10. CONICAL PROJECTIONS

In elementary texts on map projections, the projection surfaces are often described as developable surfaces, such as the cylinder (cylindrical projections) and the cone (conical projections), or a plane (azimuthal projections). These surfaces are imagined as enveloping or touching the datum surface and by some means, usually geometric, the meridians, parallels and features are projected onto these surfaces. In the case of the cone, a plane containing the axis of the cone cuts the cone on a line joining the base and the apex. If the cone is cut along this line (a generator of the cone) it can be laid flat (developed). If the axis of the cone coincides with the axis of the Earth, the projection is said to be normal aspect, if the axis lies in the plane of the equator the projection is known as transverse and in any other orientation it is known as oblique. [It is usual that the descriptor "normal" is implied in the name of a projection, but for different orientations, the words "transverse" or "oblique" are added to the name.] This simplified approach is not adequate for developing a general theory of projections (which as we can see is quite mathematical) but is useful for describing characteristics of certain projections. In the case of conical projections, some characteristics are a common feature:

(i) Meridians are equally spaced straight lines radiating from a central point *O*.
(ii) Parallels, in general, are unequally spaced concentric circles having a centre at *O*.
(iii) Meridians of longitude and parallels of latitude form an orthogonal network of lines.

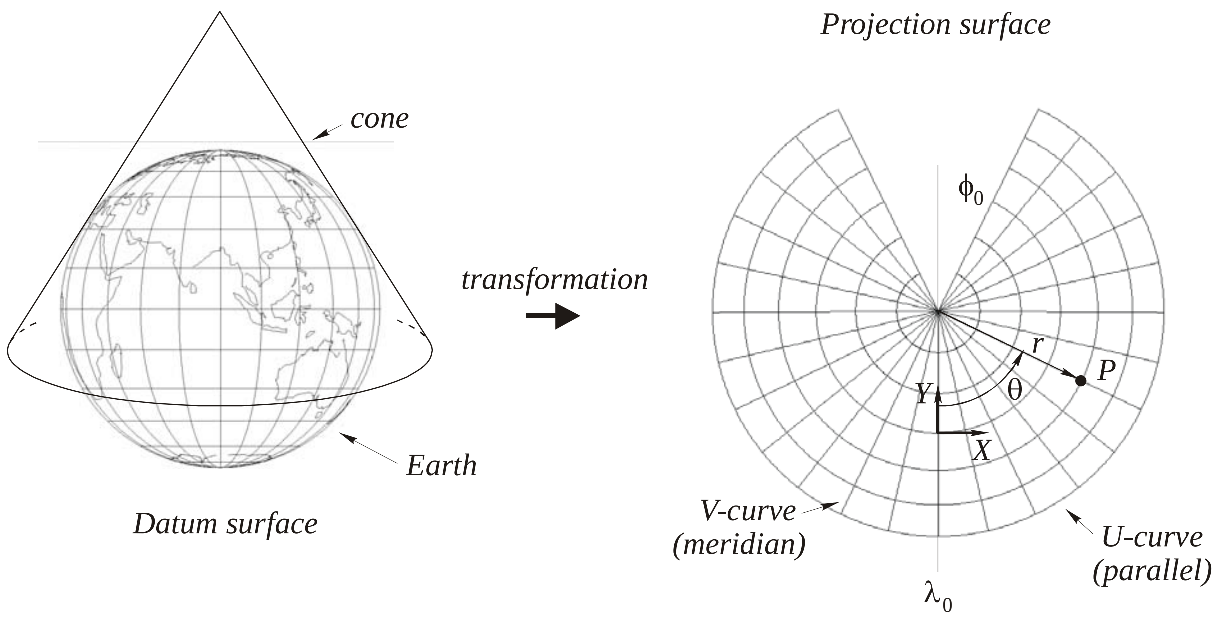

Figure 10.1 Conical projection. *u*,*v* curves on the datum surface projected as *U*,*V* curves on the projection surface

Figure 10.1 shows a schematic diagram of a conical projection demonstrating the basic characteristics common to all conical projections (normal aspect). Conical projections have a general circular shape when the whole of the Earth is displayed; the *U*-curves (parallels of latitude) are concentric circles and the *V*-curves (meridians of longitude) are equally spaced radial lines. In Figure 10.1, the centre for the circular *U*-curves and the radial *V*-curves is the pole, but this is not always the case. In some conical projections, the pole is shown as a line. The origin of the *X,Y* Cartesian coordinates is shown at the intersection of a central *V*-curve (a central meridian λ_0) and a selected *U*-curve (a parallel of latitude ϕ_0). A point *P* is shown on the projection with *polar coordinates* r,θ where *r* is a radial distance from the centre of the projection, the origin of the polar coordinate system and θ is an angle measured positive anticlockwise, negative clockwise from the central meridian. All projection equations for conical projections are given in terms of polar coordinates.

10.1. The Gaussian Fundamental Quantities of Conical Projections

For conical projections, the projection surface is a plane and the *U,V* curvilinear coordinate system is an orthogonal system of *U* and *V*-curves that are concentric circular arcs of radius *r* and straight radial lines at angles θ from a central *V*- curve. The functional relationships connecting the *U,V* coordinate system with the *X,Y,Z* Cartesian coordinate system were given previously by equations (2.2) and are restated here in more explicit form

$$\begin{aligned} X &= F_1(U,V) = F_1(r,\theta) \\ Y &= F_2(U,V) = F_2(r,\theta) \\ Z &= F_3(U,V) = 0 \end{aligned} \tag{10.1}$$

Referring to Figure 10.1, the *X,Y* Cartesian coordinates are related to the r,θ polar coordinates by the equations

$$\begin{aligned} X &= r\sin\theta \\ Y &= r_0 - r\cos\theta \end{aligned} \tag{10.2}$$

where r_0 is the radius of the *U*-curve (a parallel of latitude) passing through the *X,Y* coordinate origin.

The Gaussian Fundamental Quantities of the projection surface are given by equations (5.3) and are restated here recognising that *U*-curves are circular arcs of radius *r* and *V*-curves are radial lines at angles θ from a cental meridian and *Z* = 0

$$\begin{aligned} \bar{E} &= \left(\frac{\partial X}{\partial U}\right)^2 + \left(\frac{\partial Y}{\partial U}\right)^2 = \left(\frac{\partial X}{\partial r}\right)^2 + \left(\frac{\partial Y}{\partial r}\right)^2 \\ \bar{F} &= \frac{\partial X}{\partial U}\frac{\partial X}{\partial V} + \frac{\partial Y}{\partial U}\frac{\partial Y}{\partial V} = \frac{\partial X}{\partial r}\frac{\partial X}{\partial \theta} + \frac{\partial Y}{\partial r}\frac{\partial Y}{\partial \theta} \\ \bar{G} &= \left(\frac{\partial X}{\partial V}\right)^2 + \left(\frac{\partial Y}{\partial V}\right)^2 = \left(\frac{\partial X}{\partial \theta}\right)^2 + \left(\frac{\partial Y}{\partial \theta}\right)^2 \end{aligned} \tag{10.3}$$

The partial derivatives $\frac{\partial X}{\partial r}, \frac{\partial X}{\partial \theta}$ etc can be obtained from (10.2)

$$\begin{aligned} \frac{\partial X}{\partial r} &= \sin\theta & \frac{\partial Y}{\partial r} &= -\cos\theta \\ \frac{\partial X}{\partial \theta} &= r\cos\theta & \frac{\partial Y}{\partial r} &= r\sin\theta \end{aligned} \tag{10.4}$$

Substituting equations (10.4) into (10.3) gives

$$\begin{aligned} \bar{E} &= \left(\frac{\partial X}{\partial r}\right)^2 + \left(\frac{\partial Y}{\partial r}\right)^2 &&= \sin^2\theta + \cos^2\theta &&= 1 \\ \bar{F} &= \frac{\partial X}{\partial r}\frac{\partial X}{\partial \theta} + \frac{\partial Y}{\partial r}\frac{\partial Y}{\partial \theta} &&= r\sin\theta\cos\theta - r\sin\theta\cos\theta &&= 0 \\ \bar{G} &= \left(\frac{\partial X}{\partial \theta}\right)^2 + \left(\frac{\partial Y}{\partial \theta}\right)^2 &&= r^2\cos^2\theta + r^2\sin^2\theta &&= r^2 \end{aligned} \tag{10.5}$$

Note that $\bar{F} = 0$, which reflects the fact that the *r*-curves and θ-curves (*U* and *V*-curves that are concentric circles and radial lines) intersect at right angles.

Now, considering the datum surface to be a sphere of radius *R* and the *u*,*v* curves as parallels and meridians ϕ, λ, the Gaussian Fundamental Quantities *E*,*F*,*G* relating the functional relationships

$$\begin{aligned} X &= \bar{f}_1(\phi, \lambda) \\ Y &= \bar{f}_2(\phi, \lambda) \end{aligned} \tag{10.6}$$

are

$$\begin{aligned} E &= \left(\frac{\partial X}{\partial \phi}\right)^2 + \left(\frac{\partial Y}{\partial \phi}\right)^2 \\ F &= \frac{\partial X}{\partial \phi}\frac{\partial X}{\partial \lambda} + \frac{\partial Y}{\partial \phi}\frac{\partial Y}{\partial \lambda} \\ G &= \left(\frac{\partial X}{\partial \lambda}\right)^2 + \left(\frac{\partial Y}{\partial \lambda}\right)^2 \end{aligned} \tag{10.7}$$

Expressions for *E*,*F*,*G* can be obtained from the Transformation Matrix (6.7) in the following form

$$\begin{bmatrix} E \\ F \\ G \end{bmatrix} = \begin{bmatrix} \left(\dfrac{\partial r}{\partial \phi}\right)^2 & 2\dfrac{\partial r}{\partial \phi}\dfrac{\partial \theta}{\partial \phi} & \left(\dfrac{\partial \theta}{\partial \phi}\right)^2 \\ \dfrac{\partial r}{\partial \phi}\dfrac{\partial r}{\partial \lambda} & \dfrac{\partial r}{\partial \lambda}\dfrac{\partial \theta}{\partial \phi} + \dfrac{\partial r}{\partial \phi}\dfrac{\partial \theta}{\partial \lambda} & \dfrac{\partial \theta}{\partial \phi}\dfrac{\partial \theta}{\partial \lambda} \\ \left(\dfrac{\partial r}{\partial \lambda}\right)^2 & 2\dfrac{\partial r}{\partial \lambda}\dfrac{\partial \theta}{\partial \lambda} & \left(\dfrac{\partial \theta}{\partial \lambda}\right)^2 \end{bmatrix} \begin{bmatrix} \bar{E} \\ \bar{F} \\ \bar{G} \end{bmatrix} \tag{10.8}$$

Noting that *r* has replaced *U*, θ has replaced *V*, ϕ has replaced *u* and λ has replaced *v*.

Now from equations (10.5) we have $\bar{E}=1$, $\bar{F}=0$ and $\bar{G}=r^2$, and substituting into the Transformation Matrix (10.8) gives

$$\begin{aligned} E &= \left(\frac{\partial r}{\partial \phi}\right)^2 + r^2\left(\frac{\partial \theta}{\partial \phi}\right)^2 \\ F &= \frac{\partial r}{\partial \phi}\frac{\partial r}{\partial \lambda} + r^2\frac{\partial \theta}{\partial \phi}\frac{\partial \theta}{\partial \lambda} \\ G &= \left(\frac{\partial r}{\partial \lambda}\right)^2 + r^2\left(\frac{\partial \theta}{\partial \lambda}\right)^2 \end{aligned} \tag{10.9}$$

These expressions can be simplified if the following conditions for normal aspect Conical projections are enforced (these conditions can be "understood" by inspection of Figure 10.1)

(i) $r = f(\phi)$ i.e., the radius of a parallel of latitude (a *U*-curve) on the projection is a function of the latitude ϕ only and

(ii) $\theta = n(\lambda - \lambda_0)$ i.e., the polar angle θ (the angle between a *V*-curve and the central meridian) is a linear function of λ only. <u>*n* is a scalar quantity known as the cone constant</u>.

These two conditions mean that

$$\frac{\partial r}{\partial \lambda}=0, \quad \frac{\partial \theta}{\partial \phi}=0, \quad \frac{\partial \theta}{\partial \lambda}=n \tag{10.10}$$

Substituting these differential relationships into (10.9) gives the <u>Gaussian Fundamental Quantities *E*,*F*,*G* for normal aspect Conical projections</u>

$$\boxed{E=\left(\frac{\partial r}{\partial \phi}\right)^2, \quad F=0, \quad G=r^2\left(\frac{\partial \theta}{\partial \lambda}\right)^2} \tag{10.11}$$

Using these differential relationships and particular scale conditions we can derive *CONFORMAL*, *EQUAL AREA* and *EQUIDISTANT* Conical projections.

The scale conditions are:

For CONFORMAL Conical projections: $\frac{E}{e}=\frac{G}{g}=m^2$

EQUAL AREA Conical projections: $\frac{J}{j}=1$

EQUIDISTANT Conical projections: $\frac{E}{e}=1$

In addition, since the datum surface is a sphere of radius *R*, the Gaussian Fundamental Quantities of the datum surface are

$$\begin{aligned} e &= R^2 \\ f &= 0 \\ g &= R^2\cos^2\phi \end{aligned} \tag{10.12}$$

10.2. Conformal Conical Projections

For a Conformal Conical projection the scale condition to be enforced is

$$\frac{E}{e} = \frac{G}{g} = m^2 \tag{10.13}$$

Alternatively, using the notation for meridian and parallel scale factors we may write the scale condition as

$$h = k \tag{10.14}$$

where
$$h = \frac{\sqrt{E}}{\sqrt{e}} = \frac{\partial r}{R\partial\phi} \tag{10.15}$$

and
$$k = \frac{\sqrt{G}}{\sqrt{g}} = \frac{r\partial\theta}{R\cos\phi\,\partial\lambda} = \frac{nr}{R\cos\phi} \tag{10.16}$$

Note that in equation (10.16) the cone constant

$$n = \frac{\partial\theta}{\partial\lambda} \tag{10.17}$$

is used, see equations (10.10) and the conditions for normal aspect Conical projections.

Enforcing the scale condition $h = k$ gives the differential relationship

$$\frac{\partial r}{R\partial\phi} = \frac{nr}{R\cos\phi}$$

and rearranging gives

$$\frac{1}{r}dr = \frac{-n}{\cos\phi}d\phi \tag{10.18}$$

Note: The minus sign is introduced to reflect the fact that the radius *r* increases as the latitude ϕ decreases.

Integrating (10.18) gives

$$\int \frac{1}{r}\,dr = -n\int \frac{1}{\cos\phi}\,d\phi$$

$$\ln r + C_1 = -n\ln\left\{\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\right\} + C_2$$

$$\ln r = -n\ln\left\{\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\right\} + \ln C$$

where $\ln C = \left(C_2 - C_1\right)$ is the natural logarithm of the constants of integration. Using the *laws of logarithms,* $\log_a M^p = p\log_a M$ and $\log_a MN = \log_a M + \log_a N$ we may write

$$\ln r = \ln\left\{\left[\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\right]^{-n}\right\} + \ln C$$

$$= \ln\left\{C\left[\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\right]^{-n}\right\}$$

Taking antilogarithms of both sides gives

$$\boxed{r = C\left[\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\right]^{-n}} \qquad (10.19)$$

and from condition (ii) above

$$\boxed{\theta = n\left(\lambda - \lambda_0\right)} \qquad (10.20)$$

Equations (10.19) and (10.20) are the <u>general equations for Conformal Conic projections, but the constants *n* and *C* must be determined</u>.

To determine the constants *n* (the cone constant) and *C,* geometric constraints relating to *standard parallels* are employed, remembering that a standard parallel is defined as a parallel of latitude along which the scale factor is constant and equal to unity.

Consider the case of a single standard parallel

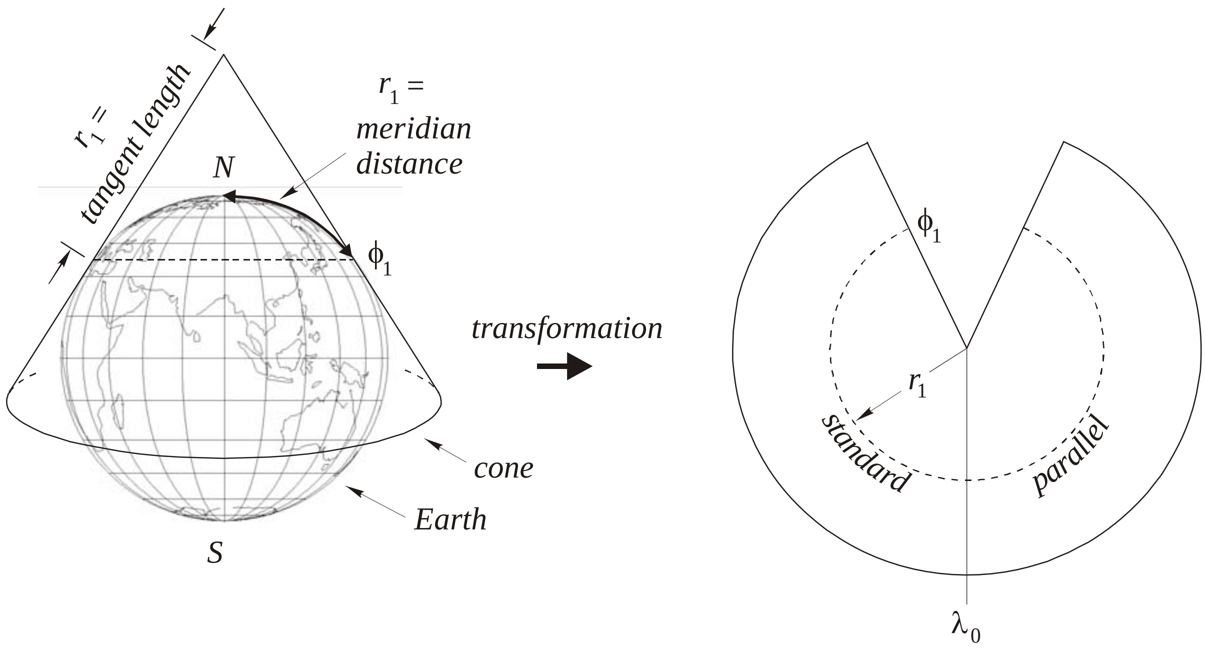

Figure 10.2 Schematic diagram of a Conical projection with a single standard parallel ϕ_1 having a radius r_1.

Setting the scale factor *k* in equation (10.16) equal to unity along a standard parallel ϕ_1 gives

$$k_1 = \frac{nr_1}{R\cos\phi_1} = 1$$

That can be rearranged to give an equation for the cone constant *n* as

$$n = \frac{R\cos\phi_1}{r_1} \tag{10.21}$$

Inspection of equation (10.21) shows that if the radius of the standard parallel r_1 is fixed then the cone constant *n* can be determined. Referring to Figure 10.2, the two choices for fixing r_1 are:

(a) Make the radius of the standard parallel equal to the tangent length of the cone.

$$r_1 = R\cot\phi_1 \tag{10.22}$$

(b) Make the radius of the standard parallel equal to the meridian distance on the Earth from the pole to the tangent point of the cone.

$$r_1 = R\left(\frac{\pi}{2} - \phi_1\right) \tag{10.23}$$

10.2.1. Conformal Conic Projection with a single standard parallel (radius = tangent length)

Projection equations:

$$r = C\left[\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\right]^{-n} \qquad (10.24)$$

$$\theta = n\left(\lambda-\lambda_0\right)$$

Cone constant:

$$n = \frac{R\cos\phi_1}{r_1} = \frac{R\cos\phi_1}{R\cot\phi_1} = \sin\phi_1 \qquad (10.25)$$

To determine the constant *C*, consider the radius r_1

$$r_1 = C\left[\tan\left(\frac{\pi}{4}+\frac{\phi_1}{2}\right)\right]^{-n} = \frac{R}{\tan\phi_1}$$

$$C = \frac{R}{\tan\phi_1}\left[\tan\left(\frac{\pi}{4}+\frac{\phi_1}{2}\right)\right]^{n} \qquad (10.26)$$

Figure 10.3 shows a Conformal Conic projection of the northern hemisphere. The projection has a single standard parallel ϕ_1 whose radius is equal to the tangent length of the cone. The graticule interval is 15°, central meridian $\lambda_0 = 30^\circ$ and the *X,Y* coordinate origin is at λ_0 and $\phi_0 = 45^\circ$. A point *P* is shown whose coordinates are $\phi_P = 0^\circ$ and $\lambda_P = 90^\circ$.

The projection parameters, scale of the projection and the *X,Y* coordinates of *P* can be computed in the following way.

$$r = C\left[\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\right]^{-n}$$

$$\theta = n\left(\lambda-\lambda_0\right)$$

$$R = 6371000 \text{ m}$$

$$n = \sin\phi_1 \qquad = 0.707107$$

$$r_1 = \frac{R}{\tan\phi_1} \qquad = 6371000.000 \text{ m}$$

$$C = r_1\left[\tan\left(\frac{\pi}{4}+\frac{\phi_1}{2}\right)\right]^{n} \qquad = 11881489.4811 \text{ m}$$

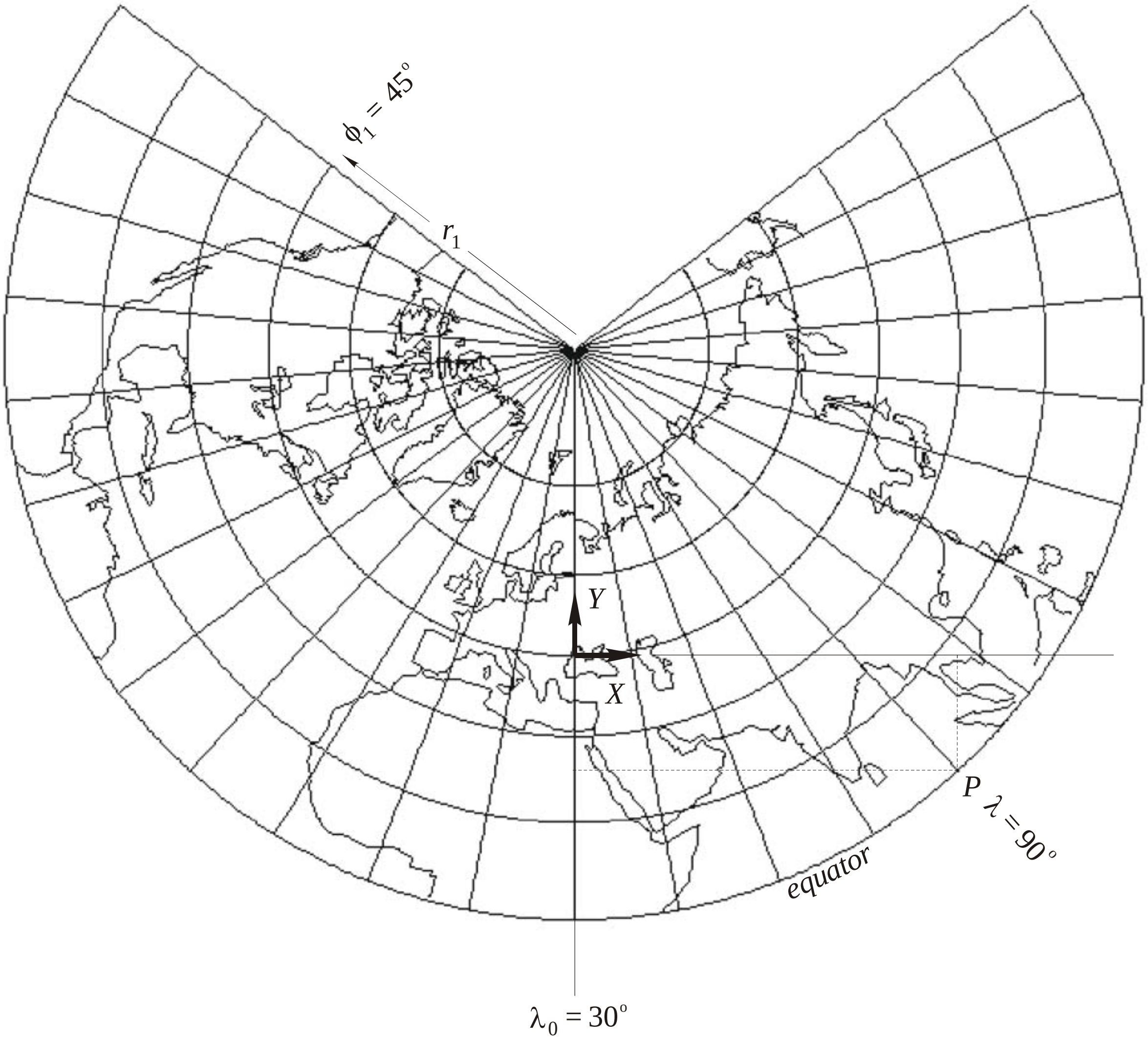

Figure 10.3 Conformal Conic projection, single standard parallel $\phi_1 = 45^\circ$ (radius of standard parallel = tangent length of cone). Graticule interval 15°, central meridian $\lambda_0 = 30^\circ$, *X,Y* coordinate origin at λ_0 and $\phi_0 = 45^\circ$

The scale of the projection can be obtained from the general relationship

$$scale = \frac{\text{distance on projection}}{\text{distance on Earth}} = \frac{r_1(map)}{r_1(Earth)}$$

From measurements on Figure 10.3, the length $r_1 = 41.5$ mm and from the previous calculations $r_1 = 6371000000$ mm on the Earth. Note that this is really the tangent length of a cone touching the Earth at latitude $\phi_1 = 45^\circ$. The scale of the projection is then

$$scale = \frac{r_1(map)}{r_1(Earth)} = \frac{41.5}{6371000000} = \frac{1}{153518072} \quad \text{or} \quad \approx 1{:}155 \text{ million}$$

The Cartesian equations for Conical projections (of the northern hemisphere) are

$$X = r\sin\theta$$
$$Y = r_0 - r\cos\theta$$

The radius r_0 of the parallel ϕ_0 is given by (10.24) and since the Cartesian origin is at the intersection of the central meridian λ_0 and the standard parallel ϕ_0 then $r_0 = r_1$.

For the point *P* at $\phi = 0^\circ$ and $\lambda = 90^\circ$ and with *C* and *n* for the projection

$$r = C\left[\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\right]^{-n} = 11881489.4811 \text{ m}$$
$$\theta = n\left(\lambda - \lambda_0\right) = 42^\circ\, 25'\, 35.06''$$
$$X = 8015759.672 \text{ m} \quad = \quad 52.2 \text{ mm}$$
$$Y = -2399255.935 \text{ m} \quad = -15.6 \text{ mm}$$