MERIDIAN DISTANCE

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March 2006

ABSTRACT

These notes provide a detailed derivation of the equations for computing meridian distance on an ellipsoid of revolution given ellipsoid parameters and latitude. Computation of meridian distance is a requirement for geodetic calculations on the ellipsoid, in particular, computations to do with conversion of geodetic coordinates ϕ, λ (latitude, longitude) to Universal Transverse Mercator (UTM) projection coordinates *E,N* (East,North) that are used in Australia for survey coordination. The "opposite" transformation, *E,N* to ϕ, λ requires latitude given meridian distance and these note show how series equations for meridian distances are "reversed" to give series equations for latitude. The derivation of equations follow methods set out in *Lehrbuch Der Geodäsie* (Baeschlin, 1948), *Handbuch der Vermessungskunde* (Jordan/Eggert/Kneissl, 1958), *Geometric Geodesy* (Rapp, 1982) and *Geodesy and Map Projections* (Lauf, 1983) and some of the formula derived are used in the *Geocentric Datum of Australia Technical Manual* (ICSM, 2002) – an on-line reference manual available from Geoscience Australia. An understanding of the methods introduced in the following pages, in particular the solution of elliptic integrals by series expansion and reversion of a series, will give the student an insight into other geodetic calculations.
MATLAB functions for computing (i) meridian distance given latitude (function *mdist.m*) and (ii) latitude given meridian distance (function *latitude.m*) are also given.

INTRODUCTION

For an ellipsoid defined by the parameter pairs (a, f), semi-major axis and flattening or (a, e^2), semi-major axis and first-eccentricity squared or (a, b), semi-major and semi-minor axes, let m be the length of an arc of the meridian from the equator to a point in latitude ϕ then

$$dm = \rho \, d\phi \tag{1}$$

where ρ is the radius of curvature in the meridian plane and

$$\rho = \frac{a\left(1-e^2\right)}{\left(1-e^2\sin^2\phi\right)^{\frac{3}{2}}} = \frac{a\left(1-e^2\right)}{W^3} \tag{2}$$

where the first-eccentricity squared e^2 and W are

$$\begin{aligned} e^2 &= \frac{a^2-b^2}{a^2} = f\,(2-f) \\ f &= \frac{a-b}{a} \\ W &= \left(1-e^2\sin^2\phi\right)^{\frac{1}{2}} \end{aligned} \tag{3}$$

Alternatively, the radius of curvature in the meridian plane is also given by

$$\rho = \frac{a^2}{b\left(1+e'^2\cos^2\phi\right)^{\frac{3}{2}}} = \frac{c}{V^3} \tag{4}$$

where the polar radius c, the second-eccentricity squared e'^2 and V are

$$\begin{aligned} c &= \frac{a^2}{b} = \frac{a}{1-f} \\ e'^2 &= \frac{a^2-b^2}{b^2} = \frac{f\,(2-f)}{(1-f)^2} \\ V &= \left(1+e'^2\cos^2\phi\right)^{\frac{1}{2}} \end{aligned} \tag{5}$$

Substituting the first group of equations [equations (2) and (3)] into equation (1) lead to series formula for the meridian distance m as a function of latitude ϕ and powers of e^2. Substituting the second group, [(4) and (5)] lead to series formula for m as a function of ϕ and powers of another ellipsoid constant $n = \dfrac{a-b}{a+b}$. Series formula for m involving powers of e^2 are more commonly found in the geodetic literature but as will be shown in the following sections, series formula for m involving powers of n are more compact and give identical results at the level of the 5th decimal place of a metre. A further advantage of the series formula involving powers of n, is that they are easier to "reverse", i.e., given m as a function of latitude ϕ and powers of n develop a series formula (by reversion of a series) that gives ϕ as a function m. This is very useful in the conversion of UTM projection coordinates E,N to geodetic coordinates ϕ, λ.

MERIDIAN DISTANCE AS A SERIES FORMULA IN POWERS OF e^2

Using equations (1), (2) and (3) the meridian distance is given by the integral

$$m = \int_0^\phi \frac{a\left(1-e^2\right)}{W^3}\, d\phi = a\left(1-e^2\right)\int_0^\phi \frac{1}{W^3}\, d\phi = a\left(1-e^2\right)\int_0^\phi \left(1-e^2\sin^2\right)^{-\frac{3}{2}} d\phi \qquad (6)$$

This is an elliptic integral of the second kind that cannot be evaluated directly; instead, the integrand $\frac{1}{W^3} = \left(1-e^2\sin^2\phi\right)^{-\frac{3}{2}}$ is expanded in a series and then evaluated by term-by-term integration.

The integrand $\frac{1}{W^3} = \left(1-e^2\sin^2\phi\right)^{-\frac{3}{2}}$ can be expanded by use of the *binomial series*

$$(1+x)^\beta = \sum_{n=0}^{\infty} \binom{\beta}{n} x^n \qquad (7)$$

An <u>infinite</u> series where n is a positive integer, β is any real number, $-1 < x < 1$ and the binomial coefficients $B_n^\beta = \binom{\beta}{n}$ are given by

$$B_n^\beta = \binom{\beta}{n} = \frac{\Gamma(\beta+1)}{\Gamma(n+1)\,\Gamma(\beta-n+1)} \qquad (8)$$

In equation (8) the *gamma function* Γ satisfies $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ with $\Gamma(\beta+1) = \beta\,\Gamma(\beta)$ for all $\beta \neq 0, -1, -2, \cdots$ and for integer values of n, $\Gamma(n+1) = n!$ with $0! = 1$.

In the case where β is a positive integer, say k, and $-1 < x < 1$, the binomial series (7) can be expressed as the <u>finite</u> sum

$$(1+x)^k = \sum_{n=0}^{k} \binom{k}{n} x^n \qquad (9)$$

where the binomial coefficients $B_n^k = \binom{k}{n}$ in series (9) are given by

$$B_n^k = \binom{k}{n} = \frac{k!}{n!(k-n)!} \tag{10}$$

The binomial coefficients $B_n^{-\frac{3}{2}} = \binom{-\frac{3}{2}}{n}$ for the series (7) are given by equation (8) as

$$B_n^{-\frac{3}{2}} = \frac{\Gamma\left(-\frac{1}{2}\right)}{n!\,\Gamma\left(-\frac{1}{2}-n\right)}$$ with the following results for $n = 0, 1, 2$ and 3

$$n = 0 \qquad B_0^{-\frac{3}{2}} = \frac{\Gamma\left(-\frac{1}{2}\right)}{0!\,\Gamma\left(-\frac{1}{2}\right)} = 1$$

$$n = 1 \qquad B_1^{-\frac{3}{2}} = \frac{\Gamma\left(-\frac{1}{2}\right)}{1!\,\Gamma\left(-\frac{3}{2}\right)} = \frac{-\frac{3}{2}\,\Gamma\left(-\frac{3}{2}\right)}{1!\,\Gamma\left(-\frac{3}{2}\right)} = -\frac{3}{2}$$

$$n = 2 \qquad B_2^{-\frac{3}{2}} = \frac{\Gamma\left(-\frac{1}{2}\right)}{2!\,\Gamma\left(-\frac{5}{2}\right)} = \frac{-\frac{3}{2}\,\Gamma\left(-\frac{3}{2}\right)}{2!\,\Gamma\left(-\frac{5}{2}\right)} = \frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\Gamma\left(-\frac{5}{2}\right)}{2!\,\Gamma\left(-\frac{5}{2}\right)} = \frac{15}{8}$$

$$n = 3 \qquad B_3^{-\frac{3}{2}} = \frac{\Gamma\left(-\frac{1}{2}\right)}{3!\,\Gamma\left(-\frac{7}{2}\right)} = \frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)\Gamma\left(-\frac{7}{2}\right)}{3!\,\Gamma\left(-\frac{7}{2}\right)} = -\frac{105}{48}$$

Inspecting the results above, we can see that the binomial coefficients $B_n^{-\frac{3}{2}} = \binom{-\frac{3}{2}}{n}$ are a sequence of the form

$$1,\ -\frac{3}{2},\ \frac{3\cdot 5}{2\cdot 4},\ -\frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6},\ \frac{3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8},\ -\frac{3\cdot 5\cdot 7\cdot 9\cdot 11}{2\cdot 4\cdot 6\cdot 8\cdot 10},\ \cdots$$

Using these coefficients gives (Baeschlin 1948, p.48; Jordan/Eggert/Kneissl 1958, p.75; Rapp 1982, p.26)

$$\begin{aligned}\frac{1}{W^3} = \left(1 - e^2 \sin^2\phi\right)^{-\frac{3}{2}} &= 1 + \frac{3}{2}e^2\sin^2\phi + \frac{3\cdot 5}{2\cdot 4}e^4\sin^4\phi + \frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}e^6\sin^6\phi \\ &\quad + \frac{3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8}e^8\sin^8\phi + \frac{3\cdot 5\cdot 7\cdot 9\cdot 11}{2\cdot 4\cdot 6\cdot 8\cdot 10}e^{10}\sin^{10}\phi + \cdots\end{aligned} \tag{11}$$

To simplify this expression, and make the eventual integration easier, the powers of $\sin\phi$ can be expressed in terms of multiple angles using the standard form

$$\begin{aligned}\sin^{2n}\phi = \frac{1}{2^{2n}}\binom{2n}{n} + \frac{(-1)^n}{2^{2n-1}}\Bigg\{&\binom{2n}{0}\cos(2n-0)\phi - \binom{2n}{1}\cos(2n-2)\phi + \binom{2n}{2}\cos(2n-4)\phi \\ &- \binom{2n}{3}\cos(2n-6)\phi + \cdots (-1)^n\binom{2n}{n-1}\cos 2\phi\Bigg\}\end{aligned} \tag{12}$$

Using equation (12) and the binomial coefficients $B_n^{2n} = \binom{2n}{n}$ computed using equation (10) gives

$$\sin^2\phi = \frac{1}{2} - \frac{1}{2}\cos 2\phi$$

$$\sin^4\phi = \frac{3}{8} + \frac{1}{8}\cos 4\phi - \frac{1}{2}\cos 2\phi$$

$$\sin^6\phi = \frac{5}{16} - \frac{1}{32}\cos 6\phi + \frac{3}{16}\cos 4\phi - \frac{15}{32}\cos 2\phi$$

$$\sin^8\phi = \frac{35}{128} + \frac{1}{128}\cos 8\phi - \frac{1}{16}\cos 6\phi + \frac{7}{32}\cos 4\phi - \frac{7}{16}\cos 2\phi$$

$$\sin^{10}\phi = \frac{63}{256} - \frac{1}{512}\cos 10\phi + \frac{5}{256}\cos 8\phi - \frac{45}{512}\cos 6\phi + \frac{15}{64}\cos 4\phi - \frac{105}{256}\cos 2\phi \qquad (13)$$

Substituting equations (13) into equation (11) and arranging according to $\cos 2\phi$, $\cos 4\phi$, etc, we obtain (Baeschlin 1948, p.48; Jordan/Eggert/Kneissl 1958, p.75; Rapp 1982, p.27)

$$\frac{1}{W^3} = \left(1 - e^2\sin^2\phi\right)^{-\frac{3}{2}} = A - B\cos 2\phi + C\cos 4\phi - D\cos 6\phi + E\cos 8\phi - F\cos 10\phi + \cdots \quad (14)$$

where the coefficients *A, B, C,* etc., are

$$\begin{aligned}
A &= 1 + \frac{3}{4}e^2 + \frac{45}{64}e^4 + \frac{175}{256}e^6 + \frac{11025}{16384}e^8 + \frac{43659}{65536}e^{10} + \cdots \\
B &= \frac{3}{4}e^2 + \frac{15}{16}e^4 + \frac{525}{512}e^6 + \frac{2205}{2048}e^8 + \frac{72765}{65536}e^{10} + \cdots \\
C &= \frac{15}{64}e^4 + \frac{105}{256}e^6 + \frac{2205}{4096}e^8 + \frac{10395}{16384}e^{10} + \cdots \\
D &= \frac{35}{512}e^6 + \frac{315}{2048}e^8 + \frac{31185}{131072}e^{10} + \cdots \\
E &= \frac{315}{16384}e^8 + \frac{3465}{65536}e^{10} + \cdots \\
F &= \frac{693}{131072}e^{10} + \cdots
\end{aligned} \qquad (15)$$

Substituting equation (14) into equation (6) gives the meridian distance as

$$m = a\left(1 - e^2\right)\int_0^\phi \{A - B\cos 2\phi + C\cos 4\phi - D\cos 6\phi + E\cos 8\phi - F\cos 10\phi + \cdots\}\, d\phi$$

Integrating term-by-term using the standard integral result $\int_0^x \cos ax\, dx = \frac{\sin ax}{a}$ gives the meridian distance m from the equator to a point in latitude ϕ as

$$\boxed{m = a\left(1 - e^2\right)\left\{A\phi - \frac{B}{2}\sin 2\phi + \frac{C}{4}\sin 4\phi - \frac{D}{6}\sin 6\phi + \frac{E}{8}\sin 8\phi - \frac{F}{10}\sin 10\phi + \cdots\right\}} \qquad (16)$$

where ϕ is in radians and the coefficients A, B, C, etc., are given by equations (15)

For the Geodetic Reference System 1980 (GRS80) ellipsoid $\left(a = 6378137,\ f = 1/298.257222101\right)$ the coefficients A, B, C, etc., have the numeric values

$$\begin{aligned} A &= 1.005\ 052\ 501\ 813\ 087 \\ B &= 0.005\ 063\ 108\ 622\ 224 \\ C &= 0.000\ 010\ 627\ 590\ 263 \\ D &= 0.000\ 000\ 020\ 820\ 379 \\ E &= 0.000\ 000\ 000\ 039\ 324 \\ F &= 0.000\ 000\ 000\ 000\ 071 \end{aligned} \tag{17}$$

Multiplying each of the coefficients in equation (16) by $a\left(1-e^2\right)$ and for latitude ϕ in degrees, meridian distance on the GRS80 ellipsoid can be expressed as

$$\begin{aligned} m = 111\ 132.952\ 546\ 998\,\phi^\circ &- 16\ 038.508\ 741\ 268\ \sin 2\phi \\ &+ 16.832\ 613\ 327\ \sin 4\phi \\ &- 0.021\ 984\ 374\ \sin 6\phi \\ &+ 0.000\ 031\ 142\ \sin 8\phi \\ &- 0.000\ 000\ 045\ \sin 10\phi \end{aligned} \tag{18}$$

and the meridian distance at latitude $\phi = 50^\circ = \dfrac{5\pi}{18}$ radians is

$$m_{50} = 5\ 540\ 847.041\ 560\ 963 \text{ metres} \tag{19}$$

From equation (16), the quadrant distance Q, the meridian distance from the equator to the pole, for the GRS80 ellipsoid is

$$Q = a\left(1-e^2\right)A\left(\tfrac{1}{2}\pi\right) = 10\ 001\ 965.729\ 229\ 864 \text{ metres} \tag{20}$$

Equation (16) may be simplified by multiplying the coefficients by $\left(1-e^2\right)$ and expressing the meridian distance as

$$m = a\left\{A_0\phi - A_2\sin 2\phi + A_4\sin 4\phi - A_6\sin 6\phi + A_8\sin 8\phi - A_{10}\sin 10\phi + \cdots\right\} \tag{21}$$

where $A_0 = (1-e^2)A$, $A_2 = (1-e^2)\dfrac{B}{2}$, $A_4 = (1-e^2)\dfrac{C}{4}$, etc., and

$$\begin{aligned}
A_0 &= 1 - \frac{1}{4}e^2 - \frac{3}{64}e^4 - \frac{5}{256}e^6 - \frac{175}{16384}e^8 - \frac{441}{65536}e^{10} + \cdots \\
A_2 &= \frac{3}{8}\left(e^2 + \frac{1}{4}e^4 + \frac{15}{128}e^6 + \frac{35}{512}e^8 + \frac{735}{16384}e^{10} + \cdots\right) \\
A_4 &= \frac{15}{256}\left(e^4 + \frac{3}{4}e^6 + \frac{35}{64}e^8 + \frac{105}{256}e^{10} + \cdots\right) \\
A_6 &= \frac{35}{3072}\left(e^6 + \frac{5}{4}e^8 + \frac{315}{256}e^{10} + \cdots\right) \\
A_8 &= \frac{315}{131072}\left(e^8 + \frac{7}{4}e^{10} + \cdots\right) \\
A_{10} &= \frac{693}{131072}\left(e^{10} + \cdots\right)
\end{aligned} \tag{22}$$

For the GRS80 ellipsoid, the coefficients A_0, A_2, A_4, etc., have the numeric values

$$\begin{aligned}
A_0 &= 0.998\ 324\ 298\ 423\ 043 \\
A_2 &= 0.002\ 514\ 607\ 124\ 555 \\
A_4 &= 0.000\ 002\ 639\ 111\ 298 \\
A_6 &= 0.000\ 000\ 003\ 446\ 837 \\
A_8 &= 0.000\ 000\ 000\ 004\ 883 \\
A_{10} &= 0.000\ 000\ 000\ 000\ 071
\end{aligned} \tag{23}$$

Multiplying each of the coefficients in equation (21) by a and for latitude ϕ in degrees, meridian distance m on the GRS80 ellipsoid can be expressed as

$$\begin{aligned}
m = 111\ 132.952\ 547\ 005\ \phi^\circ &- 16\ 038.508\ 741\ 587 \sin 2\phi \\
&+ 16.832\ 613\ 418 \sin 4\phi \\
&- 0.021\ 984\ 397 \sin 6\phi \\
&+ 0.000\ 031\ 145 \sin 8\phi \\
&- 0.000\ 000\ 453 \sin 10\phi
\end{aligned} \tag{24}$$

and the meridian distance at latitude $\phi = 50^\circ = \dfrac{5\pi}{18}$ radians is

$$m_{50} = 5\ 540\ 847.041\ 560\ 711 \text{ metres}$$

From equation (21), the quadrant distance Q on the GRS80 ellipsoid is

$$Q = aA_0\left(\tfrac{1}{2}\pi\right) = 10\ 001\ 965.729\ 230\ 469 \text{ metres}$$

Inspection of equations (18) and (24) for the meridian distance m and the values for the distances m_{50} and the quadrant distances Q computed from equations (16) and (21) show that for all practical purposes these equations [(16) and (21)] give identical results at the 5th decimal place of a metre. The differences between the meridian distances m_{50} is $0.000\ 000\ 252$ metres and the quadrant distances Q is $0.000\ 000\ 605$ metres.

THE GDA TECHNICAL MANUAL FORMULA FOR MERIDIAN DISTANCE

In the *Geocentric Datum of Australia Technical Manual* (ICSM 2002) the formula for meridian distance is given in the form

$$m = a\left\{B_0\phi - B_2 \sin 2\phi + B_4 \sin 4\phi - B_6 \sin 6\phi\right\} \tag{25}$$

where

$$\begin{aligned}
B_0 &= 1 - \frac{1}{4}e^2 - \frac{3}{64}e^4 - \frac{5}{256}e^6 \\
B_2 &= \frac{3}{8}\left(e^2 + \frac{1}{4}e^4 + \frac{15}{128}e^6\right) \\
B_4 &= \frac{15}{256}\left(e^4 + \frac{3}{4}e^6\right) \\
B_6 &= \frac{35}{3072}e^6
\end{aligned} \tag{26}$$

This is a contraction of equation (21) and the coefficients B_0, B_2, B_4 and B_6 exclude all terms involving powers of the eccentricity greater than e^6 in the coefficients A_0, A_2, A_4 and A_6. Equations (25) and (26) are the same formula given in Lauf (1983, p. 36, eq'n 3.55), and using these equations the meridian distance m for the GRS80 ellipsoid can be expressed as

$$\begin{aligned}
m = 111\ 132.952\ 549\ 403\ \phi^\circ &- 16\ 038.508\ 411\ 773 \sin 2\phi \\
&+ 16.832\ 200\ 893 \sin 4\phi \\
&- 0.021\ 800\ 767 \sin 6\phi
\end{aligned} \tag{27}$$

The meridian distance on the GRS80 ellipsoid at latitude $\phi = 50^\circ = \frac{5\pi}{18}$ radians is

$$m_{50} = 5\ 540\ 847.041\ 967\ 753 \text{ metres}$$

and the quadrant distance Q is

$$Q = 10\ 001\ 965.729\ 446\ 292 \text{ metres}$$

The differences between the meridian distance m_{50} and the quadrant distance Q computed using equation (25) and the previously computed values using equation (16) are $0.000\ 41$ metres and $0.000\ 22$ metres respectively. We can conclude from this that the formula for meridian distance given in the GDA Technical Manual will give millimetre accuracy for latitudes covering Australia.

MERIDIAN DISTANCE AS A SERIES EXPANSION IN POWERS OF *n*

The German geodesist F.R. Helmert (1880) gave a series formula for meridian distance m as a function of latitude ϕ and powers of an ellipsoid constant n that requires fewer terms than the meridian distance formula involving powers of e^2.

Using equations (1), (4) and (5), the differentially small meridian distance dm is given by

$$dm = \frac{c}{V^3}\, d\phi \tag{28}$$

With the ellipsoid constant n defined as

$$n = \frac{a-b}{a+b} = \frac{f}{2-f} \tag{29}$$

the following relationships can be derived

$$c = \frac{a^2}{b} = a\left(\frac{1+n}{1-n}\right), \qquad e^2 = \frac{4n}{(1+n)^2}, \qquad e'^2 = \frac{4n}{(1-n)^2} \tag{30}$$

Using the last member of equations (30) we may write

$$V^2 = 1 + e'^2 \cos^2\phi = \frac{(1-n)^2 + 4n\cos^2\phi}{(1-n)^2}$$

and using the trigonometric relationship $\cos 2\phi = 2\cos^2\phi - 1$

$$\begin{aligned} V^2 &= \frac{(1-n)^2 + 2n\cos 2\phi + 2n}{(1-n)^2} \\ &= \frac{1}{(1-n)^2}\left(1 + n^2 + 2n\cos 2\phi\right) \end{aligned} \tag{31}$$

Now we can make use of *Euler's identities:* $e^{i\phi} = \cos\phi + i\sin\phi$, $e^{-i\phi} = \cos\phi - i\sin\phi$ in simplifying equation (31) Note that i is the imaginary unit $\left(i^2 = -1\right)$ and

$e = 2.718281828\ldots$ is the base of the natural logarithms. e in Euler's identities should not be confused with the eccentricity of the ellipsoid. Adding Euler's identities gives $2\cos\phi = e^{i\phi} + e^{-i\phi}$ and replacing ϕ with 2ϕ gives $2\cos 2\phi = e^{i2\phi} + e^{-i2\phi}$. Substituting this result into equation (31) gives

$$\begin{aligned} V^2 &= \frac{1}{(1-n)^2}\left(1 + n^2 + n\left(e^{i2\phi} + e^{-i2\phi}\right)\right) \\ &= \frac{1}{(1-n)^2}\left(1 + n^2 + ne^{i2\phi} + ne^{-i2\phi}\right) \\ &= \frac{1}{(1-n)^2}\left(1 + ne^{i2\phi}\right)\left(1 + ne^{-i2\phi}\right) \end{aligned}$$

Now an expression for $\frac{1}{V^3}$ in equation (28) can be developed as

$$\begin{aligned} \frac{1}{V^3} &= \left(V^2\right)^{-\frac{3}{2}} \\ &= \left((1-n)^{-2}\right)^{-\frac{3}{2}}\left(1 + ne^{i2\phi}\right)^{-\frac{3}{2}}\left(1 + ne^{-i2\phi}\right)^{-\frac{3}{2}} \\ &= (1-n)^3\left(1 + ne^{i2\phi}\right)^{-\frac{3}{2}}\left(1 + ne^{-i2\phi}\right)^{-\frac{3}{2}} \end{aligned} \tag{32}$$

Using equation (32) and the first member of equations (30) in equation (28) gives

$$dm = a\left(\frac{1+n}{1-n}\right)(1-n)^3\left(1 + ne^{i2\phi}\right)^{-\frac{3}{2}}\left(1 + ne^{-i2\phi}\right)^{-\frac{3}{2}}\,d\phi \tag{33}$$

Now $\left(\frac{1+n}{1-n}\right)(1-n)^3 = (1+n)(1-n)^2 = (1-n)\left(1-n^2\right)$ and equation (33) becomes (Lauf 1983, p. 36, eq'n 3.57)

$$dm = a(1-n)\left(1-n^2\right)\left(1 + ne^{i2\phi}\right)^{-\frac{3}{2}}\left(1 + ne^{-i2\phi}\right)^{-\frac{3}{2}}\,d\phi \tag{34}$$

Using the binomial series as previously developed [see equation (11)] we may write

$$\begin{aligned} \left(1 + ne^{i2\phi}\right)^{-\frac{3}{2}} = 1 &- \frac{3}{2}ne^{i2\phi} + \frac{3\cdot 5}{2\cdot 4}n^2e^{i4\phi} - \frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}n^3e^{i6\phi} \\ &+ \frac{3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8}n^4e^{i8\phi} - \frac{3\cdot 5\cdot 7\cdot 9\cdot 11}{2\cdot 4\cdot 6\cdot 8\cdot 10}n^5e^{i10\phi} + \cdots \end{aligned}$$

and

$$\begin{aligned} \left(1 + ne^{i2\phi}\right)^{-\frac{3}{2}} = 1 &- \frac{3}{2}ne^{-i2\phi} + \frac{3\cdot 5}{2\cdot 4}n^2e^{-i4\phi} - \frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}n^3e^{-i6\phi} \\ &+ \frac{3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8}n^4e^{-i8\phi} - \frac{3\cdot 5\cdot 7\cdot 9\cdot 11}{2\cdot 4\cdot 6\cdot 8\cdot 10}n^5e^{-i10\phi} + \cdots \end{aligned}$$

The product of these two series, after gathering terms, will be a series in terms $\left(e^{i2\phi} + e^{-i2\phi}\right) = \cos 2\phi$, $\left(e^{i4\phi} + e^{-i4\phi}\right) = \cos 4\phi$, $\left(e^{i6\phi} + e^{-i6\phi}\right) = \cos 6\phi$, etc.; each term having coefficients involving powers of n. Using this product in equation (34) and simplifying gives

$$\begin{aligned} dm = a(1-n)\left(1-n^2\right)\Big\{&1+\frac{9}{4}n^2+\frac{225}{64}n^4+\cdots \\ &-2\cos 2\phi \left(\frac{3}{2}n+\frac{45}{16}n^3+\frac{525}{128}n^5+\cdots\right) \\ &+2\cos 4\phi \left(\frac{15}{8}n^2+\frac{105}{32}n^4+\cdots\right) \\ &-2\cos 6\phi \left(\frac{35}{16}n^3+\frac{945}{256}n^5+\cdots\right) \\ &+2\cos 8\phi \left(\frac{315}{128}n^4+\cdots\right) \\ &-2\cos 10\phi \left(\frac{693}{256}n^5+\cdots\right)+\cdots\Big\}\,d\phi \end{aligned}$$

Integrating term-by-term using the standard integral result $\int_0^x \cos ax\,dx = \frac{\sin ax}{a}$ gives the meridian distance m from the equator to a point in latitude ϕ as

$$m = a(1-n)\left(1-n^2\right)\left\{a_0\phi - a_2\sin 2\phi + a_4\sin 4\phi - a_6\sin 6\phi + a_8\sin 8\phi - a_{10}\sin 10\phi + \cdots\right\} \quad (35)$$

where

$$\begin{aligned} a_0 &= 1+\frac{9}{4}n^2+\frac{225}{64}n^4+\cdots \\ a_2 &= \frac{3}{2}n+\frac{45}{16}n^3+\frac{525}{128}n^5+\cdots \\ a_4 &= \frac{1}{2}\left(\frac{15}{8}n^2+\frac{105}{32}n^4+\cdots\right) \\ a_6 &= \frac{1}{3}\left(\frac{35}{16}n^3+\frac{945}{256}n^5+\cdots\right) \\ a_8 &= \frac{1}{4}\left(\frac{315}{128}n^4+\cdots\right) \\ a_{10} &= \frac{1}{5}\left(\frac{693}{256}n^5+\cdots\right) \end{aligned} \quad (36)$$

For the GRS80 ellipsoid, the coefficients a_0, a_2, a_4, etc., have the numeric values

$$\begin{aligned}
a_0 &= 1.000\ 006\ 344\ 535\ 504 \\
a_2 &= 0.002\ 518\ 843\ 909\ 281 \\
a_4 &= 0.000\ 002\ 643\ 557\ 858 \\
a_6 &= 0.000\ 000\ 003\ 452\ 645 \\
a_8 &= 0.000\ 000\ 000\ 004\ 892 \\
a_{10} &= 0.000\ 000\ 000\ 000\ 007
\end{aligned} \tag{37}$$

Multiplying each of the coefficients in equation (35) by $a(1-n)\left(1-n^2\right)$ and for latitude ϕ in degrees, meridian distance m on the GRS80 ellipsoid can be expressed as

$$\begin{aligned}
m = 111\ 132.952\ 547\ 005\ \phi^\circ &- 16\ 038.508\ 741\ 594 \sin 2\phi \\
&+ 16.832\ 613\ 428 \sin 4\phi \\
&- 0.021\ 984\ 404 \sin 6\phi \\
&+ 0.000\ 031\ 148 \sin 8\phi \\
&- 0.000\ 000\ 046 \sin 10\phi
\end{aligned} \tag{38}$$

with the meridian distance at latitude $\phi = 50^\circ = \dfrac{5\pi}{18}$ radians is

$$m_{50} = 5\ 540\ 847.041\ 560\ 969 \text{ metres}$$

and the quadrant distance Q on the GRS80 ellipsoid is

$$Q = a(1-n)\left(1-n^2\right) a_0 \left(\tfrac{1}{2}\pi\right) = 10\ 001\ 965.729\ 230\ 464 \text{ metres}$$

The differences between the meridian distance m_{50} and the quadrant distance Q computed using equation (35) and the previously computed values using equation (16) are 0.000 000 006 metres and 0.000 000 600 metres respectively. We can conclude from this that for all practical purposes equations (16), (21) and (35) give identical results at the 5th decimal place of a metre.

HELMERT'S FORMULA FOR MERIDIAN DISTANCE

Jordan/Eggert/Kneissl (1958, p.83) in a section titled *Helmertsche Formeln zur Rektifikation des Meridianbogens* (Helmert's formula for meridian distance) outlines a method of derivation attributed to Helmert (1880) that is similar to the derivation in the previous section. Their starting point (and presumably Helmert's) was $\rho = \dfrac{a\left(1-e^2\right)}{W^3}$ and $\left(1-e^2\right) = \dfrac{(1-n)^2}{(1+n)^2}$ rather than $\rho = \dfrac{c}{V^3}$ and $dm = \dfrac{c}{V^3}\,d\phi$ as above but the end result (Jordan/Eggert/Kneissl 1958, eq'n 38, p.83) is similar in form to equation (35) but without the term $-a_{10}\sin 10\phi$ and the coefficients exclude all terms involving powers of n greater than n^4. With these restrictions we give *Helmert's formula* as (Lauf 1983, p. 36, eq'n 3.55)

$$m = a(1-n)\left(1-n^2\right)\left\{b_0\phi - b_2\sin 2\phi + b_4\sin 4\phi - b_6\sin 6\phi + b_8\sin 8\phi - \cdots\right\} \tag{39}$$

where

$$\begin{aligned}
b_0 &= 1 + \frac{9}{4}n^2 + \frac{225}{64}n^4 + \cdots \\
b_2 &= \frac{3}{2}n + \frac{45}{16}n^3 + \cdots \\
b_4 &= \frac{1}{2}\left(\frac{15}{8}n^2 + \frac{105}{32}n^4 + \cdots\right) \\
b_6 &= \frac{1}{3}\left(\frac{35}{16}n^3 + \cdots\right) \\
b_8 &= \frac{1}{4}\left(\frac{315}{128}n^4 + \cdots\right)
\end{aligned} \tag{40}$$

For the GRS80 ellipsoid, the coefficients $b_0,\ b_2,\ b_4,$ etc., have the numeric values

$$\begin{aligned} b_0 &= 1.000\ 006\ 344\ 535\ 504 \\ b_2 &= 0.002\ 518\ 843\ 909\ 226 \\ b_4 &= 0.000\ 002\ 643\ 557\ 858 \\ b_6 &= 0.000\ 000\ 003\ 452\ 629 \\ b_8 &= 0.000\ 000\ 000\ 004\ 892 \end{aligned} \tag{41}$$

Multiplying each of the coefficients in equation (39) by $a(1-n)\left(1-n^2\right)$ and for latitude ϕ in degrees, meridian distance m on the GRS80 ellipsoid can be expressed as

$$\begin{aligned} m = 111\ 132.952\ 547\ 005\ \phi^\circ &- 16\ 038.508\ 741\ 245 \sin 2\phi \\ &+ \quad 16.832\ 613\ 428 \sin 4\phi \\ &- \quad 0.021\ 984\ 300 \sin 6\phi \\ &+ \quad 0.000\ 031\ 148 \sin 8\phi \end{aligned} \tag{42}$$

Using equation (39), the meridian distance at latitude $\phi = 50^\circ = \dfrac{5\pi}{18}$ radians on the GRS80 ellipsoid is

$$m_{50} = 5\ 540\ 847.041\ 561\ 252 \text{ metres}$$

and the quadrant distance Q is

$$Q = 10\ 001\ 965.729\ 230\ 464 \text{ metres}$$

The differences between the meridian distance m_{50} and the quadrant distance Q computed using equation (39) and the previously computed values using equation (16) are 0.000 000 289 metres and 0.000 000 600 metres respectively.

AN ALTERNATIVE FORM OF HELMERT'S FORMULA

An alternative form of *Helmert's formula* [equation (39)] can be developed by noting that

$$\begin{aligned}(1-n)\left(1-n^2\right) &= \frac{1+n}{1+n}(1-n)\left(1-n^2\right)\\ &= \frac{\left(1-n^2\right)\left(1-n^2\right)}{1+n}\end{aligned}$$

Multiplying the coefficients $b_0,\ b_2,\ b_4,\ b_6$ and b_8 by $\left(1-n^2\right)\left(1-n^2\right)$ gives

$$m = \frac{a}{1+n}\left\{c_0\phi - c_2 \sin 2\phi + c_4 \sin 4\phi - c_6 \sin 6\phi + c_8 \sin 8\phi - \cdots\right\} \tag{43}$$

where

$$\begin{aligned}c_0 &= 1 + \frac{1}{4}n^2 + \frac{1}{64}n^4 + \cdots\\ c_2 &= \frac{3}{2}\left(n - \frac{1}{8}n^3 - \cdots\right)\\ c_4 &= \frac{15}{16}\left(n^2 - \frac{1}{4}n^4 - \cdots\right)\\ c_6 &= \frac{35}{48}\left(n^3 - \cdots\right)\\ c_8 &= \frac{315}{512}\left(n^4 - \cdots\right)\end{aligned} \tag{44}$$

Equation (43) with expressions for the coefficients $c_0,\ c_2,\ c_4$ etc., is, except for a slight change in notation, the same as Rapp (1982, p. 30, eq'n 95) who cites Helmert (1880) and is essentially the same as Baeschlin (1948, p. 50, eq'n 5.5) and Jordan/Eggert/Kneissl (1958, p.83-2, eq'ns 38 and 42)

For the GRS80 ellipsoid, the coefficients $c_0,\ c_2,\ c_4,$ etc., have the numeric values

$$\begin{aligned}c_0 &= 1.000\ 000\ 704\ 945\ 408\\ c_2 &= 0.002\ 518\ 829\ 704\ 124\\ c_4 &= 0.000\ 002\ 643\ 542\ 949\\ c_6 &= 0.000\ 000\ 003\ 452\ 629\\ c_8 &= 0.000\ 000\ 000\ 004\ 892\end{aligned} \tag{45}$$

Multiplying each of the coefficients in equation (43) by $\frac{a}{1+n}$ and for latitude ϕ in degrees, meridian distance m on the GRS80 ellipsoid can be expressed as

$$
\begin{aligned}
m = 111\ 132.952\ 547\ 005\ \phi^\circ &- 16\ 038.508\ 741\ 596 \sin 2\phi \\
&+ \quad 16.832\ 613\ 428 \sin 4\phi \\
&- \quad 0.021\ 984\ 424 \sin 6\phi \\
&+ \quad 0.000\ 031\ 148 \sin 8\phi
\end{aligned} \tag{46}
$$

Using equation (43), the meridian distance at latitude $\phi = 50^\circ = \dfrac{5\pi}{18}$ radians on the GRS80 ellipsoid is

$$m_{50} = 5\ 540\ 847.041\ 561\ 015 \text{ metres}$$

and the quadrant distance Q is

$$Q = 10\ 001\ 965.729\ 230\ 464 \text{ metres}$$

The difference between the meridian distances m_{50} computed using equation (39) [*Helmert's formula*] and equation (43) (its alternative form) is 0.000 000 237 metres and the quadrant distances Q are identical. The differences between m_{50} and Q computed using equation (43) and the previously computed values using equation (16) are 0.000 000 052 metres and 0.000 000 600 metres respectively. We can conclude from this and other comparisons made in previous sections that (i) for all practical purposes, equations (16), (21), (35), (39) and (43) give identical results at the 5th decimal place of a metre and (ii) equations (39) (*Helmert's formula*) or (43) (the alternative form of *Helmert's formula*) are the simplest, having fewer terms in the expressions for the coefficients than the other three equations that give comparable accuracy.

LATITUDE FROM HELMERT'S FORMULA BY REVERSION OF A SERIES

Helmert's formula [equation (39)] gives meridian distance m as a function of latitude ϕ and powers of n and this formula (or another involving ϕ and e^2 developed above) is necessary for the conversion of ϕ, λ to UTM projection coordinates *E,N.* The reverse operation, *E,N* to ϕ, λ requires a method of computing ϕ given *m.* This could be done by a computer program implementing the Newton-Raphson scheme of iteration (described in a following section), or as it was in pre-computer days, by inverse interpolation of printed tables of latitudes and meridian distances. An efficient direct formula can be obtained by "reversing" *Helmert's formula* using *Lagrange's Theorem* to give a series formula for ϕ as a function of an angular quantity σ and powers of *n;* and σ, as we shall see, is directly connected to the meridian distance *m.* We thus have a direct way of computing ϕ given m that is extremely useful in map projection computations.

The following pages contain an expanded explanation of the very concise derivation set out in Lauf (1983); the only text on Geodesy where (to my knowledge) this useful technique and formula is set down.

Using *Helmert's formula* [equation (39)] and substituting the value $\phi = \frac{1}{2}\pi$ gives a formula for the quadrant distance Q as

$$Q = a(1-n)\left(1-n^2\right)\left(1+\frac{9}{4}n^2+\frac{225}{64}n^4+\cdots\right)\frac{\pi}{2} \qquad (47)$$

[The quadrant distance is the length of the meridian arc from the equator to the pole and the ten-millionth part of this distance was originally intended to have defined the metre when that unit was introduced. For those interested in the history of geodesy, *The Measure Of All Things* (Adler 2002) has a detailed account of the measurement of the French Arc (an arc of the meridian from Dunkerque, France to Barcelona, Spain and passing through Paris) by John-Baptiste-Joseph Delambre and Pierre-François-André Méchain in 1792-9 during the French Revolution. The analysis of their measurements enabled the computation of the dimensions of the earth that lead to the definitive metre platinum bar of 1799.]

Also, we can establish two quantities:

(i) *G,* the mean length of a meridian arc of one radian

$$G = \frac{Q}{\frac{1}{2}\pi} = a(1-n)\left(1-n^2\right)\left(1+\frac{9}{4}n^2+\frac{225}{64}n^4+\cdots\right) \tag{48}$$

(ii) σ , an angular quantity in radians and

$$\sigma = \frac{m}{G} \tag{49}$$

An expression for σ as a function of ϕ and powers of n is obtained by dividing equation (48) into *Helmert's formula* [equation (39)] giving

$$\begin{aligned}\sigma &= \phi - \left\{\frac{\frac{3}{2}n+\frac{45}{16}n^3+\cdots}{1+\frac{9}{4}n^2+\frac{225}{64}n^4+\cdots}\right\}\sin 2\phi + \frac{1}{2}\left\{\frac{\frac{15}{8}n^2+\frac{105}{32}n^4+\cdots}{1+\frac{9}{4}n^2+\frac{225}{64}n^4+\cdots}\right\}\sin 4\phi \\ &\quad -\frac{1}{3}\left\{\frac{\frac{35}{16}n^3+\cdots}{1+\frac{9}{4}n^2+\frac{225}{64}n^4+\cdots}\right\}\sin 6\phi + \frac{1}{4}\left\{\frac{\frac{315}{128}n^4+\cdots}{1+\frac{9}{4}n^2+\frac{225}{64}n^4+\cdots}\right\}\sin 8\phi - \cdots\end{aligned} \tag{50}$$

Using a special case of the binomial theorem

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots$$

the numerator of each coefficient in the equation for σ can be written as

$$\begin{aligned}\left(1+\frac{9}{4}n^2+\frac{225}{64}n^4+\cdots\right)^{-1} &= 1-\left(\frac{9}{4}n^2+\frac{225}{64}n^4+\cdots\right)+\left(\frac{9}{4}n^2+\frac{225}{64}n^4+\cdots\right)^2 \\ &\quad -\left(\frac{9}{4}n^2+\frac{225}{64}n^4+\cdots\right)^3+\cdots\end{aligned}$$

and expanding the right-hand side and simplifying gives

$$\left(1+\frac{9}{4}n^2+\frac{225}{64}n^4+\cdots\right)^{-1} = 1-\frac{9}{4}n^2+\frac{99}{64}n^4-\cdots \tag{51}$$

Substituting equation (51) into equation (50), multiplying the terms and simplifying gives the equation for σ as (Lauf 1983, p. 37, eq'n 3.67)

$$\begin{aligned}\sigma = \frac{m}{G} &= \phi-\left(\frac{3}{2}n-\frac{9}{16}n^3-\cdots\right)\sin 2\phi+\left(\frac{15}{16}n^2-\frac{15}{32}n^4-\cdots\right)\sin 4\phi \\ &\quad -\left(\frac{35}{48}n^3-\cdots\right)\sin 6\phi+\left(\frac{315}{512}n^4-\cdots\right)\sin 8\phi-\cdots\end{aligned} \tag{52}$$

If we require the value of ϕ corresponding to a particular value of σ, then the series (52) needs to be reversed. This can be done using *Lagrange's Theorem* (or *Lagrange's expansion*) a proof of which can be found in Carr (1970).

Suppose that

$$y = z + xF(y) \quad \text{or} \quad z = y - xF(y) \tag{53}$$

then *Lagrange's Theorem* states that

$$\begin{aligned} f(y) &= f(z) + xF(z)f'(z) + \frac{x^2}{2!}\frac{d}{dz}\left[\{F(z)\}^2 f'(z)\right] + \frac{x^3}{3!}\frac{d^2}{dz^2}\left[\{F(z)\}^3 f'(z)\right] + \cdots \\ &\cdots + \frac{x^n}{n!}\frac{d^{n-1}}{dz^{n-1}}\left[\{F(z)\}^n f'(z)\right] \end{aligned} \tag{54}$$

In our case, comparing the variables in equations (52) and (53), $z = \sigma$, $y = \phi$ and $x = 1$, and if we choose $f(y) = y$ then $f(z) = z$ and $f'(z) = 1$. So, in our case equation (52) can be expressed as

$$\sigma = \phi - F(\phi) \tag{55}$$

and *Lagrange's Theorem* gives

$$\phi = \sigma + F(\sigma) + \frac{1}{2}\frac{d}{d\sigma}\left[\{F(\sigma)\}^2\right] + \frac{1}{6}\frac{d^2}{d\sigma^2}\left[\{F(\sigma)\}^3\right] + \cdots + \frac{1}{n!}\frac{d^{n-1}}{d\sigma^{n-1}}\left[\{F(\sigma)\}^n\right] \tag{56}$$

Now, comparing equations (55) and (52) the function $F(\phi)$ is

$$\begin{aligned} F(\phi) = &\left(\frac{3}{2}n - \frac{9}{16}n^3 - \cdots\right)\sin 2\phi - \left(\frac{15}{16}n^2 - \frac{15}{32}n^4 - \cdots\right)\sin 4\phi \\ &+ \left(\frac{35}{48}n^3 - \cdots\right)\sin 6\phi - \left(\frac{315}{512}n^4 - \cdots\right)\sin 8\phi - \cdots \end{aligned}$$

and so replacing ϕ with σ gives the function $F(\sigma)$ in equation (56) as

$$\begin{aligned} F(\sigma) = &\left(\frac{3}{2}n - \frac{9}{16}n^3 - \cdots\right)\sin 2\sigma - \left(\frac{15}{16}n^2 - \frac{15}{32}n^4 - \cdots\right)\sin 4\sigma \\ &+ \left(\frac{35}{48}n^3 - \cdots\right)\sin 6\sigma - \left(\frac{315}{512}n^4 - \cdots\right)\sin 8\sigma - \cdots \end{aligned} \tag{57}$$

Squaring $F(\sigma)$ gives

$$\begin{aligned} \{F(\sigma)\}^2 = &\left(\frac{9}{4}n^2 - \frac{27}{16}n^4 + \cdots\right)\sin^2 2\sigma - \left(\frac{45}{16}n^3 - \cdots\right)\sin 2\sigma \sin 4\sigma \\ &+ \left(\frac{35}{16}n^4 - \cdots\right)\sin 2\sigma \sin 6\sigma + \left(\frac{225}{256}n^4 - \cdots\right)\sin^2 4\sigma - \cdots \end{aligned}$$

and expressing powers and products of trigonometric functions as multiple angles using $\sin^2 A = \frac{1}{2} - \frac{1}{2}\cos 2A$ and $\sin A \sin B = \frac{1}{2}\left\{\cos(A-B) - \cos(A+B)\right\}$ gives, after some simplification

$$\{F(\sigma)\}^2 = \left(\frac{9}{8}n^2 - \frac{207}{512}n^4 - \cdots\right) - \left(\frac{45}{32}n^3 + \cdots\right)\cos 2\sigma - \left(\frac{9}{8}n^2 - \frac{31}{16}n^4 + \cdots\right)\cos 4\sigma$$
$$+\left(\frac{45}{32}n^3 - \cdots\right)\cos 6\sigma - \left(\frac{785}{512}n^4 + \cdots\right)\cos 8\sigma + \cdots$$

Differentiating with respect to σ and then dividing by 2 gives the 3rd term in equation (56) as

$$\frac{1}{2}\frac{d}{d\sigma}\left[\{F(\sigma)\}^2\right] = \left(\frac{45}{32}n^3 - \cdots\right)\sin 2\sigma + \left(\frac{9}{4}n^2 - \frac{31}{8}n^4 - \cdots\right)\sin 4\sigma$$
$$-\left(\frac{135}{32}n^3 - \cdots\right)\sin 6\sigma + \left(\frac{785}{128}n^4 - \cdots\right)\sin 8\sigma + \cdots \tag{58}$$

Using similar methods the 4th and 5th terms in equation (56) are

$$\frac{1}{6}\frac{d^2}{d\sigma^2}\left[\{F(\sigma)\}^3\right] = -\left(\frac{27}{16}n^3 - \cdots\right)\sin 2\sigma + \left(\frac{135}{16}n^4 - \cdots\right)\sin 4\sigma$$
$$+\left(\frac{81}{16}n^3 - \cdots\right)\sin 6\sigma - \left(\frac{135}{8}n^4 - \cdots\right)\sin 8\sigma + \cdots \tag{59}$$

$$\frac{1}{24}\frac{d^3}{d\sigma^3}\left[\{F(\sigma)\}^4\right] = -\left(\frac{27}{4}n^4 - \cdots\right)\sin 4\sigma + \left(\frac{27}{2}n^4 - \cdots\right)\sin 8\sigma + \cdots \tag{60}$$

Substituting equations (57) to (60) into equation (56) and simplifying gives an equation for ϕ as a function of σ and powers of n as (Lauf 1983, p. 38, eq'n 3.72)

$$\phi = \sigma + \left(\frac{3}{2}n - \frac{27}{32}n^3 - \cdots\right)\sin 2\sigma + \left(\frac{21}{16}n^2 - \frac{55}{32}n^4 + \cdots\right)\sin 4\sigma$$
$$+\left(\frac{151}{96}n^3 + \cdots\right)\sin 6\sigma + \left(\frac{1097}{512}n^4 - \cdots\right)\sin 8\sigma - \cdots \tag{61}$$

where $\sigma = \dfrac{m}{G}$ radians and G is given by equation (48). This very useful series now gives a direct way of computing the latitude given a meridian distance.

LATITUDE FROM HELMERT'S FORMULA USING NEWTON-RAPHSON ITERATION

In the preceding section, *Helmert's formula* was "reversed" using *Lagrange's Theorem* to give equation (61), a direct solution for the latitude ϕ given the meridian distance m and the ellipsoid parameters. As an alternative, a value for ϕ can be computed using the Newton-Raphson method for the real roots of the equation $f(\phi) = 0$ given in the form of an iterative equation

$$\phi_{n+1} = \phi_n - \frac{f(\phi_n)}{f'(\phi_n)} \tag{62}$$

where n denotes the n^{th} iteration and $f(\phi)$ can be obtained from *Helmert's formula* [equation (39)] as

$$f(\phi) = a(1-n)(1-n^2)\{b_0\phi - b_2 \sin 2\phi + b_4 \sin 4\phi - b_6 \sin 6\phi + b_8 \sin 8\phi\} - m \tag{63}$$

and the derivative $f'(\phi) = \frac{d}{d\phi}\{f(\phi)\}$ is given by

$$f'(\phi) = a(1-n)(1-n^2)\{b_0 - 2b_2 \cos 2\phi + 4b_4 \cos 4\phi - 6b_6 \cos 6\phi + 8b_8 \cos 8\phi\} \tag{64}$$

An initial value for ϕ (for $n = 1$) can be computed from $\phi_1 = \frac{m}{a}$ and the functions $f(\phi_1)$ and $f'(\phi_1)$ evaluated from equations (63) and (64) using ϕ_1. ϕ_2 (ϕ for $n = 2$) can now be computed from equation (62) and this process repeated to obtain values $\phi_3, \phi_4, \ldots$. This iterative process can be concluded when the difference between ϕ_{n+1} and ϕ_n reaches an acceptably small value.

MATLAB FUNCTIONS FOR: (i) MERIDIAN DISTANCE USING HELMERT'S FORMULA, (ii) LATITUDE FROM HELMERT'S FORMULA USING REVERSION OF A SERIES AND (iii) LATITUDE FROM HELMERT'S FORMULA USING NEWTON-RAPHSON ITERATION

Three MATLAB functions are given below. The first function *mdist.m* uses *Helmert's formula* [equation (39)] to compute the meridian distance m given the ellipsoid parameters a (semi-major axis of ellipsoid)and *flat* (the denominator of the flattening f) and the latitude *lat* in the form ddd.mmss, where a latitude of $-37^\circ\,48'\,33.1234''$ would be input into the function as -37.48331234. The function is designed to be run from the MATLAB command window with output from the function printed in the MATLAB command window.

The second function *latitude.m* computes the latitude ϕ given the meridian distance m and the ellipsoid parameters a (semi-major axis of ellipsoid), *flat* (the denominator of the flattening f). The function uses equation (61), the series formula developed by reversing *Helmert's formula,* and is designed to be run from the MATLAB command window with output from the function printed in the MATLAB command window.

The third function *latitude2.m* computes the latitude ϕ given the meridian distance m and the ellipsoid parameters a (semi-major axis of ellipsoid), *flat* (the denominator of the flattening f). The function uses the Newton-Raphson iterative scheme to compute the latitude from *Helmert's formula* [equation (39)] and is designed to be run from the MATLAB command window with output from the function printed in the MATLAB command window.

MATLAB FUNCTION *mdist.m*

```
function mdist(a,flat,lat)
%
% MDIST(A,FLAT,LAT)  Function computes the meridian distance on an
%  ellipsoid defined by semi-major axis (A) and denominator of flattening
%  (FLAT) from the equator to a point having latitude (LAT) in d.mmss format.
%  For example: mdist(6378137, 298.257222101, -37.48331234) will compute the
%  meridian distance for a point having latitude -37 degrees 48 minutes
%  33.1234 seconds on the GRS80 ellipsoid (a = 6378137, f = 1/298.257222101)

%--------------------------------------------------------------------------
% Function:  mdist()
%
% Usage:     mdist(a,flat,lat)
%
% Author:    R.E.Deakin,
%            School of Mathematical & Geospatial Sciences, RMIT University
%            GPO Box 2476V, MELBOURNE, VIC 3001, AUSTRALIA.
%            email: rod.deakin@rmit.edu.au
%            Version  1.0  22 March 2006
%
% Purpose:   Function mdist(a,f,lat) will compute the meridian distance on
%            an ellipsoid defined by semi-major axis a and flat, the
%            denominator of the flattening f where f = 1/flat.  Latitude is
%            given in d.mmss format.
%
% Functions required:
%            decdeg = dms2deg(dms)
%            [D,M,S] = DMS(DecDeg)
%
% Variables: a      - semi-major axis of spheroid
%            b0,b1,b2, - coefficients
%            d2r    - degree to radian conversion factor 57.29577951...
%            n,n2,n3, etc - powers of n
%            f      - f = 1/flat is the flattening of ellipsoid
%            flat   - denominator of flattening of ellipsoid
%
% Remarks:   Helmert's formula for meridian distance is given in
%            Lauf, G.B., 1983, Geodesy and Map Projections,
%            TAFE Publications Unit, Collingwood, p. 36, eq'n 3.58.
%            A derivation can also be found in Deakin, R.E., Meridian
%            Distance, Lecture Notes, School of Mathematical and
%            Geospatial Sciences, RMIT University, March 2006.
%--------------------------------------------------------------------------

% degree to radian conversion factor
d2r = 180/pi;

% compute flattening f and ellipsoid constant n
f = 1/flat;
n = f/(2-f);

% powers n
n2 = n*n;
n3 = n2*n;
n4 = n3*n;

% coefficients in Helmert's series expansion for meridian distance
b0  = 1+(9/4)*n2+(225/64)*n4;
b2  = (3/2)*n+(45/16)*n3;
b4  = (1/2)*((15/8)*n2+(105/32)*n4);
b6  = (1/3)*((35/16)*n3);
b8  = (1/4)*((315/128)*n4);

% compute meridian distance
x = abs(dms2deg(lat)/d2r);
term1 = b0*x;
term2 = b2*sin(2*x);
```

```
term3 = b4*sin(4*x);
term4 = b6*sin(6*x);
term5 = b8*sin(8*x);

mdist = a*(1-n)*(1-n2)*(term1-term2+term3-term4+term5);

% print result to screen
fprintf('\n a  = %12.4f',a);
fprintf('\n f  = 1/%13.9f',flat);
[D,M,S] = DMS(x*d2r);
if D == 0 && lat < 0
    fprintf('\nLatitude =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLatitude      = %4d %2d %9.6f (D M S)',D,M,S);
end
    fprintf('\nMeridian dist = %15.6f',mdist);

fprintf('\n\n');
```

OUTPUT FROM MATLAB FUNCTION *mdist.m*

```
>> help mdist


  MDIST(A,FLAT,LAT)  Function computes the meridian distance on an
   ellipsoid defined by semi-major axis (A) and denominator of flattening
   (FLAT) from the equator to a point having latitude (LAT) in d.mmss format.
   For example: mdist(6378137, 298.257222101, -37.48331234) will compute the
   meridian distance for a point having latitude -37 degrees 48 minutes
   33.1234 seconds on the GRS80 ellipsoid (a = 6378137, f = 1/298.257222101)

>> mdist(6378137,298.257222101,-37.48331234)

 a  = 6378137.0000
 f  = 1/298.257222101
Latitude      =   37 48 33.123400 (D M S)
Meridian dist =  4186320.340377

>>
```

MATLAB FUNCTION *latitude.m*

```
function latitude(a,flat,mdist)
%
% LATITUDE(A,FLAT,MDIST)  Function computes the latitude of a point
%   on an ellipsoid defined by semi-major axis (A) and denominator of
%   flattening (FLAT) given the meridian distance (MDIST) from the
%   equator to the point.
%   For example: latitude(6378137,298.257222101,5540847.041561) should
%   return a latitude of 50 degrees 00 minutes 00 seconds for a meridian
%   distance of 5540847.041561m on the GRS80 ellipsoid (a = 6378137, f =
%   1/298.257222101)

%----------------------------------------------------------------------------
% Function:  latitude()
%
% Usage:     latitude(a,f,mdist)
%
% Author:    R.E.Deakin,
%            School of Mathematical & Geospatial Sciences, RMIT University
%            GPO Box 2476V, MELBOURNE, VIC 3001, AUSTRALIA.
%            email: rod.deakin@rmit.edu.au
%            Version  1.0  23 March 2006
%
% Functions required:
%            [D,M,S] = DMS(DecDeg)
%
% Purpose:
%    Function latitude() will compute the latitude of a point on on an
%    ellipsoid defined by semi-major axis (a) and denominator of
%    flattening (flat) given meridian distance (m_dist) from the
%    equator to the point.
%
% Variables:
%    a       - semi-major axis of spheroid
%    d2r     - degree to radian conversion factor 57.29577951...
%    f       - flattening of ellipsoid
%    flat    - denominator of flattening f = 1/flat
%    lat     - latitude (degrees)
%    g       - mean length of an arc of one radian of the meridian
%    mdist   - meridian distance
%    n       - eta, n = f/(2-f)
%    n2,n4,  - powers of eta
%    s       - sigma s = m_dist/g
%    s2,s3,  - powers of sigma
%
% Remarks:
%    For an ellipsoid defined by semi-major axis (a) and flattening (f) the
%    meridian distance (mdist) can be computed by series expansion
%    formulae (see function mdist.m).  The reverse operation, given a
%    meridian distance on a defined ellipsoid to calculate the latitude,
%    can be achieved by series formulae published in THE AUSTRALIAN GEODETIC
%    DATUM Technical Manual Special Publication 10, National Mapping Council
%    of Australia, 1986 (section 4.4, page 24-25).  The development of these
%    formulae are given in Lauf, G.B., 1983, GEODESY AND MAP PROJECTIONS,
%    Tafe Publications, Vic., pp.35-38.
%    This function is generally used to compute the "footpoint latitude"
%    which is the latitude for which the meridian distance is equal to the
%    y-coordinate divided by the central meridian scale factor, i.e.,
%    latitude for m_dist = y/k0.
%----------------------------------------------------------------------------

% degree to radian conversion factor
d2r = 180/pi;

% calculate flatteninf f and ellipsoid constant n and powers of n
f  = 1/flat;
n  = f/(2.0-f);
n2 = n*n;
```

```
n3 = n2*n;
n4 = n3*n;

% calculate the mean length an arc of one radian on the meridian
g = a*(1-n)*(1-n2)*(1+9/4*n2+225/64*n4);

% calculate sigma (s) and powers of sigma
s  = mdist/g;
s2 = 2.0*s;
s4 = 4.0*s;
s6 = 6.0*s;
s8 = 8.0*s;

% calculate the latitude (in radians)
lat = s + (3*n/2 - 27/32*n3)*sin(s2)...
        + (21/16*n2 - 55/32*n4)*sin(s4)...
        + (151/96*n3)*sin(s6)...
        + (1097/512*n4)*sin(s8);

% convert latitude to degrees
lat = lat*d2r;

% print result to screen
fprintf('\n a  = %12.4f',a);
fprintf('\n f  = 1/%13.9f',flat);
[D,M,S] = DMS(lat);
if D == 0 && lat < 0
    fprintf('\nLatitude =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLatitude      = %4d %2d %9.6f (D M S)',D,M,S);
end
    fprintf('\nMeridian dist = %15.6f',mdist);

fprintf('\n\n');
```

OUTPUT FROM MATLAB FUNCTION *latitude.m*

```
>> help latitude


  LATITUDE(A,FLAT,MDIST)  Function computes the latitude of a point
    on an ellipsoid defined by semi-major axis (A) and denominator of
    flattening (FLAT) given the meridian distance (MDIST) from the
    equator to the point.
    For example: latitude(6378137,298.257222101,5540847.041561) should
    return a latitude of 50 degrees 00 minutes 00 seconds for a meridian
    distance of 5540847.041561m on the GRS80 ellipsoid (a = 6378137, f =
    1/298.257222101)

>> latitude(6378137,298.257222101,4186320.340377)

 a  = 6378137.0000
 f  = 1/298.257222101
Latitude      =   37 48 33.123400 (D M S)
Meridian dist =  4186320.340377

>>
```

MATLAB FUNCTION *latitude2.m*

```
function latitude2(a,flat,mdist)
%
% LATITUDE2(A,FLAT,MDIST)  Function computes the latitude of a point
%   on an ellipsoid defined by semi-major axis (A) and denominator of
%   flattening (FLAT) given the meridian distance (MDIST) from the
%   equator to the point.
%   For example: latitude(6378137,298.257222101,5540847.041561) should
%   return a latitude of 50 degrees 00 minutes 00 seconds for a meridian
%   distance of 5540847.041561m on the GRS80 ellipsoid (a = 6378137, f =
%   1/298.257222101)

%--------------------------------------------------------------------------
% Function:  latitude2()
%
% Usage:     latitude2(a,f,mdist)
%
% Author:    R.E.Deakin,
%            School of Mathematical & Geospatial Sciences, RMIT University
%            GPO Box 2476V, MELBOURNE, VIC 3001, AUSTRALIA.
%            email: rod.deakin@rmit.edu.au
%            Version  1.0  23 March 2006
%
% Functions required:
%            [D,M,S] = DMS(DecDeg)
%
% Purpose:
%    Function latitude2() will compute the latitude of a point on on an
%    ellipsoid defined by semi-major axis (a) and denominator of
%    flattening (flat) given meridian distance (mdist) from the
%    equator to the point.
%
% Variables:
%    a       - semi-major axis of spheroid
%    b0,b1,b2,... coefficients in Helmert's formula
%    corrn   - correction term in Newton-Raphson iteration
%    count   - iteration number
%    d2r     - degree to radian conversion factor 57.29577951...
%    F       - a function of latitude (Helmert's formula)
%    Fdash   - the derivative of F
%    f       - flattening of ellipsoid
%    flat    - denominator of flattening f = 1/flat
%    lat     - latitude
%    mdist   - meridian distance
%    n       - eta, n = f/(2-f)
%    n2,n4,  - powers of eta
%
% Remarks:
%    For an ellipsoid defined by semi-major axis (a) and flattening (f) the
%    meridian distance (mdist) can be computed by series expansion
%    formulae (see function mdist.m).  The reverse operation, given a
%    meridian distance on a defined ellipsoid to calculate the latitude,
%    can be achieved by using Newton's Iterative scheme.
%--------------------------------------------------------------------------

% degree to radian conversion factor
d2r = 180/pi;

% calculate flattening f and ellipsoid constant n and powers of n
f  = 1/flat;
n  = f/(2.0-f);
n2 = n*n;
n3 = n2*n;
n4 = n3*n;

% coefficients in Helmert's series expansion for meridian distance
b0  = 1+(9/4)*n2+(225/64)*n4;
b2  = (3/2)*n+(45/16)*n3;
```

```
b4  = (1/2)*((15/8)*n2+(105/32)*n4);
b6  = (1/3)*((35/16)*n3);
b8  = (1/4)*((315/128)*n4);

% set the first approximation of the latitude and then Newton's iterative
% scheme where F is the function of latitude and Fdash is the derivative of
% the function F
lat   = mdist/a;
corrn = 1;
count = 0;
while (abs(corrn)>1e-10)
    F = a*(1-n)*(1-n2)*(b0*lat...
                        - b2*sin(2*lat)...
                        + b4*sin(4*lat)...
                        - b6*sin(6*lat)...
                        + b8*sin(8*lat)) - mdist;
    Fdash = a*(1-n)*(1-n2)*(b0...
                            - 2*b2*cos(2*lat)...
                            + 4*b4*cos(4*lat)...
                            - 6*b6*cos(6*lat)...
                            + 8*b8*cos(8*lat));
    corrn  = -F/Fdash;
    lat = lat + corrn;
    count = count+1;
end

% convert latitude to degrees
lat = lat*d2r;

% print result to screen
fprintf('\n a  = %12.4f',a);
fprintf('\n f  = 1/%13.9f',flat);
[D,M,S] = DMS(lat);
if D == 0 && lat < 0
    fprintf('\nLatitude =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLatitude      = %4d %2d %9.6f (D M S)',D,M,S);
end
fprintf('\nMeridian dist = %15.6f',mdist);
fprintf('\niterations    = %d',count);

fprintf('\n\n');
```

OUTPUT FROM MATLAB FUNCTION *latitude2.m*

```
>> help latitude2


  LATITUDE2(A,FLAT,MDIST)  Function computes the latitude of a point
    on an ellipsoid defined by semi-major axis (A) and denominator of
    flattening (FLAT) given the meridian distance (MDIST) from the
    equator to the point.
    For example: latitude(6378137,298.257222101,5540847.041561) should
    return a latitude of 50 degrees 00 minutes 00 seconds for a meridian
    distance of 5540847.041561m on the GRS80 ellipsoid (a = 6378137, f =
    1/298.257222101)

>> latitude2(6378137,298.257222101,4186320.340377)

 a  = 6378137.0000
 f  = 1/298.257222101
Latitude      =   37 48 33.123400 (D M S)
Meridian dist =  4186320.340377
iterations    = 3

>>
```

MATLAB FUNCTIONS *DMS.m* and *dms2deg.m*

MATLAB functions *mdist.m, latitude.m* and *latitude2.m* call functions *DMS.m* and *dms2deg.m* to convert decimal degrees to degrees, minutes and seconds (for printing) and ddd.mmss format to decimal degrees. These functions are shown below.

```
function [D,M,S] = DMS(DecDeg)
% [D,M,S] = DMS(DecDeg)  This function takes an angle in decimal degrees and returns
%   Degrees, Minutes and Seconds

val = abs(DecDeg);
D = fix(val);
M = fix((val-D)*60);
S = (val-D-M/60)*3600;
if(DecDeg<0)
  D = -D;
end
return
```

```
function DecDeg=dms2deg(DMS)
% DMS2DEG
%  Function to convert from DDD.MMSS format to decimal degrees

x = abs(DMS);
D = fix(x);
x = (x-D)*100;
M = fix(x);
S = (x-M)*100;
DecDeg = D + M/60 + S/3600;
if(DMS<0)
  DecDeg = -DecDeg;
end
return
```

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