Now, the bearing θ between two points *A* and *B* on Mercator's projection having *X*,*Y* coordinates is given by

$$\tan\theta = \frac{\Delta X_{AB}}{\Delta Y_{AB}}$$

and using the projection equations (9.22) the bearing θ can be written as

$$\begin{aligned}\tan\theta = \frac{\Delta X_{AB}}{\Delta Y_{AB}} &= \frac{R\left(\lambda_B - \lambda_0\right) - R\left(\lambda_A - \lambda_0\right)}{R\ln\left\{\tan\left(\frac{\pi}{4} - \frac{\phi_B}{2}\right)\right\} - R\ln\left\{\tan\left(\frac{\pi}{4} - \frac{\phi_A}{2}\right)\right\}} \\ &= \frac{\lambda_B - \lambda_A}{\ln\left\{\tan\left(\frac{\pi}{4} - \frac{\phi_B}{2}\right)\right\} - \ln\left\{\tan\left(\frac{\pi}{4} - \frac{\phi_A}{2}\right)\right\}}\end{aligned} \tag{9.31}$$

<u>Inspection of equations (9.31) and (9.29) reveals that the bearing on the map projection between two points is identical to the azimuth of the loxodrome between the same two points on the Earth's spherical surface.</u> This is the unique property possessed by Mercator's projection, which has made it such an invaluable projection for navigation and exploration.

9.3. EQUAL AREA CYLINDRICAL PROJECTION

For an Equal Area Cylindrical projection the scale condition to be enforced is

$$\frac{J}{j} = 1 \tag{9.32}$$

where $J^2 = EG - F^2$ and $j = eg - f^2$. Since the meridians and parallels on the datum surface (a sphere) intersect at right angles then $f = 0$ and the *U* and *V* curves on the projection surface also intersect at right angles (a property of normal aspect cylindrical projections), hence $F = 0$. Therefore, the scale condition (9.32) can be written as

$$\frac{J}{j}=\frac{\sqrt{EG-F^2}}{\sqrt{eg-f^2}}=\frac{\sqrt{EG}}{\sqrt{eg}}=\frac{\sqrt{E}}{\sqrt{e}}\frac{\sqrt{G}}{\sqrt{g}}=1$$

Now, the meridian scale factor $h=\frac{\sqrt{E}}{\sqrt{e}}$ and the parallel scale factor $k=\frac{\sqrt{G}}{\sqrt{g}}$; this leads to the scale condition for Equal Area Cylindrical projections being expressed as

$$h \times k = 1 \tag{9.33}$$

Where, from (9.7) and (9.8)

$$h=\frac{\sqrt{E}}{\sqrt{e}}=\frac{dY}{R\,d\phi} \tag{9.34}$$

$$k=\frac{\sqrt{G}}{\sqrt{g}}=\frac{dX}{R\cos\phi\,d\lambda} \tag{9.35}$$

Using (9.33), (9.35) and (9.35) gives this scale condition as

$$\frac{dY}{R\,d\phi}\frac{dX}{R\cos\phi\,d\lambda}=1 \tag{9.36}$$

In a similar way to the derivation of the projection equations for the Conformal Cylindrical projection we simplify this equation by enforcing a particular scale condition: <u>that the scale factor along the equator be unity</u>. Using (9.35), this condition can be written as

$$k_0=\frac{dX}{R\cos\phi_0\,d\lambda}=1 \tag{9.37}$$

Now since $\phi_0=0^\circ$ and $\cos\phi_0=1$ this particular scale condition gives the differential equation

$$dX = R\,d\lambda \tag{9.38}$$

Integrating (9.38) gives

$$X = R\int d\lambda = R\lambda + C_1$$

C_1 is a constant of integration that can be evaluated by considering that when $X=0$, $\lambda=\lambda_0$ and $C_1=-R\lambda_0$ giving

$$\boxed{X = R\left(\lambda-\lambda_0\right)} \tag{9.39}$$

Substituting (9.38) into (9.36) gives the differential equation

$$dY = R\cos\phi\,d\phi \tag{9.40}$$

Integration gives

$$\begin{aligned} Y &= R\int \cos\phi \, d\phi \\ &= R\sin\phi + C_2 \end{aligned}$$

C_2 is a constant of integration that can be evaluated by considering that when $Y = 0$, $\phi = 0^\circ$, $\sin\phi_0 = 0$. Hence $C_2 = 0$ giving

$$\boxed{Y = R\sin\phi} \tag{9.41}$$

Equations (9.39) and (9.41) are the projection equations for an Equal Area Cylindrical projection.

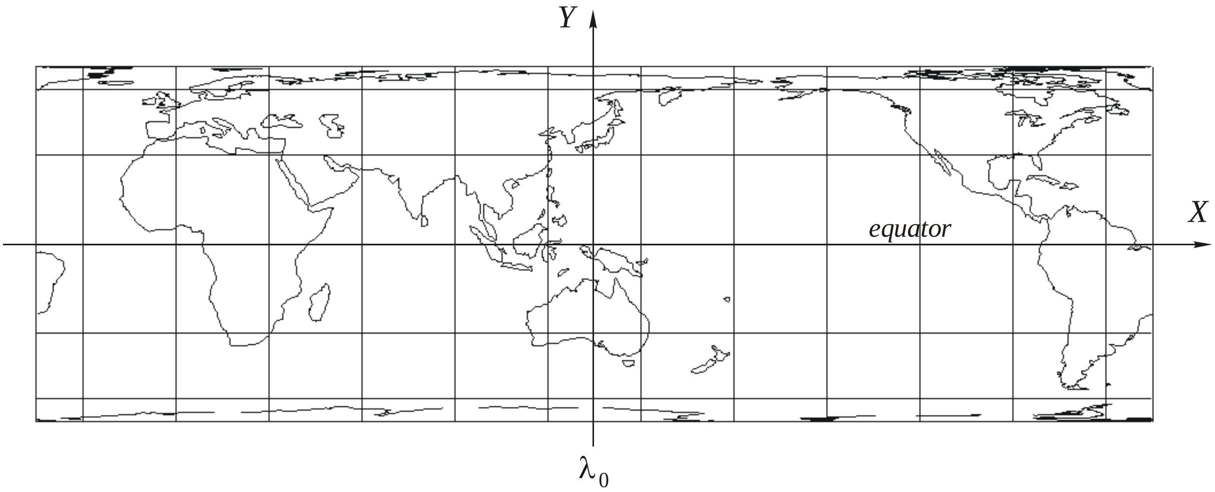


Figure 9.5 Equal Area Cylindrical projection.
Scale 1:270 million, graticule interval 30º, central meridian $\lambda_0 = 135^\circ$

The particular scale condition (scale factor along equator equal to unity) used to simplify the differential equation (9.36) can have a more general meaning and we may write that the scale factor along a particular parallel of latitude equals unity

$$k_0 = \frac{dX}{R\cos\phi_0 \, d\lambda} = 1 \tag{9.42}$$

where ϕ_0 denotes a *standard parallel*. This leads to a more general differential equation

$$dX = R\cos\phi_0 \, d\lambda \tag{9.43}$$

Solving this equation, treating $\cos\phi_0$ as a constant gives

$$\boxed{X = R\cos\phi_0 \left(\lambda - \lambda_0\right)} \tag{9.44}$$

Substituting (9.43) into (9.36) gives the differential equation

$$dY = \frac{R}{\cos\phi_0}\cos\phi \, d\phi \tag{9.45}$$

Integrating gives

$$\boxed{Y = \frac{R\sin\phi}{\cos\phi_0}} \tag{9.46}$$

Equations (9.44) and (9.46) are the *general equations* for an Equal Area Cylindrical projection.

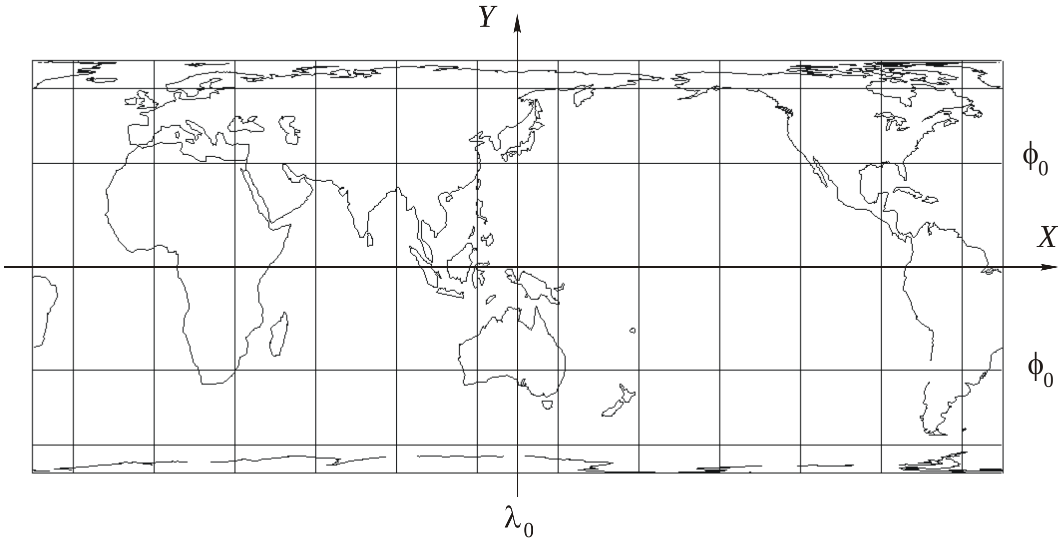


Figure 9.6 Equal Area Cylindrical projection.
Scale 1:270 million, graticule interval 30º, central meridian $\lambda_0 = 135^\circ$
Standard parallels $\phi_N = \phi_S = \phi_0$ at 30º

9.3.1. Properties of an Equal Area Cylindrical Projection

(i) Projection is Equal Area

(ii) Gaussian Fundamental Quantities E,F,G

The projection equations are:

$$\boxed{\begin{aligned} X &= R\cos\phi_0\left(\lambda - \lambda_0\right) \\ Y &= \frac{R\sin\phi}{\cos\phi_0} \end{aligned}} \tag{9.47}$$

where ϕ_0 is the latitude of standard parallels $\phi_N = \phi_S = \phi_0$

Using these equations, we may determine expressions for the derivatives $\frac{\partial X}{\partial\phi}, \frac{\partial X}{\partial\lambda}, \frac{\partial Y}{\partial\phi}, \frac{\partial Y}{\partial\lambda}$

$$\frac{\partial X}{\partial \phi} = 0 \qquad \frac{\partial X}{\partial \lambda} = R\cos\phi_0 \tag{9.48}$$

$$\frac{\partial Y}{\partial \phi} = \frac{R\cos\phi}{\cos\phi_0} \qquad \frac{\partial Y}{\partial \lambda} = 0 \tag{9.49}$$

Substituting (9.48) and (9.49) into the general equations for the Gaussian Fundamental Quantities E,F,G gives

$$\begin{aligned} E &= \left(\frac{\partial X}{\partial \phi}\right)^2 + \left(\frac{\partial Y}{\partial \phi}\right)^2 && = \frac{R^2\cos^2\phi}{\cos^2\phi_0} \\ F &= \frac{\partial X}{\partial \phi}\frac{\partial X}{\partial \lambda} + \frac{\partial Y}{\partial \phi}\frac{\partial Y}{\partial \lambda} && = 0 \\ G &= \left(\frac{\partial X}{\partial \lambda}\right)^2 + \left(\frac{\partial Y}{\partial \lambda}\right)^2 && = R^2\cos^2\phi_0 \\ J &= \sqrt{EG - F^2} && = R^2\cos\phi \end{aligned} \tag{9.50}$$

Note that $\frac{J}{j} = \frac{R^2\cos\phi}{R^2\cos\phi} = 1$ which satisfies the equal area scale condition

(iii) Scale Factors h (meridian) and k (parallel)

Using equations (9.34) and (9.35)

$$\begin{aligned} h &= \frac{\sqrt{E}}{\sqrt{e}} = \frac{R\cos\phi}{\cos\phi_0}\frac{1}{R} && = \frac{\cos\phi}{\cos\phi_0} \\ k &= \frac{\sqrt{G}}{\sqrt{g}} = R\cos\phi_0\frac{1}{R\cos\phi} && = \frac{\cos\phi_0}{\cos\phi} \end{aligned} \tag{9.51}$$

Note that the scale factors h and k multiplied together equal unity and the projection is equal area since $f = F = 0$.

(iv) Scale factor along equator $\qquad k_{equator} = \cos\phi_0$

9.3.2. Pseudocylindrical Equal Area projection

Imposing the scale condition $\frac{J}{j} = 1$ leads to the general differential equation for equal area cylindrical given by (9.36)

$$\frac{dY}{R\,d\phi}\frac{dX}{R\cos\phi\,d\lambda} = 1 \tag{9.52}$$

To simplify this equation we may impose the scale condition: <u>that the scale factor along **every** parallel is unity</u>. Using (9.35), this condition can be written as

$$k = \frac{dX}{R\cos\phi\, d\lambda} = 1 \tag{9.53}$$

This gives the differential equation

$$dX = R\cos\phi\, d\lambda \tag{9.54}$$

Integrating and solving for the constant of integration gives

$$\boxed{X = R\cos\phi\left(\lambda - \lambda_0\right)} \tag{9.55}$$

Substituting (9.54) into (9.52) gives the differential equation

$$dY = R\, d\phi$$

Integrating and solving for the constant of integration gives

$$\boxed{Y = R\phi} \tag{9.56}$$

Equations (9.55) and (9.56) are the projection equations for an Equal Area Pseudocylindrical projection known as the *Sinusoidal projection* or the *Sanson-Flamsteed projection*

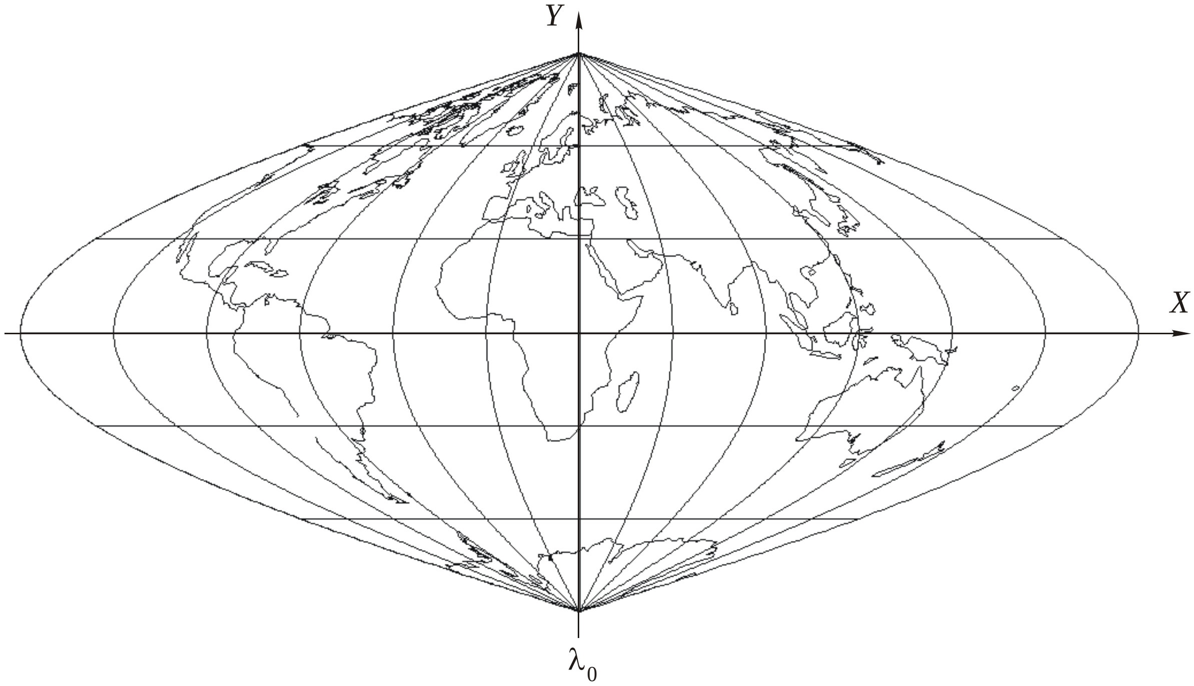


Figure 9.7 Sinusoidal Pseudocylindrical Equal Area projection.
Scale 1:270 million, graticule interval 30º, central meridian $\lambda_0 = 30^\circ$

9.3.3. Properties of Sinusoidal Pseudocylindrical Projection

(i) Projection is Equal Area

(ii) Gaussian Fundamental Quantities E,F,G

The projection equations are:

$$\boxed{\begin{aligned} X &= R\cos\phi\left(\lambda-\lambda_0\right) \\ Y &= R\phi \end{aligned}} \tag{9.57}$$

Using these equations, we may determine expressions for the derivatives $\dfrac{\partial X}{\partial \phi}, \dfrac{\partial X}{\partial \lambda}, \dfrac{\partial Y}{\partial \phi}, \dfrac{\partial Y}{\partial \lambda}$

$$\frac{\partial X}{\partial \phi} = -R\sin\phi\left(\lambda-\lambda_0\right) \qquad \frac{\partial X}{\partial \lambda} = R\cos\phi \tag{9.58}$$

$$\frac{\partial Y}{\partial \phi} = R \qquad \frac{\partial Y}{\partial \lambda} = 0 \tag{9.59}$$

Substituting (9.48) and (9.49) into the general equations for the Gaussian Fundamental Quantities E,F,G gives

$$\begin{aligned} E &= \left(\frac{\partial X}{\partial \phi}\right)^2 + \left(\frac{\partial Y}{\partial \phi}\right)^2 &&= R^2\left(1+\sin^2\phi\left(\lambda-\lambda_0\right)^2\right) \\ F &= \frac{\partial X}{\partial \phi}\frac{\partial X}{\partial \lambda} + \frac{\partial Y}{\partial \phi}\frac{\partial Y}{\partial \lambda} &&= -R^2\sin\phi\cos\phi\left(\lambda-\lambda_0\right) \\ G &= \left(\frac{\partial X}{\partial \lambda}\right)^2 + \left(\frac{\partial Y}{\partial \lambda}\right)^2 &&= R^2\cos^2\phi \\ J &= \sqrt{EG-F^2} &&= R^2\cos\phi \end{aligned} \tag{9.60}$$

Note that $\dfrac{J}{j} = \dfrac{R^2\cos\phi}{R^2\cos\phi} = 1$ which satisfies the equal area scale condition

(iii) Scale Factors h (meridian) and k (parallel)

Using equations (9.34) and (9.35)

$$\begin{aligned} h &= \frac{\sqrt{E}}{\sqrt{e}} = \frac{R\sin\phi\left(\lambda-\lambda_0\right)}{R} &&= \sin\phi\left(\lambda-\lambda_0\right) \\ k &= \frac{\sqrt{G}}{\sqrt{g}} = \frac{R\cos\phi}{R\cos\phi} &&= 1 \end{aligned} \tag{9.61}$$

Note that the scale factors h and k multiplied together <u>do not</u> equal unity since $F \neq 0$ but the projection is still equal area as we can see from the more general expression $\dfrac{J}{j} = 1$

(iv) Scale factor along equator $k_{equator} = 1$

9.4. EQUIDISTANT CYLINDRICAL PROJECTION

For an Equidistant Cylindrical projection the scale condition to be enforced is

$$\frac{E}{e} = 1 \tag{9.62}$$

Alternatively, using the notation for meridian and parallel scale factors we may write the scale condition as

$$h = 1 \tag{9.63}$$

where using (9.7) and (9.8)

$$h = \frac{\sqrt{E}}{\sqrt{e}} = \frac{dY}{R\,d\phi} = 1 \tag{9.64}$$

This leads to the differential equation

$$dY = R\,d\phi \tag{9.65}$$

Integrating this equation gives

$$\begin{aligned} Y &= R\int d\phi \\ &= R\phi + C_2 \end{aligned}$$

C_2 is a constant of integration that can be evaluated by considering that when $Y = 0$, $\phi = 0^\circ$ and $C_2 = 0$ giving

$$\boxed{Y = R\phi} \tag{9.66}$$

In a similar way to the Conformal and Equal Area Cylindrical projections a particular scale condition: <u>that the scale factor along the equator be unity</u>. Using (9.35), this condition can be written as

$$k_0 = \frac{\sqrt{G}}{\sqrt{g}} = \frac{dX}{R\cos\phi_0\,d\lambda} = 1$$

Now since $\phi_0 = 0^\circ$ and $\cos\phi_0 = 1$ this particular scale condition gives the differential equation

$$dX = R\,d\lambda \tag{9.67}$$

Integrating (9.67) gives

$$X = R\int d\lambda = R\lambda + C_1$$

C_1 is a constant of integration that can be evaluated by considering that when $X = 0$, $\lambda = \lambda_0$ and $C_1 = -R\lambda_0$ giving

$$\boxed{X = R\left(\lambda - \lambda_0\right)} \tag{9.68}$$

Equations (9.68) and (9.66) are the projection equations for an Equidistant Cylindrical projection.

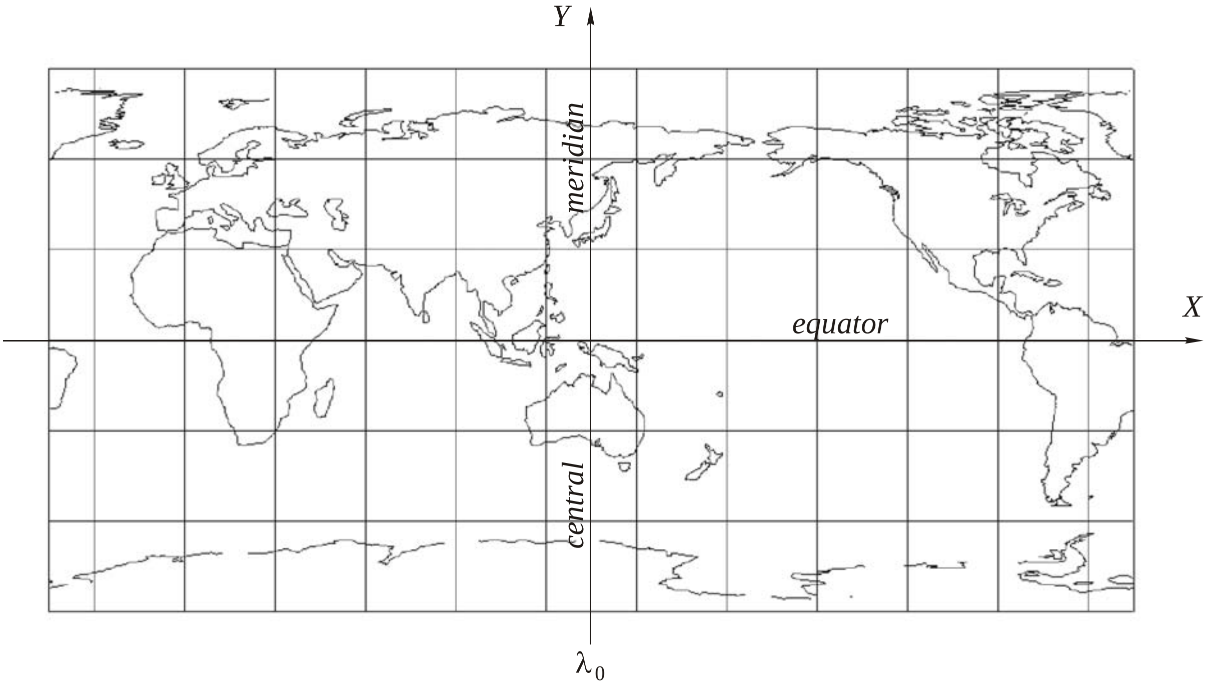


Figure 9.8 Equidistant Cylindrical projection.
Scale 1:270 million, graticule interval 30º, central meridian $\lambda_0 = 135^\circ$

The particular scale condition (scale factor along equator equal to unity) used to obtain the equation for the meridians can have a more general meaning and we may write that the scale factor along a particular parallel of latitude equals unity

$$k_0 = \frac{dX}{R\cos\phi_0\, d\lambda} = 1 \tag{9.69}$$

where ϕ_0 denotes a *standard parallel*. This leads to a more general differential equation

$$dX = R\cos\phi_0\, d\lambda \tag{9.70}$$

Solving this equation, treating $\cos\phi_0$ as a constant gives

$$\boxed{X = R\cos\phi_0\left(\lambda - \lambda_0\right)} \tag{9.71}$$

Equations (9.71) and (9.66) are the *general equations* for an Equidistant Cylindrical projection

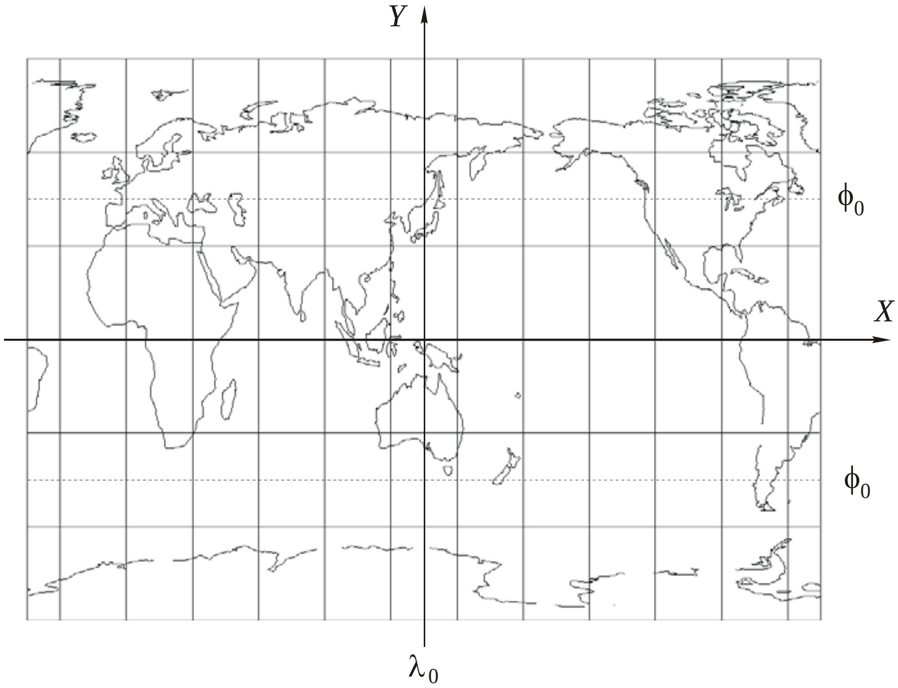


Figure 9.9 Equidistant Cylindrical projection.
Scale 1:270 million, graticule interval 30º, central meridian $\lambda_0 = 135^\circ$
Standard parallels $\phi_N = \phi_S = \phi_0$ at 45º

9.4.1. Properties of an Equidistant Cylindrical Projection

(i) Projection is Equidistant (along meridians)

(ii) Gaussian Fundamental Quantities *E,F,G*

The projection equations are:

$$\boxed{\begin{aligned} X &= R\cos\phi_0\left(\lambda - \lambda_0\right) \\ Y &= R\phi \end{aligned}} \tag{9.72}$$

where ϕ_0 is the latitude of standard parallels $\phi_N = \phi_S = \phi_0$

Using these equations, we may determine expressions for the derivatives $\dfrac{\partial X}{\partial \phi}, \dfrac{\partial X}{\partial \lambda}, \dfrac{\partial Y}{\partial \phi}, \dfrac{\partial Y}{\partial \lambda}$

$$\frac{\partial X}{\partial \phi} = 0 \qquad \frac{\partial X}{\partial \lambda} = R\cos\phi_0 \tag{9.73}$$

$$\frac{\partial Y}{\partial \phi} = R \qquad \frac{\partial Y}{\partial \lambda} = 0 \tag{9.74}$$

Substituting (9.73) and (9.74) into the general equations for the Gaussian Fundamental Quantities E,F,G gives

$$\begin{aligned} E &= \left(\frac{\partial X}{\partial \phi}\right)^2 + \left(\frac{\partial Y}{\partial \phi}\right)^2 &&= R^2 \\ F &= \frac{\partial X}{\partial \phi}\frac{\partial X}{\partial \lambda} + \frac{\partial Y}{\partial \phi}\frac{\partial Y}{\partial \lambda} &&= 0 \\ G &= \left(\frac{\partial X}{\partial \lambda}\right)^2 + \left(\frac{\partial Y}{\partial \lambda}\right)^2 &&= R^2\cos^2\phi_0 \\ J &= \sqrt{EG - F^2} &&= R^2\cos\phi_0 \end{aligned} \tag{9.75}$$

(iii) Scale Factors h (meridian) and k (parallel)

$$\begin{aligned} h &= \frac{\sqrt{E}}{\sqrt{e}} = \frac{R}{R} &&= 1 \\ k &= \frac{\sqrt{G}}{\sqrt{g}} = \frac{R\cos\phi_0}{R\cos\phi} &&= \frac{\cos\phi_0}{\cos\phi} \end{aligned} \tag{9.76}$$

Note that the scale factor $h = 1$, hence the projection is equidistant (along the meridians).

(iv) Scale factor along equator $k_{equator} = \cos\phi_0$