ENGINEERING SURVEYING 1

PLANES LINES AND DIRECTION COSINES

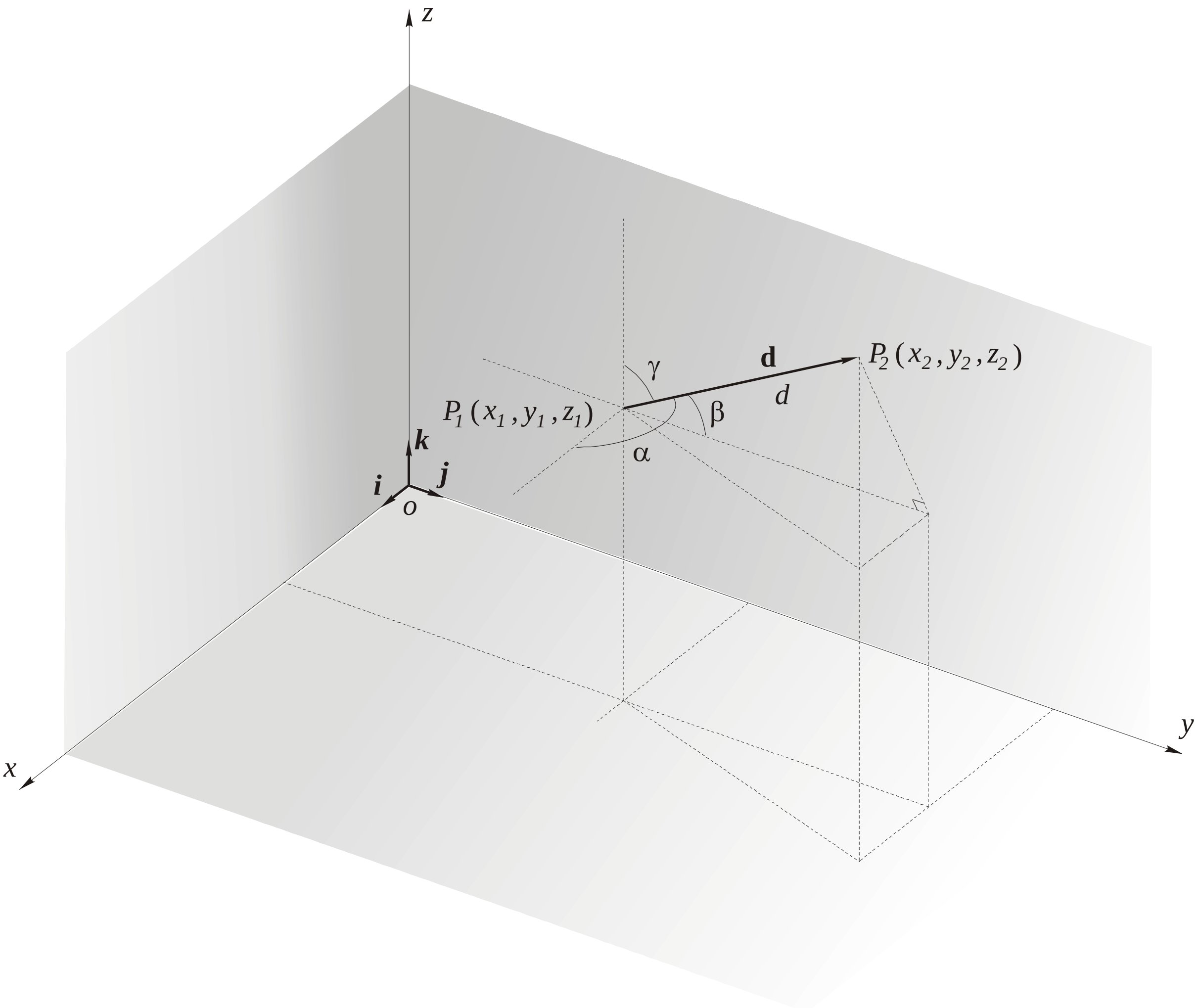


Figure 1

VECTOR BETWEEN TWO POINTS IN SPACE

The vector $\overrightarrow{P_1P_2}$ between the two points $P_1\left(x_1, y_1, z_1\right)$ and $P_2\left(x_2, y_2, z_2\right)$ is denoted as

$$\begin{aligned} \mathbf{d} &= d_1\mathbf{i} + d_2\mathbf{j} + d_3\mathbf{j} \\ &= \left(x_2 - x_1\right)\mathbf{i} + \left(y_2 - y_1\right)\mathbf{k} + \left(z_2 - z_1\right)\mathbf{k} \end{aligned} \tag{1}$$

$d_1\mathbf{i}$, $d_2\mathbf{j}$ and $d_3\mathbf{k}$ are the *component vectors* of **d.**

i, **j**, **k** are *unit vectors* in the directions of the positive *x*, *y* and *z*-axes respectively.

$d_1 = x_2 - x_1$, $d_2 = y_2 - y_1$, $d_3 = z_2 - z_1$ are the *Cartesian components* of **d.**

THE DISTANCE *d* BETWEEN TWO POINTS IN SPACE

The *magnitude* of vector **d** is denoted by $|\mathbf{d}|$ or d and is the distance between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$

$$|\mathbf{d}| = d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{2}$$

DIRECTION COSINES OF A LINE BETWEEN TWO POINTS IN SPACE

The direction of the line P_1P_2 is defined by the angles α, β and γ which are the angles the line makes with the positive *x, y* and *z*-axes respectively.

The *direction cosines* of the line P_1P_2, denoted by *l, m* and *n* are

$$l = \cos\alpha = \frac{x_2 - x_1}{d}, \quad m = \cos\beta = \frac{y_2 - y_1}{d}, \quad n = \cos\gamma = \frac{z_2 - z_1}{d} \tag{3}$$

Direction cosines have the following property

$$\begin{aligned}\cos^2\alpha + \cos^2\beta + \cos^2\gamma &= \frac{(x_2 - x_1)^2}{d^2} + \frac{(y_2 - y_1)^2}{d^2} + \frac{(z_2 - z_1)^2}{d^2} \\ &= \frac{1}{d^2}\left((x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right) \\ &= \frac{1}{d^2}\left(d^2\right)\end{aligned}$$

hence

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \quad \text{or} \quad l^2 + m^2 + n^2 = 1 \tag{4}$$

Also

$$(x_2 - x_1)\cos\alpha + (y_2 - y_1)\cos\beta + (z_2 - z_1)\cos\gamma = d \tag{5a}$$

or

$$l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1) = d \tag{5b}$$

and

$$\mathbf{d} = dl\mathbf{i} + dm\mathbf{j} + dn\mathbf{k} \tag{5c}$$

UNIT VECTOR BETWEEN TWO POINTS IN SPACE

The unit vector in the direction of **d** is denoted by $\hat{\mathbf{d}}$ and is defined as

$$\hat{\mathbf{d}} = \frac{\mathbf{d}}{|\mathbf{d}|} = \frac{d_1}{|\mathbf{d}|}\mathbf{i} + \frac{d_2}{|\mathbf{d}|}\mathbf{j} + \frac{d_3}{|\mathbf{d}|}\mathbf{k} = \frac{(x_2 - x_1)}{d}\mathbf{i} + \frac{(y_2 - y_1)}{d}\mathbf{j} + \frac{(z_2 - z_1)}{d}\mathbf{k} \tag{6}$$

Unit vectors can be defined in terms of their direction cosines, e.g.

$$\hat{\mathbf{d}} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k} \tag{7}$$

DIRECTION NUMBERS

Say that the distance *d* is scaled by a factor *k* then

$$kd = \sqrt{\left\{k\left(x_2 - x_1\right)\right\}^2 + \left\{k\left(y_2 - y_1\right)\right\}^2 + \left\{k\left(z_2 - z_1\right)\right\}^2}$$

Letting the numbers $L = k\left(x_2 - x_1\right)$, $M = k\left(y_2 - y_1\right)$, $N = k\left(z_2 - z_1\right)$ then

$$l = \cos\alpha = \frac{x_2 - x_1}{d} = \frac{k\left(x_2 - x_1\right)}{kd} = \frac{L}{\sqrt{L^2 + M^2 + N^2}}$$

and similarly for *m* and *n*. Hence the direction numbers *L*, *M*, *N* are proportional to the direction cosines *l*, *m*, *n*

$$l = \frac{L}{\sqrt{L^2 + M^2 + N^2}}, \quad m = \frac{M}{\sqrt{L^2 + M^2 + N^2}}, \quad n = \frac{N}{\sqrt{L^2 + M^2 + N^2}} \tag{8}$$

THE NORMAL EQUATION OF A PLANE

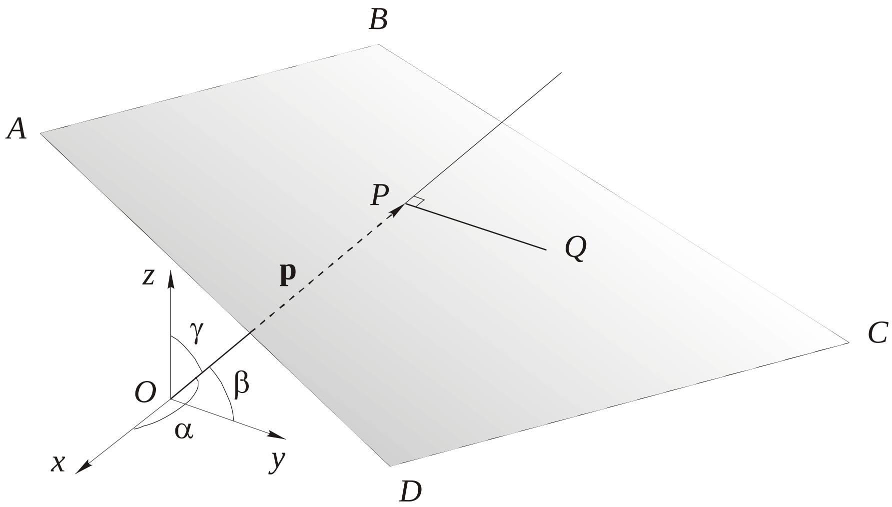


Figure 2

ABCD is a plane. The normal to the plane from the origin *O* cuts the plane at *P* and the position vector of *P* is $\overrightarrow{OP} = \mathbf{p}$. The distance *OP* = *p*. *Q* is any other point in the plane.

The vectors $\overrightarrow{OP}$ and $\overrightarrow{PQ}$ are perpendicular so their *dot product* will be zero

$$\left(x_P\mathbf{i} + y_P\mathbf{j} + z_P\mathbf{k}\right)\bullet\left(\left(x_Q - x_P\right)\mathbf{i} + \left(y_Q - y_P\right)\mathbf{j} + \left(z_Q - z_P\right)\mathbf{k}\right) = 0$$

With *l*, *m*, *n* as the direction cosines of **p** and bearing in mind equations (3) and (5) we may write

$$\begin{aligned}(pl\mathbf{i} + pm\mathbf{j} + pn\mathbf{k}) \bullet \left((x_Q - pl)\mathbf{i} + (y_Q - pm)\mathbf{j} + (z_Q - pn)\mathbf{k}\right) &= 0 \\ pl(x_Q - pl) + pm(y_Q - pm) + pn(z_Q - pn) &= 0 \\ plx_Q + pmy_Q + pnz_Q - p^2\left(l^2 + m^2 + n^2\right) &= 0\end{aligned}$$

But $l^2 + m^2 + n^2 = 1$ and x_Q, y_Q, z_Q are the coordinates of any point in the plane, thus with some rearrangement, we may write the equation of a plane in terms of the direction cosines of the normal to the plane or

The Normal equation of the plane	$lx + my + nz = p$	(9)

Note that the set of direction cosines which when used as coefficients for *x*, *y*, *z* gives a *positive* right-hand-side to the equation, is the set of direction cosines of the normal directed from the origin to the plane.

Note also

$$\left(\frac{l}{p}\right)x + \left(\frac{m}{p}\right)y + \left(\frac{n}{p}\right)z = 1 \tag{10}$$

$$\left(\frac{l}{p}\right)^2 + \left(\frac{m}{p}\right)^2 + \left(\frac{n}{p}\right)^2 = \frac{1}{p^2}\left(l^2 + m^2 + n^2\right) = \frac{1}{p^2} \tag{11}$$

GENERAL EQUATION OF A PLANE

The general equation of a plane is $Ax + By + Cz = D$	(12)

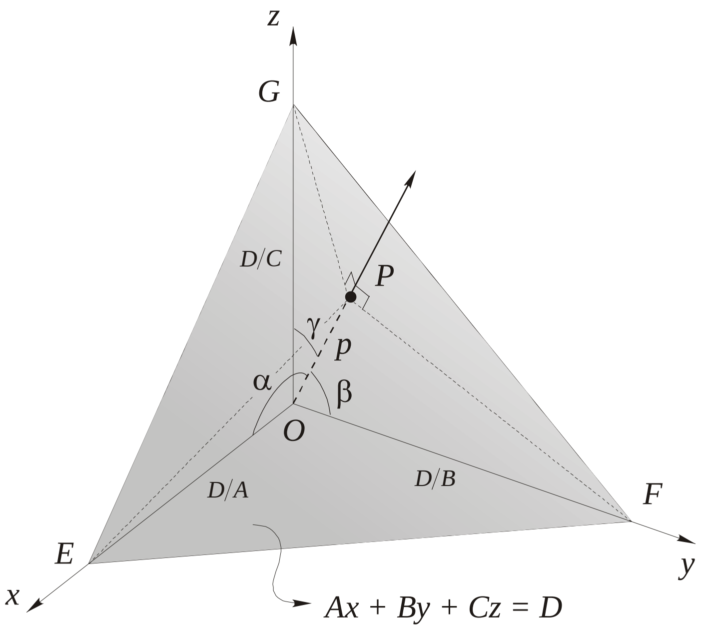


Figure 3

The connection between the General equation of a plane $Ax + By + Cz = D$ and the Normal equation of the plane $lx + my + nz = p$ may be established in the following manner.

In Figure 3, the plane $Ax + By + Cz = D$ intersects the *x*, *y* and *z*-axes at *E*, *F* and *G*. The intercept on the *x*-axis can be found by setting $y = 0$ and $z = 0$ in (12) giving the distance $OE = x = D/A$. Similarly, the plane cuts the *y* and *z*-axes at $y = D/B$ and $z = D/C$ respectively. A normal to the plane from the origin *O* passes through the plane at *P* and the distance *OP* = *p*. α, β and γ are the angles the normal makes with the positive *x*, *y* and *z*-axes respectively.

In the plane *OEP* containing the *x*-axis and the normal

$$\cos\alpha = \frac{p}{x} = \frac{p}{D/A} = \frac{p}{D}A$$

and similarly

$$\cos\beta = \frac{p}{D}B, \quad \cos\gamma = \frac{p}{D}C$$

Now $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ hence

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\frac{p}{D}\right)^2\left(A^2 + B^2 + C^2\right) = 1$$

and $$\frac{p}{D} = \frac{1}{\sqrt{A^2 + B^2 + C^2}}$$

Substituting into the equations above gives the direction cosines *l*, *m* and *n* of the normal to the plane and the perpendicular distance *p* from the origin to the plane as

$$l = \cos\alpha = \frac{A}{\sqrt{A^2 + B^2 + C^2}} \tag{13a}$$

$$m = \cos\beta = \frac{B}{\sqrt{A^2 + B^2 + C^2}} \tag{13b}$$

$$n = \cos\gamma = \frac{C}{\sqrt{A^2 + B^2 + C^2}} \tag{13c}$$

$$p = \frac{D}{\sqrt{A^2 + B^2 + C^2}} \tag{13d}$$

These equations demonstrate the connection between the General equation of the plane and the Normal equation of the plane. Note that the coefficients *A*, *B* and *C* are identical to direction numbers *L*, *M* and *N* and the General equation of the plane $Ax + By + Cz = D$ is often expressed as $Lx + My + Nz = P$ where *P* is identical to *D*.

DISTANCE FROM POINT (x_0, y_0, z_0) TO PLANE $Ax + By + Cz = D$

The perpendicular distance between the origin and the plane $Ax + By + Cz = D$ is given by (13d)

$$p = \frac{D}{\sqrt{A^2 + B^2 + C^2}}$$

For the *parallel* plane passing through the point (x_0, y_0, z_0) we have $Ax_0 + By_0 + Cz_0 = D_0$ and the perpendicular distance between the origin and this parallel plane is

$$p_0 = \frac{D_0}{\sqrt{A^2 + B^2 + C^2}}$$

The perpendicular distance *d* between the two parallel planes is $p_0 - p$ hence

$$\begin{aligned} d &= \frac{D_0}{\sqrt{A^2 + B^2 + C^2}} - \frac{D}{\sqrt{A^2 + B^2 + C^2}} \\ &= \frac{D_0 - D}{\sqrt{A^2 + B^2 + C^2}} \end{aligned}$$

hence

$$d = \frac{Ax_0 + By_0 + Cz_0 - D}{\sqrt{A^2 + B^2 + C^2}} \tag{14}$$

Note that *d* is considered as a positive quantity and any negative sign attached to it may be disregarded.

In addition, if the General equation of a plane is given as $Ax + By + Cz + D = 0$, which is very common, then (14) would be written as

$$d = \frac{Ax_0 + By_0 + Cz_0 + D}{\sqrt{A^2 + B^2 + C^2}}$$

EQUATION OF PLANE PASSING THROUGH POINTS (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)

There are three methods of computing a plane passing through three points and each of these methods will be compared by way of an example. Figure 4 shows a plane defined by three points, 1, 2 and 3 where the coordinates of each point are shown as, for example $x_1 = 5.0$, $y_1 = 6.7$ and $z_1 = 1.5$

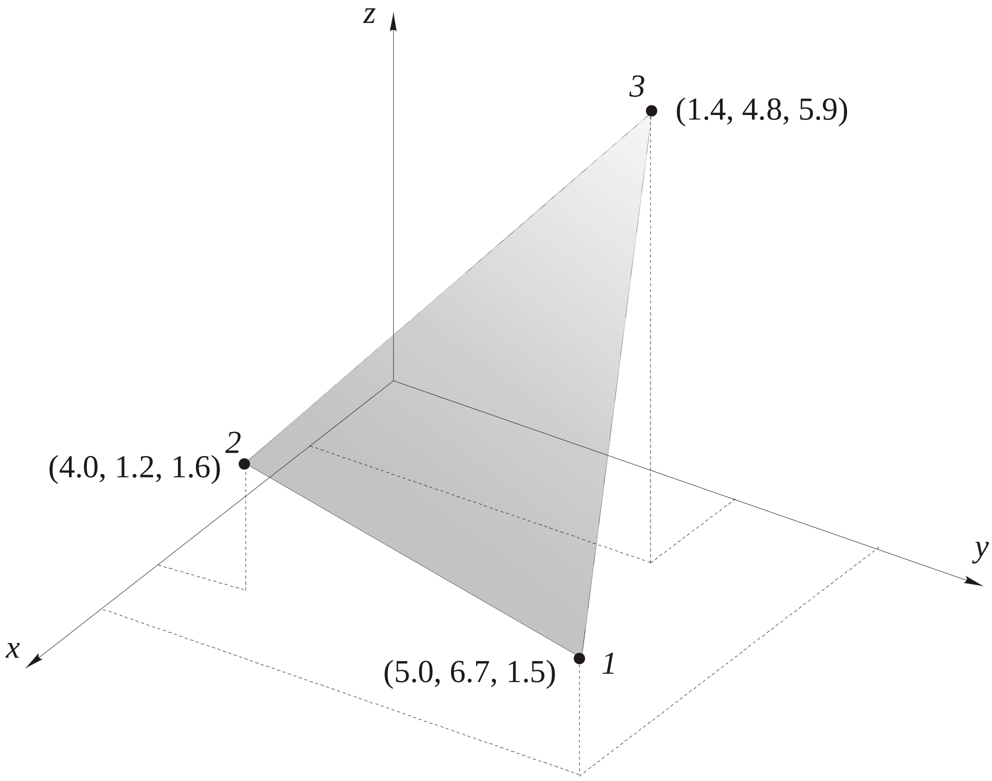


Figure 4

Method 1: Determinant

The General equation of a plane may be written as

$$Ax + By + Cz + D = 0 \tag{14}$$

This is slightly different from equation (12), where D is a positive quantity on the right-hand-side rather than a positive quantity on the left-hand-side of the equation. The equation of the plane (14) passing through three points is given in the form of a 3rd order determinant

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_2 & y_3-y_2 & z_3-z_2 \end{vmatrix} = 0 \tag{15}$$

or expanded into 2nd order determinants

$$\begin{vmatrix} y_2-y_1 & z_2-z_1 \\ y_3-y_2 & z_3-z_2 \end{vmatrix}(x-x_1) - \begin{vmatrix} x_2-x_1 & z_2-z_1 \\ x_3-x_2 & z_3-z_2 \end{vmatrix}(y-y_1) + \begin{vmatrix} x_2-x_1 & y_2-y_1 \\ x_3-x_2 & y_3-y_2 \end{vmatrix}(z-z_1) = 0 \tag{16}$$

Substituting the coordinates of points 1, 2 and 3 gives

$$\begin{vmatrix} -5.5 & 0.1 \\ 3.6 & 4.3 \end{vmatrix}(x-5.0) - \begin{vmatrix} -1.0 & 0.1 \\ -2.6 & 4.3 \end{vmatrix}(y-6.7) + \begin{vmatrix} -1.0 & -5.5 \\ -2.6 & 3.6 \end{vmatrix}(z-1.5) = 0$$

Evaluating the determinants and gathering terms gives

$$\begin{aligned} (-24.01)(x-5.0)-(-4.04)(y-6.7)+(-17.09)(z-1.5) &= 0 \\ -24.01x+120.05+4.04y-27.068-17.9z+26.85 &= 0 \\ -24.01x+4.04y-17.9z+119.832 &= 0 \end{aligned}$$

Expressing the result in the form of the General equation of the plane $Ax+By+Cz=D$ with D as a positive quantity on the right-hand-side gives

$$\boxed{24.01x-4.04y+17.9z=119.832}$$

where $A=24.01,\ B=-4.04,\ C=17.9,\ D=119.832$

Using equations (13)

$$l=\cos\alpha=\frac{A}{\sqrt{A^2+B^2+C^2}}=0.794523$$

$$m=\cos\beta=\frac{B}{\sqrt{A^2+B^2+C^2}}=-0.133689$$

$$n=\cos\gamma=\frac{C}{\sqrt{A^2+B^2+C^2}}=0.592335$$

$$p=\frac{D}{\sqrt{A^2+B^2+C^2}}=3.965401$$

and the Normal equation of the plane $lx+my+nz=p$ is

$$\boxed{0.794523x-0.133689y+0.592335z=3.965401}$$

Note that equations (15) and (16) can be expressed as the determinant

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0 \tag{17}$$

or expanded into 2nd order determinants

$$\begin{vmatrix} y_2-y_1 & z_2-z_1 \\ y_3-y_1 & z_3-z_1 \end{vmatrix}(x-x_1)-\begin{vmatrix} x_2-x_1 & z_2-z_1 \\ x_3-x_1 & z_3-z_1 \end{vmatrix}(y-y_1)+\begin{vmatrix} x_2-x_1 & y_2-y_1 \\ x_3-x_1 & y_3-y_1 \end{vmatrix}(z-z_1)=0 \qquad (18)$$

Substituting the coordinates of points 1, 2 and 3 gives

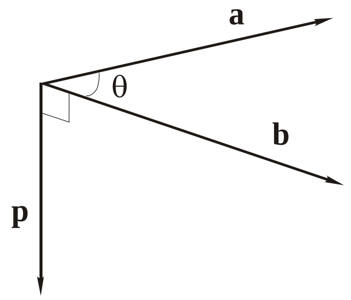
$$\begin{vmatrix} -5.5 & 0.1 \\ -1.9 & 4.4 \end{vmatrix}(x-5.0)-\begin{vmatrix} -1.0 & 0.1 \\ -3.6 & 4.4 \end{vmatrix}(y-6.7)+\begin{vmatrix} -1.0 & -5.5 \\ -3.6 & -1.9 \end{vmatrix}(z-1.5)=0$$

Evaluating the determinants gives

$$(-24.01)(x-5.0)-(-4.04)(y-6.7)+(-17.09)(z-1.5)=0$$

This is exactly the same result as before. Hence equations (15) and (17) give identical results.

Method 2: Vector cross product



The vector cross product of vectors **a** and **b** is

$$\mathbf{a}\times\mathbf{b}=|\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{p}}=\mathbf{p} \qquad (19)$$

The result of a vector cross product is a vector **p** perpendicular to the plane containing **a** and **b**.

$|\mathbf{a}||\mathbf{b}|\sin\theta$ is a scalar quantity and $\hat{\mathbf{p}}$ is a unit vector.

The direction of **p** is given by the *right-hand-screw rule*, i.e. if **a** and **b** are in the plane of the head of a screw then a clockwise rotation of **a** to **b** through an angle θ would mean that the direction of **p** would be the same as the direction of advance of a right-handed screw turned clockwise.

If $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ and $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$ then the vector cross product can be written as

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix} \overset{+}{\mathbf{i}} & \overset{-}{\mathbf{j}} & \overset{+}{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}=(a_2b_3-a_3b_2)\,\mathbf{i}-(a_1b_3-a_3b_1)\,\mathbf{j}+(a_1b_2-a_2b_1)\,\mathbf{k}=\mathbf{p} \qquad (20)$$

The perpendicular vector $\mathbf{p}=p_1\mathbf{i}+p_2\mathbf{j}+p_3\mathbf{k}$ where $p_1=(a_2b_3-a_3b_2)$, $p_2=-(a_1b_3-a_3b_1)$ and $p_3=(a_1b_2-a_2b_1)$ are the components.

The perpendicular unit vector $\hat{\mathbf{p}}=\dfrac{\mathbf{p}}{|\mathbf{p}|}=\dfrac{p_1}{|\mathbf{p}|}\mathbf{i}+\dfrac{p_2}{|\mathbf{p}|}\mathbf{j}+\dfrac{p_3}{|\mathbf{p}|}\mathbf{k}=l\mathbf{i}+m\mathbf{j}+n\mathbf{k}$.

The direction cosines *l, m, n* of the perpendicular vector **p** are also the direction cosines of the normal to the plane and hence are the coefficients in the Normal equation of the plane (9). Substituting the coordinates of one of the three points defining the vectors **a** and **b** into the Normal equation of the plane will yield the perpendicular distance *p* from the origin to the plane. Note that *p* in the Normal equation of the plane (9) is not the magnitude of the perpendicular vector **p** resulting from the cross product (19).

The method of computation is set out below using the example data of Figure 4

Let **a** be the vector from point 1 to point 2.

$$\begin{aligned}\mathbf{a} &= a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}\\ &= (x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\,\mathbf{j} + (z_2 - z_1)\,\mathbf{k}\\ &= -1.0\mathbf{i} - 5.5\mathbf{j} + 0.1\mathbf{k}\end{aligned}$$

Let **b** be the vector from point 1 to point 3.

$$\begin{aligned}\mathbf{b} &= b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}\\ &= (x_3 - x_1)\,\mathbf{i} + (y_3 - y_1)\,\mathbf{j} + (z_3 - z_1)\,\mathbf{k}\\ &= -3.6\mathbf{i} - 1.9\mathbf{j} + 4.4\mathbf{k}\end{aligned}$$

The perpendicular vector **p** is given by the vector cross product

$$\mathbf{p} = \mathbf{a}\times\mathbf{b} = \begin{vmatrix} \overset{+}{\mathbf{i}} & \overset{-}{\mathbf{j}} & \overset{+}{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\,\mathbf{i} - (a_1b_3 - a_3b_1)\,\mathbf{j} + (a_1b_2 - a_2b_1)\,\mathbf{k}$$

$$= \begin{vmatrix} \overset{+}{\mathbf{i}} & \overset{-}{\mathbf{j}} & \overset{+}{\mathbf{k}} \\ -1.0 & -5.5 & 0.1 \\ -3.6 & -1.9 & 4.4 \end{vmatrix} = -24.01\mathbf{i} + 4.04\mathbf{j} - 17.9\mathbf{k}$$

The magnitude $|\mathbf{p}| = 30.219393$ and the unit vector $\hat{\mathbf{p}} = -0.794523\mathbf{i} + 0.133689\mathbf{j} - 0.592335\mathbf{k}$

The direction cosines of the normal are $l = -0.794523,\ m = 0.133689,\ n = -0.592335$ and the Normal equation of the plane is $lx + my + nz = p$. Substituting the coordinates of point 1 gives

$$(-0.794523)(5.0) + (0.133689)(6.7) + (-0.592335)(1.5) = -3.965401$$

Since p is a negative quantity, the signs of l, m and n are reversed to give the direction from the origin to the plane and the Normal equation of the plane is

$$\boxed{0.794523x - 0.133689y + 0.592335z = 3.965401}$$

This is identical to the result obtained in Method 1

Method 3: Solution of simultaneous equations

The Normal equation of a plane may be written in the form of equation (10)

$$\left(\frac{l}{p}\right)x+\left(\frac{m}{p}\right)y+\left(\frac{n}{p}\right)z=1$$

With $\left(\frac{l}{p}\right)$, $\left(\frac{m}{p}\right)$ and $\left(\frac{n}{p}\right)$ as unknown coefficients, the *x, y* and *z* coordinates of three points defining a plane will yield three simultaneous equations, written in matrix form as $\mathbf{Ax}=\mathbf{b}$.

$$\mathbf{A}=\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix},\quad \mathbf{x}=\begin{bmatrix} (l/p) \\ (m/p) \\ (n/p) \end{bmatrix},\quad \mathbf{b}=\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The solution for $\mathbf{x}=\mathbf{A}^{-1}\mathbf{b}$ and *p* can be determined from equation (11)

$$\left(\frac{l}{p}\right)^2+\left(\frac{m}{p}\right)^2+\left(\frac{n}{p}\right)^2=\frac{1}{p^2}.$$

The direction cosines *l*, *m*, *n* then follow and the Normal equation of the plane is determined.

The method of computation is set out below using the example data of Figure 4

$$\mathbf{A}=\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}=\begin{bmatrix} 5.0 & 6.7 & 1.5 \\ 4.0 & 1.2 & 1.6 \\ 1.4 & 4.8 & 5.9 \end{bmatrix},\quad \mathbf{x}=\begin{bmatrix} (l/p) \\ (m/p) \\ (n/p) \end{bmatrix},\quad \mathbf{b}=\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The matrix inverse $\mathbf{A}^{-1}=\begin{bmatrix} 0.005007 & 0.269794 & -0.074438 \\ 0.178250 & -0.228653 & 0.016690 \\ -0.146205 & 0.122004 & 0.173576 \end{bmatrix}$

The solution vector $\mathbf{x}=\begin{bmatrix} (l/p) \\ (m/p) \\ (n/p) \end{bmatrix}=\begin{bmatrix} 0.200364 \\ -0.033714 \\ 0.149376 \end{bmatrix}$

$$\left(\frac{l}{p}\right)^2+\left(\frac{m}{p}\right)^2+\left(\frac{n}{p}\right)^2=\frac{1}{p^2}=0.063595 \text{ and } p=3.965401$$

Multiplying the numeric values in the vector **x** by *p* gives $l=0.794523$, $m=-0.133689$ and $n=0.592335$. The Normal equation of the plane is

$$\boxed{0.794523x-0.133689y+0.592335z=3.965401}$$

This is identical to the result obtained in Method 1

DIRECTION OF *STRIKE* AND *MAXIMUM DIP* ON AN INCLINED PLANE

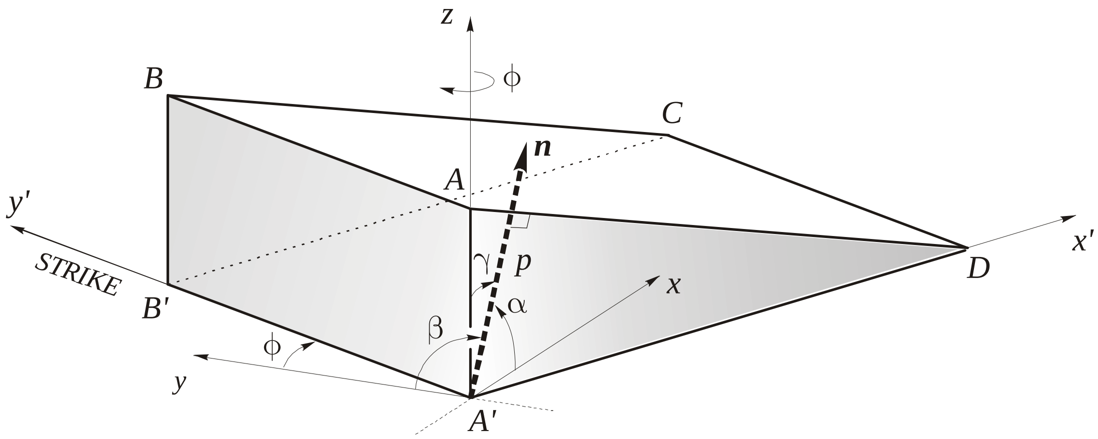
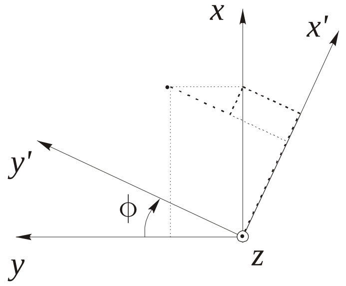


Figure 5

In Figure 5, *ABCD* is a portion of an inclined plane. *A′* and *B′* are vertical projections of *A* and *B* onto a horizontal *x*-*y* plane (*A'B'CD*) and the line *CD* is the intersection of the inclined and horizontal planes. The *xyz* Cartesian coordinate origin is at *A′* with the *z*-axis vertical.

The line *AB* is a level line on the inclined plane and is known as the *strike line.* The line *CD,* which is parallel to *AB,* is also a strike line as is any other parallel line in the inclined plane. The line perpendicular to the strike line is the direction of *maximum dip.* In Figure 5, the *y*-axis and the *x*-axis are the North and East directions respectively the *y′*-axis is the direction of strike and the *x′*-axis is the direction of maximum dip.

The direction of strike can be determined from the normal equation of the plane by considering a clockwise rotation of the *X*-*Y* axes about the *Z*-axis by an angle ϕ. If the *Y*-axis is the direction of north then ϕ will be the bearing of the strike line of the inclined plane. A clockwise rotation about the *Z*-axis can be represented by the matrix equation



$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad (21)$$

Referring to Figure 5, when the *y′*-axis is the direction of strike, the *y′* coordinate of any point along the normal to the inclined plane will be zero, i.e., the normal will lie in the *z*-*x′* plane. Hence, from equation (21)

$$y' = x\sin\phi + y\cos\phi = 0 \qquad (22)$$

Now the x and y coordinates of the point where the normal pierces the inclined plane are $p\cos\alpha$ and $p\cos\beta$ respectively, giving

$$\tan\phi = \frac{-p\cos\beta}{p\cos\alpha} = \frac{-m}{l} \tag{23}$$

The "whole circle" bearing ϕ $\left(0^\circ \le \phi < 360^\circ\right)$ of the strike line must be determined by resolving the correct quadrant for the angle ϕ given by equation (23).

DIRECTION OF *STRIKE* AND *MAXIMUM DIP* ON AN INCLINED PLANE (ALTERNATIVE DERIVATION)

As an alternative to the derivation above, consider the normal equation of the plane $lx + my + nz = p$ expressed as the surface $\varphi\left(x, y, z\right) = \text{constant}$, i.e., to each point $\left(x, y, z\right)$ of a region in space there corresponds a number or scalar $\varphi\left(x, y, z\right)$ equal to a constant value. φ is called a *scalar function of position*. If we apply the vector differential operator $\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$ (∇ is known as *del* or *nabla*) to the scalar function φ we obtain the *gradient* $\nabla\varphi = \frac{\partial\varphi}{\partial x}\mathbf{i} + \frac{\partial\varphi}{\partial y}\mathbf{j} + \frac{\partial\varphi}{\partial z}\mathbf{k}$.
$\nabla\varphi$ is a vector perpendicular to the surface $\varphi\left(x, y, z\right) = \text{constant}$ and it points in the direction in which φ increases at its greatest. Hence if $\varphi\left(x, y, z\right) = \text{constant}$ defines a plane then $\nabla\varphi$ is in the direction of maximum rise, the opposite direction of maximum dip. The direction of strike will be ±90° from this direction.

If $\varphi\left(x, y, z\right) = lx + my + nz = p$ then $\nabla\varphi = l\,\mathbf{i} + m\,\mathbf{j} + n\,\mathbf{k}$ and l, m and n are the direction cosines $l = \cos\alpha,\ m = \cos\beta,\ n = \cos\gamma$. Referring to Figure 6, $\nabla\varphi$ is in the direction of maximum rise and its projection on the x-y plane has a bearing θ. This is the bearing of maximum rise. The bearing of the strike line will be $\phi = \theta + 90^\circ$. From Figure 6 the bearing of the line of maximum rise can be found from

$$\tan\theta = \frac{\cos\alpha}{\cos\beta} = \frac{l}{m}$$

and since $\tan A = -\cot\left(90 + A\right)$ and $\cot B = \frac{1}{\tan B}$ then the bearing of the strike line can be found from

$$\tan\phi = \frac{-m}{l}$$

This is the same as equation (23).

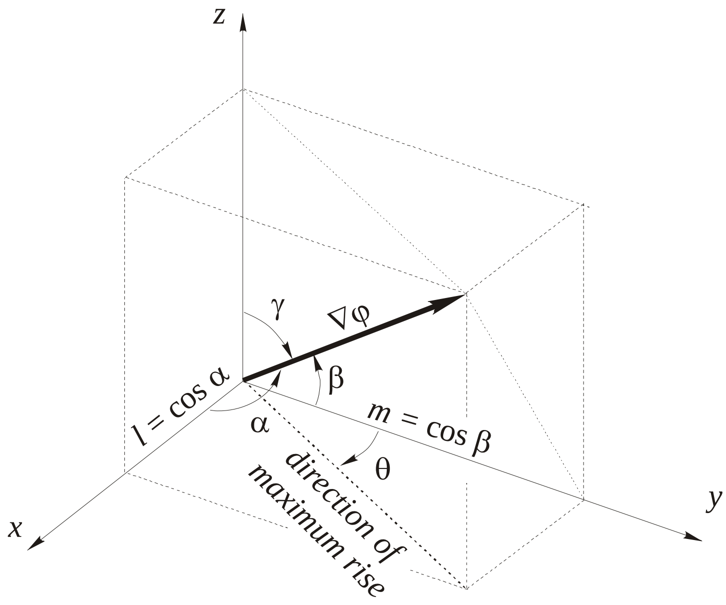


Figure 6