A NOTE ON THE BURSA-WOLF AND MOLODENSKY-BADEKAS TRANSFORMATIONS

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ABSTRACT

The *Bursa–Wolf* and *Molodensky*[1]*–Badekas* transformations are conformal three-dimensional (3D) Cartesian coordinate transformations commonly used in surveying, photogrammetry and geodesy. They are also called similarity or seven-parameter transformations and they combine a scale change, three axes-rotations and three origin-shifts in a practical mathematical model of the relationships between points in two different 3D coordinate systems. They differ slightly in their operation; the Molodensky–Badekas transformation uses a centroid but the Bursa-Wolf transformation does not, hence additional information (the centroid coordinates) is required when using the Molodensky–Badekas transformation; a factor that makes the Bursa–Wolf transformation more popular. This paper aims to provide an explanation of both transformations.

INTRODUCTION

3D conformal transformations, also known as similarity transformations (since conformal transformations preserve shape and angles between vectors in space remain unchanged) are commonly used in surveying, photogrammetry and geodesy. For instance, in engineering surveying applications 3D transformations are used measure objects (e.g., sections of elevated roadways) off-site before they are moved on-site to make sure they will fit with

[1] Molodensky is also spelt as Molodenskii

existing construction, and in tunnelling operations, 3D transformations are used to control the direction and orientation of tunnel boring machines. In photogrammetry they are used in the (interior and exterior) orientation of digital images of structures and aerial photographs. In geodesy, the main thrust of this paper, 3D transformations are used to convert coordinates related to one geodetic datum to another, and this operation is commonly known as datum transformation. In such applications, the rotations between the two 3D coordinate axes are small (usually less than 1 second of arc) and certain approximations are used to simplify rotation matrices; these simplified matrices are a common feature of the Bursa–Wolf and the Molodensky–Badekas transformations.

The names of the two transformations are an acknowledgement to the authors M. Bursa (1962), H. Wolf (1963), M.S. Molodensky et al. (1962) and J. Badekas (1969) of technical papers and reports on transformation methods related to the orientation of reference ellipsoids and 3D geodetic networks.

3D conformal transformations are often given in the form

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_2 = s\,\mathbf{R}_{ZYX} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_1 + \begin{bmatrix} t_X \\ t_Y \\ t_Z \end{bmatrix}_2 \tag{1}$$

The subscripts $[\;]_1$ and $[\;]_2$ refer to the X, Y, Z Cartesian coordinates of systems 1 and 2 respectively. s is a scale factor, $\mathbf{R}_{XYZ}$ is a 3×3 rotation matrix (the product of rotations[2] r_X, r_Y, r_Z about the coordinate axes) and t_X, t_Y, t_Z are translations between the origins of the two systems measured in the directions of the system 2 coordinate axes.

Alternatively, the transformation can be given in the form of a vector equation

$$\mathbf{l}_2 = s\mathbf{R}_{ZYX}\mathbf{l}_1 + \mathbf{t}_2 \tag{2}$$

$\mathbf{l}_1 = \begin{bmatrix} X & Y & Z \end{bmatrix}_1^T$ and $\mathbf{l}_2 = \begin{bmatrix} X & Y & Z \end{bmatrix}_2^T$ are position vectors or vectors of coordinates in systems 1 and 2 respectively and $\mathbf{t}_2 = \begin{bmatrix} t_X & t_Y & t_Z \end{bmatrix}_2^T$ is a vector of translations measured in the directions of system 2 coordinates axes. In these notes, both forms will be used where appropriate.

[2] [In this paper, rotations are considered positive anticlockwise when looking along the axis towards the origin; the positive sense of rotation being determined by the right-hand-grip rule where an imaginary right hand grips the axis with the thumb pointing in the positive direction of the axis and the natural curl of the fingers indicating the positive direction of rotation.]

In geodetic datum transformations the *Z-axis* is in the direction of the minor axis of a reference ellipsoid (an ellipsoid of revolution) passing through the north pole; the X-Z plane is the Greenwich meridian plane (the origin of longitudes); the X-Y plane is the equatorial plane of the ellipsoid (the origin of latitudes); the *X-axis* is in the direction of the intersection of the Greenwich meridian and equatorial planes and the *Y-axis* is advanced 90° east along the equator.

The right-handed coordinate system and positive anticlockwise rotations (given by the right-hand-grip rule[3]) are consistent with conventions used in mathematics and physics and will be used in these notes.

The Bursa–Wolf and the Molodensky–Badekas transformations have a modified form of equation (1) where

(i) the rotation matrix $\mathbf{R}_{XYZ}$ has the approximated form $\mathbf{R}_S$ where the subscript s refers to <u>small</u> rotation angles ε_X, ε_Y, ε_Z about the coordinate axes and

$$\mathbf{R}_{ZYX} \cong \mathbf{R}_S = \begin{bmatrix} 1 & \varepsilon_Z & -\varepsilon_Y \\ -\varepsilon_Z & 1 & \varepsilon_X \\ \varepsilon_Y & -\varepsilon_X & 1 \end{bmatrix} \tag{3}$$

(ii) the scale factor s is expressed in the form

$$s = 1 + ds \tag{4}$$

where ds is a small value usually expressed in ppm[4].

The two transformations are then given in the form

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[3] The right-hand-grip rule is a useful rule for determining the positive direction of rotations. An imaginary right hand grips the coordinate axis with the thumb pointing in the positive direction of the axis and the natural curl of the fingers indicate the positive direction of rotation. There is also a left-hand-grip rule to define positive clockwise rotations, but this will not be used in these notes.

[4] ppm is parts-per-million. A scale factor of $s = 1.000045$ expressed as $1 + ds$ has $ds = 0.000045$ or $ds = 45$ ppm. ppm is also "mm per km" since there are 1 million millimetres in 1 kilometre.

BURSA–WOLF TRANSFORMATION

Figure 1 shows the geometry of the Bursa–Wolf transformation. the X,Y,Z axes of system 1 are rotated by very small angles ε_X, ε_Y, ε_Z from the X,Y,Z axes of system 2, and the origins of the two systems are displaced by translations t_X, t_Y, t_Z in the directions of the X,Y,Z axes of system 2. $\mathbf{l}_1$ and $\mathbf{l}_2$ are vectors of coordinates in both systems and $\mathbf{t}$ is a vector of translations.

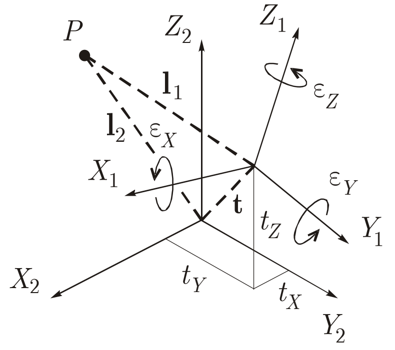

Figure 1:
Geometry of Bursa-Wolf transformation

The mathematical relationship between coordinates in both systems can be written in the form of a vector equation

$$\mathbf{l}_2 = \mathbf{t}_2 + (1+ds)\mathbf{R}_s\mathbf{l}_1 \tag{5}$$

Alternatively, the Bursa–Wolf transformation may be written as

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_2 = (1+ds)\begin{bmatrix} 1 & \varepsilon_Z & -\varepsilon_Y \\ -\varepsilon_Z & 1 & \varepsilon_X \\ \varepsilon_Y & -\varepsilon_X & 1 \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_1 + \begin{bmatrix} t_X \\ t_Y \\ t_Z \end{bmatrix}_2 \tag{6}$$

MOLODENSKY–BADEKAS TRANSFORMATION

Figure 2 shows the geometry of the Molodensky–Badekas transformation that makes use of a centroid. The X,Y,Z axes of system 1 are rotated by very small angles ε_X, ε_Y, ε_Z from the X,Y,Z axes of system 2, and the origins O_1 and O_2 of the two systems are displaced. The $\overline{X}_1, \overline{Y}_1, \overline{Z}_1$ system is a centroidal system whose origin is at a centroid G of a set of points in system 1 and whose axes are parallel to the X,Y,Z axes of system 1.

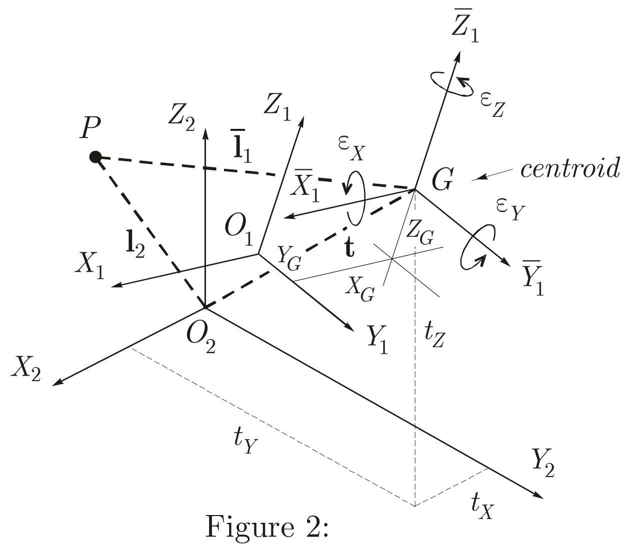

Figure 2:
Geometry of Molodensky-Badekas transformation

In Figure 2, the centroid G is displaced from O_2 by translations t_X, t_Y, t_Z measured in the directions of the X, Y, Z axes of system 2 and $\mathbf{t}_2 = \begin{bmatrix} t_X & t_Y & t_Z \end{bmatrix}_2^T$ is the position vector of the centroid.

The mathematical relationship between coordinates in both systems, including a scale factor $s = 1 + ds$, can be developed by using vector equations, where from Figure 2 we may write

$$\mathbf{l}_2 = \mathbf{t}_2 + (1 + ds)\mathbf{R}_s \overline{\mathbf{l}}_1 \tag{7}$$

where

$$\overline{\mathbf{l}}_1 = \begin{bmatrix} \overline{X} \\ \overline{Y} \\ \overline{Z} \end{bmatrix}_1 = \begin{bmatrix} X - X_G \\ Y - Y_G \\ Z - Z_G \end{bmatrix}_1 = \mathbf{l}_1 - \mathbf{g}_1 \tag{8}$$

X_G, Y_G, Z_G are the coordinates of the centroid and $\mathbf{g}_1 = \begin{bmatrix} X_G & Y_G & Z_G \end{bmatrix}_1^T$ is the position vector of the centroid in system 1 coordinates.

Alternatively, the Molodensky–Badekas transformation may be written as

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_2 = (1 + ds)\begin{bmatrix} 1 & \varepsilon_Z & -\varepsilon_Y \\ -\varepsilon_Z & 1 & \varepsilon_X \\ \varepsilon_Y & -\varepsilon_X & 1 \end{bmatrix}\begin{bmatrix} \overline{X} \\ \overline{Y} \\ \overline{Z} \end{bmatrix}_1 + \begin{bmatrix} t_X \\ t_Y \\ t_Z \end{bmatrix}_2 \tag{9}$$

SOME PROBLEMS IN THE DESCRIPTION OF THE MOLODENSKY–BADEKAS TRANSFORMATION

There are several variations of the Molodensky–Badekas transformation.

The transformation shown above [see equations (7) and (9)] is similar to the one described in Krakiwsky and Thomson (1974, p. 608-9) in the section headed **MOLODENSKII MODEL** where they state:

> "There are two different versions of the Molodenskii model: the first version exists in the literature [*Veis* 1960; *Molodenskii et al* 1962; *Badekas* 1969], and the second version is given herin."

They then go on to describe "the first version":

> "The first version of Molodenskii's model is obtained by assuming that the position vector of the initial point is known in a geodetic system that is parallel to the average terrestrial system. The model is (Figure 3)"
>
> $$\vec{F}_i = (\vec{r}_0)_{AT} + (\vec{r}_k)_{G\|AT} + (1+\kappa) R_\varepsilon (\vec{r}_{ki})_G - \vec{\rho}_i = 0$$

In Krakiwsky and Thomson (1974) the *average terrestrial system* is our system 2, the *geodetic system* is our system 1 and the *initial point* is our centroid. In their notation, $\vec{r}_0$, $\vec{r}_k, \vec{r}_{ki}$ and $\vec{\rho}_i$ are position vectors, $(1+\kappa)$ is the scale factor and R_ε is the rotation matrix. Inspection of their Figure 3 (not shown here) reveals that their vector $\left(\vec{r}_{ki}\right)_G$ is identical with our vector $\overline{\mathbf{l}}_1$; their vector $\vec{\rho}_i$ is identical to our vector $\mathbf{l}_2$ and their vector sum $(\vec{r}_0)_{AT} + \left(\vec{r}_k\right)_{G\|AT}$ is equivalent to our translation vector $\mathbf{t}$. So their transformation, that they describe as "the first version of Molodenskii's model" is effectively the same as our equations (7) and (9).

Krakiwsky and Thompson (1974) describe two Molodenskii models – where the original authors (Molodensky *et al* 1962) describe only one – and the mathematical description is entirely different from the original. This is a possible source of confusion.

In Molodensky *et al.* (1962), the authors derive a set of differential equations for transforming coordinates from one geodetic datum to another. Their equations (Molodensky *et al.,* (I.3.2), p. 14), linked changes in *x, y, z* Cartesian coordinates of a point with, (i) rotations $\varepsilon_x, \varepsilon_y, \varepsilon_z$ of the Cartesian axes about some fixed point x_0, y_0, z_0, (ii) "progressive translations" dx_0, dy_0, dz_0 of the ellipsoid origin between *x, y, z* Cartesian axes, and changes in the ellipsoid parameters δa and δf with changes in curvilinear coordinates $\delta\phi, \delta\lambda, \delta h$. Subsequent publications by other authors have described "Molodensky's" transformation in terms different from the original.

This confusion was addressed by Soler (1976, p.2) who states:

> "… the differential equations published in the English translation of [Molodensky *et al.,* 1962] are equivalent to conventional conformal transformations. This dissipates the confusion created recently by some authors [Badekas, 1969], Krakiwsky and Thomson, 1974], who credited [Molodensky *et al.,* 1962] with a model they never wrote."

It is now common in the literature to describe three Molodensky transformations:

(i) The *Molodensky–Badekas* transformation: a seven-parameter conformal transformation (or similarity transformation) linking rotations $\varepsilon_X, \varepsilon_Y, \varepsilon_Z$ and translations $\delta X, \delta Y, \delta Z$ between the X, Y, Z Cartesian axes and a scale factor δs to changes in the Cartesian coordinates.

(ii) The *standard Molodensky* transformation: a five-parameter transformation linking translations $\delta X, \delta Y, \delta Z$ between the X, Y, Z Cartesian axes, and changes in the ellipsoid parameters δa and δf with changes in curvilinear coordinates $\delta\phi, \delta\lambda, \delta h$.

(iii) The *abridged Molodensky* transformation: a modified version of the standard Molodensky transformation obtained by certain simplifying assumptions. The abridged Molodensky transformation equations do not contain the ellipsoidal heights h of points to be transformed.

There is another form of the Molodensky–Badekas transformation that is commonly cited. Referring to Figure 2, the vector $\mathbf{t}$ can be written as the sum of two vectors

$$\mathbf{t} = \overrightarrow{O_2O_1} + \overrightarrow{O_1G}$$

where O_1, O_2 are the origins of systems 1 and 2 respectively and G is the centroid and all vectors have components in system 2. Now, the vector $\overrightarrow{O_1G}$ (in system 2) is equal to the scaled and rotated position vector of the centroid in system 1, i.e.,

$$\overrightarrow{O_1G} = (1 + ds)\mathbf{R}_s\mathbf{g}_1$$

And, denoting

$$\overrightarrow{O_2O_1} = \begin{bmatrix}\Delta X & \Delta Y & \Delta Z\end{bmatrix}_2^T$$

equation (9) may be written as

$$
\begin{aligned}
\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_2 &= (1+ds)\begin{bmatrix} 1 & \varepsilon_Z & -\varepsilon_Y \\ -\varepsilon_Z & 1 & \varepsilon_X \\ \varepsilon_Y & -\varepsilon_X & 1 \end{bmatrix}\begin{bmatrix} \overline{X} \\ \overline{Y} \\ \overline{Z} \end{bmatrix}_1 + \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix}_2 + (1+ds)\begin{bmatrix} 1 & \varepsilon_Z & -\varepsilon_Y \\ -\varepsilon_Z & 1 & \varepsilon_X \\ \varepsilon_Y & -\varepsilon_X & 1 \end{bmatrix}\begin{bmatrix} X_G \\ Y_G \\ Z_G \end{bmatrix}_1 \\
&= (1+ds)\begin{bmatrix} 1 & \varepsilon_Z & -\varepsilon_Y \\ -\varepsilon_Z & 1 & \varepsilon_X \\ \varepsilon_Y & -\varepsilon_X & 1 \end{bmatrix}\begin{bmatrix} \overline{X} \\ \overline{Y} \\ \overline{Z} \end{bmatrix}_1 + \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix}_2 + \begin{bmatrix} X_G \\ Y_G \\ Z_G \end{bmatrix}_2
\end{aligned} \tag{10}
$$

The last term in equation (10) is the position vector of the centroid related to the origin O_1 <u>but with Cartesian components in system 2</u>. In equation (10), the product of the scale $(1+ds)$ and the rotation matrix for small angles $\mathbf{R}_s$ produces products of small quantities, i.e., $ds\,\varepsilon_X, ds\,\varepsilon_Y$, etc which, if ignored, give the transformation in the form

$$
\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_2 = \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix}_2 + \begin{bmatrix} X_G \\ Y_G \\ Z_G \end{bmatrix}_2 + \begin{bmatrix} (1+ds) & \varepsilon_Z & -\varepsilon_Y \\ -\varepsilon_Z & (1+ds) & \varepsilon_X \\ \varepsilon_Y & -\varepsilon_X & (1+ds) \end{bmatrix}\begin{bmatrix} X - X_G \\ Y - Y_G \\ Z - Z_G \end{bmatrix}_1 \tag{11}
$$

This is a common expression of the Molodensky–Badekas transformation but it should be noted that X_G, Y_G, Z_G in the last vector are components of the position vector of the centroid in system 1, but in the second vector X_G, Y_G, Z_G are components in system 2. <u>This is an important difference that is not mentioned in many publications.</u>

The following sections show how the rotation matrices $\mathbf{R}_{XYZ}$ and $\mathbf{R}_S$ are obtained

THE 3D ROTATION MATRIX

Figure 3 shows the rotation of the orthogonal X, Y, Z axes to new (orthogonal) axes X''', Y''', Z''' by a sequence of rotations r_X, r_Y, r_Z. The first rotation, r_X about the X-axis, rotates the X, Y, Z axes to X', Y', Z' (X and X' axes unchanged). The second rotation, r_Y about the Y'-axis, rotates the X', Y', Z' axes to X'', Y'', Z'' (Y' and Y'' unchanged). The final rotation, r_Z about the Z''-axis, rotates the X'', Y'', Z'' axes to X''', Y''', Z'''.

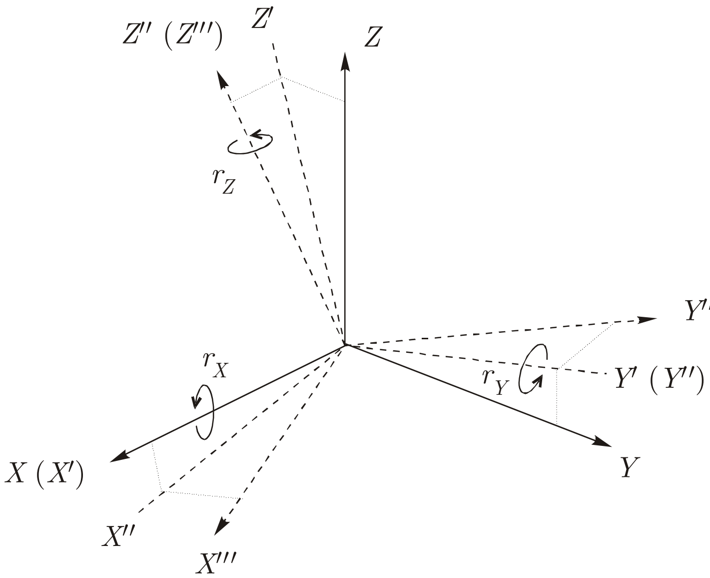

Figure 3: 3D Rotations r_X, r_Y, r_Z

The three rotations in order are:

(i) Rotation of r_X about the X-axis. This rotates the Y and Z axis to Y' and Z' with the X and X' axes coincident. Coordinates in the new system will be given by the matrix equation

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \underset{(\mathbf{R}_X)}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos r_X & \sin r_X \\ 0 & -\sin r_X & \cos r_X \end{bmatrix}} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{12}$$

(ii) Rotation of r_Y about the new Y' axis. This rotates the Y' and Z' to X'' and Z'' with the Y' and Y'' axes coincident. Coordinates in the new system will be given by the matrix equation

$$\begin{bmatrix} X'' \\ Y'' \\ Z'' \end{bmatrix} = \underset{(\mathbf{R}_Y)}{\begin{bmatrix} \cos r_Y & 0 & -\sin r_Y \\ 0 & 1 & 0 \\ \sin r_Y & 0 & \cos r_Y \end{bmatrix}} \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} \tag{13}$$

(iii) Rotation of r_Z about the new Z'' axis. This rotates the X'' and Y'' to X''' and Y''' with the Z'' and Z''' axes coincident. Coordinates in the new system will be given by the matrix equation

$$\begin{bmatrix} X''' \\ Y''' \\ Z''' \end{bmatrix} = \underset{(\mathbf{R}_Z)}{\begin{bmatrix} \cos r_Z & \sin r_Z & 0 \\ -\sin r_Z & \cos r_Z & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} X'' \\ Y'' \\ Z'' \end{bmatrix} \tag{14}$$

The coefficient matrices $\mathbf{R}_Z$, $\mathbf{R}_Y$, $\mathbf{R}_Z$ above are 3D rotation matrices which can be multiplied together (in that order) to give another rotation matrix $\mathbf{R}_{ZYX}$

$$\begin{bmatrix} X''' \\ Y''' \\ Z''' \end{bmatrix} = \mathbf{R}_Z\,\mathbf{R}_Y\,\mathbf{R}_X \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \mathbf{R}_{ZYX} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{15}$$

with

$$\mathbf{R}_{ZYX} = \begin{bmatrix} c_Y c_Z & c_X s_Z + s_X s_Y c_Z & s_X s_Z - c_X s_Y c_Z \\ -c_Y s_Z & c_X c_Z - s_X s_Y s_Z & s_X c_Z + c_X s_Y s_Z \\ s_Y & -s_X c_Y & c_X c_Y \end{bmatrix} \tag{16}$$

where, for instance, $c_Z s_Y s_X = \cos r_Z \ \sin r_Y \ \sin r_X$.

Rotation matrices, e.g. $\mathbf{R}_X$, $\mathbf{R}_Y$, $\mathbf{R}_Z$ and $\mathbf{R}_{ZYX}$ are *orthogonal,* i.e., the sum of squares of the elements of any row or column is equal to unity. They have the unique property that their inverse is equal to their transpose, i.e. $\mathbf{R}^{-1} = \mathbf{R}^T$ which will be used in later developments.

THE 3D ROTATION MATRIX FOR SMALL ANGLES

For <u>small</u> rotation angles $\varepsilon_X, \varepsilon_Y, \varepsilon_Z$ the rotation matrix $\mathbf{R}_{ZYX}$ may be simplified by the approximations

$$\begin{aligned} \cos\varepsilon_X &\simeq 1 \\ \sin\varepsilon_X &\simeq \varepsilon_X \text{ (radians)} \\ \sin\varepsilon_Z \ \sin\varepsilon_Y &\simeq 0 \end{aligned}$$

and equation (16) becomes the anti-symmetric (or skew-symmetric) matrix [equation (3)] (Harvey 1986).

$$\mathbf{R}_S = \begin{bmatrix} 1 & \varepsilon_Z & -\varepsilon_Y \\ -\varepsilon_Z & 1 & \varepsilon_X \\ \varepsilon_Y & -\varepsilon_X & 1 \end{bmatrix} \tag{17}$$

It should be noted that $\mathbf{R}_s$ is no longer orthogonal but its inverse will, nevertheless, be given by its transpose $\left(\mathbf{R}_s^{-1} = \mathbf{R}_s^T\right)$, since it is the approximate form of the orthogonal matrix $\mathbf{R}_{ZYX}^T$ (Hotine 1969, p. 263).

In the least squares development to follow it is useful to split (17) (the rotation matrix for small angles) into two parts

$$\mathbf{R}_S = \mathbf{I} + \mathbf{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & \varepsilon_Z & -\varepsilon_Y \\ -\varepsilon_Z & 0 & \varepsilon_X \\ \varepsilon_Y & -\varepsilon_X & 0 \end{bmatrix} \tag{18}$$

LEAST SQUARES SOLUTION OF TRANSFORMATION PARAMETERS

To determine the Bursa–Wolf or the Molodensky–Badekas transformation parameters; scale *ds,* rotations $\varepsilon_X, \varepsilon_Y, \varepsilon_Z$ and translations t_X, t_Y, t_Z, common points are required. Common points are those points whose coordinates are known in both Cartesian systems and each common point will yield three equations; one equation linking X-coordinates in both systems, one equation linking Y-coordinates and one equation linking Z-coordinates. For n common points there will be $3n$ equations in 7 unknowns (the parameters) and least squares can be used to obtain the best estimates of the parameters. There are two least squares techniques that may be used; they are (i) *combined least squares* that allows individual weighting of the coordinates of common points, and (ii) *parametric least squares* that assumes all pairs of common points have the same weight. The second method of solution is simpler and more commonly used.

Here, weight w is a measure of precision and is inversely proportional to variance σ^2 or $w \propto \dfrac{1}{\sigma^2}$. Hence precise observations with a small variance have a large weight

COMBINED LEAST SQUARES SOLUTION FOR PARAMETERS

BURSA–WOLF TRANSFORMATION

Using equation (18) the Bursa–Wolf transformation [Equation (5)] can be written as

$$\mathbf{l}_2 = (1 + ds)(\mathbf{I} + \mathbf{U})\mathbf{l}_1 + \mathbf{t}_2 \tag{19}$$

where $\mathbf{l}_1 = \begin{bmatrix} X & Y & Z \end{bmatrix}_1^T$ and $\mathbf{l}_2 = \begin{bmatrix} X & Y & Z \end{bmatrix}_2^T$ are vectors of coordinates in both systems and may be regarded as observations, ds is a very small quantity usually expressed in ppm, $\mathbf{t}_2 = \begin{bmatrix} t_X & t_Y & t_Z \end{bmatrix}_2^T$ is a vector of translations (in system 2), $\mathbf{I}$ is the Identity matrix and $\mathbf{U}$, defined in equation (18), contains the small rotations $\varepsilon_X, \varepsilon_Y, \varepsilon_Z$.

Expanding equation (19) gives

$$\begin{aligned} \mathbf{l}_2 &= (\mathbf{I} + \mathbf{U})\mathbf{l}_1 + ds(\mathbf{I} + \mathbf{U})\mathbf{l}_1 + \mathbf{t}_2 \\ &= \mathbf{R}_s\mathbf{l}_1 + ds\,\mathbf{I}\,\mathbf{l}_1 + ds\,\mathbf{U}\,\mathbf{l}_1 + \mathbf{t}_2 \end{aligned}$$

Now $ds\,\mathbf{I}\,\mathbf{l}_1 = ds\,\mathbf{l}_1$ since a vector pre-multiplied by the Identity matrix is equal to the vector and $ds\,\mathbf{U}\,\mathbf{l}_1 \simeq \mathbf{0}$ since ds is small (usually $<$ 1ppm) and the off-diagonal elements of $\mathbf{U}$ (the small rotations $\varepsilon_X, \varepsilon_Y, \varepsilon_Z$) are usually less than 1 second of arc ($\simeq 4.8E-06$) the products will be exceedingly small and may be neglected. Hence, for practical purposes we may write

$$\mathbf{l}_2 = \mathbf{R}_s\mathbf{l}_1 + ds\,\mathbf{l}_1 + \mathbf{t}_2 \tag{20}$$

Expanding equation (20) gives, for a single common point, the expanded matrix equation

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_2 = \begin{bmatrix} 1 & \varepsilon_Z & -\varepsilon_Y \\ -\varepsilon_Z & 1 & \varepsilon_X \\ \varepsilon_Y & -\varepsilon_X & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_1 + ds \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_1 + \begin{bmatrix} t_X \\ t_Y \\ t_Z \end{bmatrix}_2$$

Expressed as three separate equations, we have

$$\begin{aligned} X_2 &= X_1 + Y_1\varepsilon_Z - Z_1\varepsilon_Y + X_1 ds + t_X \\ Y_2 &= -X_1\varepsilon_Z + Y_1 + Z_1\varepsilon_X + Y_1 ds + t_Y \\ Z_2 &= X_1\varepsilon_Y - Y_1\varepsilon_X + Z_1 + Z_1 ds + t_Z \end{aligned}$$

and these equations may be re-formed into another expanded matrix equation as

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_2 = \left[\begin{array}{ccc:ccc:c} 1 & 0 & 0 & 0 & -Z & Y & X \\ 0 & 1 & 0 & Z & 0 & -X & Y \\ 0 & 0 & 1 & -Y & X & 0 & Z \end{array}\right]_1 \left[\begin{array}{c} t_X \\ t_Y \\ t_Z \\ \hdashline \varepsilon_X \\ \varepsilon_Y \\ \varepsilon_Z \\ \hdashline ds \end{array}\right] + \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_1 \tag{21}$$

Equation (21) can be re-arranged as

$$\left[\begin{array}{ccc:ccc:c} 1 & 0 & 0 & 0 & -Z & Y & X \\ 0 & 1 & 0 & Z & 0 & -X & Y \\ 0 & 0 & 1 & -Y & X & 0 & Z \end{array}\right]_1 \left[\begin{array}{c} t_X \\ t_Y \\ t_Z \\ \hdashline \varepsilon_X \\ \varepsilon_Y \\ \varepsilon_Z \\ \hdashline ds \end{array}\right] + \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_1 - \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{22}$$

And this equation has the general form

$$F\left(\hat{\mathbf{l}}, \hat{\mathbf{x}}\right) = \mathbf{0} \tag{23}$$

$\hat{\mathbf{l}}, \hat{\mathbf{x}}$ are estimates derived from the least squares process and $\hat{\mathbf{l}} = \mathbf{l} + \mathbf{v}$ and $\hat{\mathbf{x}} = \mathbf{x} + \delta\mathbf{x}$.

$\mathbf{l}$ is the vector of observations, $\mathbf{v}$ is a vector of residuals (small corrections to observations) $\mathbf{x}$ is the vector of parameters and $\delta\mathbf{x}$ is a vector of small corrections to the parameters.

The vectors $\mathbf{l}$ and $\mathbf{x}$ are $\quad \mathbf{l} = \left[\begin{array}{c} X_1 \\ Y_1 \\ Z_1 \\ \hdashline X_2 \\ Y_2 \\ Z_2 \end{array}\right]$ and $\mathbf{x} = \left[\begin{array}{c} t_X \\ t_Y \\ t_Z \\ \hdashline \varepsilon_X \\ \varepsilon_Y \\ \varepsilon_Z \\ \hdashline ds \end{array}\right]$

The linearized form of equation (23) is the matrix equation

$$\mathbf{Av} + \mathbf{B}\delta\mathbf{x} = \mathbf{f}^0 \tag{24}$$

where the matrices $\mathbf{A}$ and $\mathbf{B}$ contain partial derivatives

$$\mathbf{A} = \frac{\partial F}{\partial \mathbf{l}} = \left[\begin{array}{ccc|ccc} \left(1 + ds^0\right) & \varepsilon_Z^0 & -\varepsilon_Y^0 & -1 & 0 & 0 \\ -\varepsilon_Z^0 & \left(1 + ds^0\right) & \varepsilon_X^0 & 0 & -1 & 0 \\ \varepsilon_Y^0 & -\varepsilon_X^0 & \left(1 + ds^0\right) & 0 & 0 & -1 \end{array}\right] \tag{25}$$

$$\mathbf{B} = \frac{\partial F}{\partial \mathbf{x}} = \left[\begin{array}{ccc|ccc|c} 1 & 0 & 0 & 0 & -Z_1 & Y_1 & X_1 \\ 0 & 1 & 0 & Z_1 & 0 & -X_1 & Y_1 \\ 0 & 0 & 1 & -Y_1 & X_1 & 0 & Z_1 \end{array}\right] \tag{26}$$

The vectors $\mathbf{v}$ and $\delta\mathbf{x}$ are

$$\mathbf{v} = \left[\begin{array}{c} v_{X_1} \\ v_{Y_1} \\ v_{Z_1} \\ \hline v_{X_2} \\ v_{Y_2} \\ v_{Z_2} \end{array}\right] \quad \text{and} \quad \delta\mathbf{x} = \left[\begin{array}{c} \delta t_X \\ \delta t_Y \\ \delta t_Z \\ \hline \delta\varepsilon_X \\ \delta\varepsilon_Y \\ \delta\varepsilon_Z \\ \hline \delta ds \end{array}\right]$$

The vector of numeric terms $\mathbf{f}^0 = -F\left(\mathbf{l}, \mathbf{x}^0\right)$ where the superscript indicates approximate values or starting estimates in an iterative sequence and

$$\mathbf{x}^0 = \begin{bmatrix} t_X^0 & t_Y^0 & t_Z^0 & \varepsilon_X^0 & \varepsilon_Y^0 & \varepsilon_Z^0 & ds^0 \end{bmatrix}^T$$

This gives the numeric terms $\mathbf{f}^0$ as

$$\mathbf{f}^0 = \begin{bmatrix} -\left(t_X^0 - Z_1\varepsilon_Y^0 + Y_1\varepsilon_Z^0 + X_1 ds^0 + X_1 - X_2\right) \\ -\left(t_Y^0 + Z_1\varepsilon_X^0 - X_1\varepsilon_Z^0 + Y_1 ds^0 + Y_1 - Y_2\right) \\ -\left(t_Z^0 - Y_1\varepsilon_X^0 + X_1\varepsilon_Y^0 + Z_1 ds^0 + Z_1 - Z_2\right) \end{bmatrix} \tag{27}$$

The least squares solution for the vector of small corrections to the parameters $\delta\mathbf{x}$ and the vector of residuals $\mathbf{v}$ is

$$\begin{aligned} \delta\mathbf{x} &= \mathbf{N}^{-1}\mathbf{t} \\ \mathbf{v} &= \mathbf{Q}\mathbf{A}^T\mathbf{k} \end{aligned} \tag{28}$$

where

$$
\begin{aligned}
\mathbf{N} &= \mathbf{B}^T \mathbf{W}_e \mathbf{B} \\
\mathbf{t} &= \mathbf{B}^T \mathbf{W}_e \mathbf{f}^0 \\
\mathbf{Q}_e &= \mathbf{A}\mathbf{Q}\mathbf{A}^T \\
\mathbf{W}_e &= \mathbf{Q}_e^{-1} \\
\mathbf{k} &= \mathbf{W}_e \left(\mathbf{f}^0 - \mathbf{B}\delta\mathbf{x}\right)
\end{aligned}
\tag{29}
$$

The adjusted parameters $\hat{\mathbf{x}}$ and adjusted observation $\hat{\mathbf{l}}$ are

$$
\begin{aligned}
\hat{\mathbf{x}} &= \mathbf{x}^0 + \delta\mathbf{x} \\
\hat{\mathbf{l}} &= \mathbf{l} + \mathbf{v}
\end{aligned}
\tag{30}
$$

The least squares solution is iterative, i.e., a set of approximate values of the parameters is determined $\mathbf{x}^0$ and these are used to compute the vector of numeric terms $\mathbf{f}^0$. The normal equations are formed and solved for $\delta\mathbf{x}$ and these values are used to determine a new set of approximate parameters $\mathbf{x}^0$ and numeric terms $\mathbf{f}^0$. The normal equations are solved and the next set of $\delta\mathbf{x}$ computed, and so on, until the elements of $\delta\mathbf{x}$ are sufficiently small in which case the last approximate vector $\mathbf{x}^0$ contains the "correct" parameters.

In equations (29), $\mathbf{Q}$ is a cofactor matrix containing estimates of the variances of the observations, which in this case are the coordinates of the common points. By definition, weight matrices $\mathbf{W} = \mathbf{Q}^{-1}$ and the subscript "e" denoted equivalent. $\mathbf{k}$ is a vector of Lagrange multipliers.

The dimensions of the matrices, shown as $(rows, cols)$, for $n =$ number of common points and $u =$ number of parameters are:

$$
\mathbf{A}_{(3n,6n)} \quad \mathbf{B}_{(3n,u)} \quad \mathbf{Q}_{(6n,6n)} \quad \mathbf{v}_{(6n,1)} \quad \delta\mathbf{x}_{(u,1)} \quad \mathbf{f}^0_{(3n,1)} \quad \mathbf{k}_{(3n,1)}
$$

$$
\begin{aligned}
\mathbf{A}_{(3n,6n)}\mathbf{Q}_{(6n,6n)}\mathbf{A}^T_{(6n,3n)} &= \mathbf{Q}_{e_{(3n,3n)}} \\
\mathbf{Q}^{-1}_{e_{(3n,3n)}} &= \mathbf{W}_{e_{(3n,3n)}} \\
\mathbf{B}^T_{(u,3n)}\mathbf{W}_{e_{(3n,3n)}}\mathbf{B}_{(3n,u)} &= \mathbf{N}_{(u,u)} \\
\mathbf{B}^T_{(u,3n)}\mathbf{W}_{e_{(3n,3n)}}\mathbf{f}^0_{(3n,1)} &= \mathbf{t}_{(u,1)}
\end{aligned}
$$

For $n = 4$ common points denoted A, B, C and D whose coordinates are known in both systems 1 and 2, the least squares solution for the $u = 7$ parameters would have the matrix equation

$$
\mathbf{A}_{(3n,6n)}\mathbf{v}_{(6n,1)} + \mathbf{B}_{(3n,u)}\delta\mathbf{x}_{(u,1)} = \mathbf{f}^0_{(3n,1)}
$$

or

$$\begin{bmatrix} \mathbf{A}_A & 0 & 0 & 0 \\ 0 & \mathbf{A}_B & 0 & 0 \\ 0 & 0 & \mathbf{A}_C & 0 \\ 0 & 0 & 0 & \mathbf{A}_D \end{bmatrix} \begin{bmatrix} \mathbf{v}_A \\ \mathbf{v}_B \\ \mathbf{v}_C \\ \mathbf{v}_D \end{bmatrix} + \begin{bmatrix} \mathbf{B}_A \\ \mathbf{B}_B \\ \mathbf{B}_C \\ \mathbf{B}_D \end{bmatrix} \delta\mathbf{x} = \begin{bmatrix} \mathbf{f}_A^0 \\ \mathbf{f}_B^0 \\ \mathbf{f}_C^0 \\ \mathbf{f}_D^0 \end{bmatrix}$$

The sub-matrices $\mathbf{A}_A, \mathbf{A}_B, \mathbf{A}_C, \mathbf{A}_D$ would be identical with dimensions $(3,6)$ or 3 rows by 6 columns and would contain elements as per equation (25).

The sub-matrices $\mathbf{B}_A, \mathbf{B}_B, \mathbf{B}_C, \mathbf{B}_D$ would all have dimensions $(3, u)$ and each would contain elements as per equation (26) relating to system 1 coordinates of the common points *A, B, C* and *D*.

The sub-vectors $\mathbf{f}_A^0, \mathbf{f}_B^0, \mathbf{f}_C^0, \mathbf{f}_D^0$ would all have dimensions $(3,1)$ and each would contain elements as per equation (27) that are functions of the system 1 and 2 coordinates of the common points *A, B, C* and *D* and the approximate values of the parameters.

COMBINED LEAST SQUARES SOLUTION FOR PARAMETERS

MOLODENSKY–BADEKAS TRANSFORMATION

Using equation (18) the Molodensky–Badekas transformation [Equation (7)] can be written as

$$\mathbf{l}_2 = (1 + ds)(\mathbf{I} + \mathbf{U})\,\overline{\mathbf{l}}_1 + \mathbf{t}_2 \tag{31}$$

where $\overline{\mathbf{l}}_1 = \begin{bmatrix} \overline{X} & \overline{Y} & \overline{Z} \end{bmatrix}_1^T$ and the other variables are as described previously, but noting that the vector of translations $\mathbf{t}$ is different from $\mathbf{t}$ in the Bursa–Wolf transformation.

Expanding equation (31) using the approximations discussed in the previous section gives

$$\begin{aligned} \mathbf{l}_2 &= (\mathbf{I} + \mathbf{U})\,\overline{\mathbf{l}}_1 + ds\,(\mathbf{I} + \mathbf{U})\,\overline{\mathbf{l}}_1 + \mathbf{t}_2 \\ &= \mathbf{R}_s\overline{\mathbf{l}}_1 + ds\,\mathbf{I}\,\overline{\mathbf{l}}_1 + ds\,\mathbf{U}\,\overline{\mathbf{l}}_1 + \mathbf{t}_2 \\ &= \mathbf{R}_s\overline{\mathbf{l}}_1 + ds\,\overline{\mathbf{l}}_1 + \mathbf{t}_2 \end{aligned} \tag{32}$$

Expanding equation (32) gives, for a single common point, the expanded matrix equation

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_2 = \begin{bmatrix} 1 & \varepsilon_Z & -\varepsilon_Y \\ -\varepsilon_Z & 1 & \varepsilon_X \\ \varepsilon_Y & -\varepsilon_X & 1 \end{bmatrix} \begin{bmatrix} \overline{X} \\ \overline{Y} \\ \overline{Z} \end{bmatrix}_1 + ds \begin{bmatrix} \overline{X} \\ \overline{Y} \\ \overline{Z} \end{bmatrix}_1 + \begin{bmatrix} t_X \\ t_Y \\ t_Z \end{bmatrix}_2$$

Expressed as three separate equations, we have

$$
\begin{aligned}
X_2 &= \overline{X}_1 + \overline{Y}_1\varepsilon_Z - \overline{Z}_1\varepsilon_Y + \overline{X}_1 ds + t_X \\
Y_2 &= -\overline{X}_1\varepsilon_Z + \overline{Y}_1 + \overline{Z}_1\varepsilon_X + \overline{Y}_1 ds + t_Y \\
Z_2 &= \overline{X}_1\varepsilon_Y - \overline{Y}_1\varepsilon_X + \overline{Z}_1 + \overline{Z}_1 ds + t_Z
\end{aligned}
$$

and these equations may be re-formed into another expanded matrix equation as

$$
\left[\begin{array}{ccc:ccc:c} 1 & 0 & 0 & 0 & -\overline{Z} & \overline{Y} & \overline{X} \\ 0 & 1 & 0 & \overline{Z} & 0 & -\overline{X} & \overline{Y} \\ 0 & 0 & 1 & -\overline{Y} & \overline{X} & 0 & \overline{Z} \end{array}\right]_1 \left[\begin{array}{c} t_X \\ t_Y \\ t_Z \\ \hdashline \varepsilon_X \\ \varepsilon_Y \\ \varepsilon_Z \\ \hdashline ds \end{array}\right] + \begin{bmatrix} \overline{X} \\ \overline{Y} \\ \overline{Z} \end{bmatrix}_1 - \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{33}
$$

Following the development for the Bursa–Wolf least squares solution we have

$$
\mathbf{Av} + \mathbf{B}\delta\mathbf{x} = \mathbf{f}^0
$$

where

$$
\mathbf{A} = \left[\begin{array}{ccc:ccc} \left(1+ds^0\right) & \varepsilon_Z^0 & -\varepsilon_Y^0 & -1 & 0 & 0 \\ -\varepsilon_Z^0 & \left(1+ds^0\right) & \varepsilon_X^0 & 0 & -1 & 0 \\ \varepsilon_Y^0 & -\varepsilon_X^0 & \left(1+ds^0\right) & 0 & 0 & -1 \end{array}\right] \tag{34}
$$

$$
\mathbf{B} = \left[\begin{array}{ccc:ccc:c} 1 & 0 & 0 & 0 & -\overline{Z}_1 & \overline{Y}_1 & \overline{X}_1 \\ 0 & 1 & 0 & \overline{Z}_1 & 0 & -\overline{X}_1 & \overline{Y}_1 \\ 0 & 0 & 1 & -\overline{Y}_1 & \overline{X}_1 & 0 & \overline{Z}_1 \end{array}\right] \tag{35}
$$

$$
\mathbf{f}^0 = \begin{bmatrix} -\left(t_X^0 - \overline{Z}_1\varepsilon_Y^0 + \overline{Y}_1\varepsilon_Z^0 + \overline{X}_1 ds^0 + \overline{X}_1 - X_2\right) \\ -\left(t_Y^0 + \overline{Z}_1\varepsilon_X^0 - \overline{X}_1\varepsilon_Z^0 + \overline{Y}_1 ds^0 + \overline{Y}_1 - Y_2\right) \\ -\left(t_Z^0 - \overline{Y}_1\varepsilon_X^0 + \overline{X}_1\varepsilon_Y^0 + \overline{Z}_1 ds^0 + \overline{Z}_1 - Z_2\right) \end{bmatrix} \tag{36}
$$

and the solution for the parameters $\delta\mathbf{x}$ is identical to the one set out in the Bursa–Wolf solution.

PARAMETRIC LEAST SQUARES SOLUTION FOR PARAMETERS

BURSA–WOLF TRANSFORMATION

In the same manner as the Combined Least Squares Solution [see equation (21)] we have the expanded matrix equation for a single common point as

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_2 = \left[\begin{array}{ccc|ccc|c} 1 & 0 & 0 & 0 & -Z & Y & X \\ 0 & 1 & 0 & Z & 0 & -X & Y \\ 0 & 0 & 1 & -Y & X & 0 & Z \end{array}\right]_1 \left[\begin{array}{c} t_X \\ t_Y \\ t_Z \\ \hline \varepsilon_X \\ \varepsilon_Y \\ \varepsilon_Z \\ \hline ds \end{array}\right] + \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_1 \tag{37}$$

Equation (37) can be re-arranged as

$$\begin{bmatrix} v_X \\ v_Y \\ v_Z \end{bmatrix} + \left[\begin{array}{ccc|ccc|c} 1 & 0 & 0 & 0 & -Z_1 & Y_1 & X_1 \\ 0 & 1 & 0 & Z_1 & 0 & -X_1 & Y_1 \\ 0 & 0 & 1 & -Y_1 & X_1 & 0 & Z_1 \end{array}\right] \left[\begin{array}{c} t_X \\ t_Y \\ t_Z \\ \hline \varepsilon_X \\ \varepsilon_Y \\ \varepsilon_Z \\ \hline ds \end{array}\right] = \begin{bmatrix} X_2 - X_1 \\ Y_2 - Y_1 \\ Z_2 - Z_1 \end{bmatrix} \tag{38}$$

and this equation has the general form

$$\mathbf{v} + \mathbf{Bx} = \mathbf{f} \tag{39}$$

where the vector of residuals $\mathbf{v}$ has been added to the left-hand-side to reflect the fact that measurements (the coordinates) contain random errors and do not fit the model exactly. The residuals are simply an acknowledgement of a slight discordance between model and reality.

The least squares solution for the vector of parameters $\mathbf{x}$ is

$$\mathbf{x} = \mathbf{N}^{-1}\mathbf{t} \tag{40}$$

where

$$\begin{aligned} \mathbf{N} &= \mathbf{B}^T\mathbf{WB} \\ \mathbf{t} &= \mathbf{B}^T\mathbf{Wf} \\ \mathbf{W} &= \mathbf{Q}^{-1} \end{aligned} \tag{41}$$

In equations (41), $\mathbf{Q}$ is a cofactor matrix containing estimates of the variances of the observations, which in this case are the coordinates of the common points and by definition, weight matrices $\mathbf{W} = \mathbf{Q}^{-1}$. It is often difficult to assess the variances of the coordinate differences and in this method of solution for the transformation parameters it is often assumed that all measurements (the coordinate differences) have the same variance, and in this case $\mathbf{W} = \mathbf{Q}^{-1} = \mathbf{I}$.

PARAMETRIC LEAST SQUARES SOLUTION FOR PARAMETERS

MOLODENSKY–BADEKAS TRANSFORMATION

In the same manner as the Combined Least Squares Solution [see equation (33)] we have the expanded matrix equation for a single common point as

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_2 = \left[\begin{array}{ccc:ccc:c} 1 & 0 & 0 & 0 & -\overline{Z} & \overline{Y} & \overline{X} \\ 0 & 1 & 0 & \overline{Z} & 0 & -\overline{X} & \overline{Y} \\ 0 & 0 & 1 & -\overline{Y} & \overline{X} & 0 & \overline{Z} \end{array}\right]_1 \left[\begin{array}{c} t_X \\ t_Y \\ t_Z \\ \hdashline \varepsilon_X \\ \varepsilon_Y \\ \varepsilon_Z \\ \hdashline ds \end{array}\right] + \begin{bmatrix} \overline{X} \\ \overline{Y} \\ \overline{Z} \end{bmatrix}_1 \tag{42}$$

Equation (42) can be re-arranged as

$$\begin{bmatrix} v_X \\ v_Y \\ v_Z \end{bmatrix} + \left[\begin{array}{ccc:ccc:c} 1 & 0 & 0 & 0 & -\overline{Z}_1 & \overline{Y}_1 & \overline{X}_1 \\ 0 & 1 & 0 & \overline{Z}_1 & 0 & -\overline{X}_1 & \overline{Y}_1 \\ 0 & 0 & 1 & -\overline{Y}_1 & \overline{X}_1 & 0 & \overline{Z}_1 \end{array}\right] \left[\begin{array}{c} t_X \\ t_Y \\ t_Z \\ \hdashline \varepsilon_X \\ \varepsilon_Y \\ \varepsilon_Z \\ \hdashline ds \end{array}\right] = \begin{bmatrix} X_2 - \overline{X}_1 \\ Y_2 - \overline{Y}_1 \\ Z_2 - \overline{Z}_1 \end{bmatrix} \tag{43}$$

and this equation has the general form

$$\mathbf{v} + \mathbf{B}\mathbf{x} = \mathbf{f} \tag{44}$$

The least squares solution for the vector of parameters $\mathbf{x}$ is identical to the parametric least squares solution for the Bursa–Wolf transformation shown above.

RELATIONSHIP BETWEEN BURSA–WOLF TRANSLATIONS AND MOLODENSKY-BADEKAS TRANSLATIONS

From equations (5) and (7) we may write the two transformations in vector form as

Bursa–Wolf $$\mathbf{l}_2 = (1 + ds)\mathbf{R}_s \mathbf{l}_1 + \mathbf{t}_B \tag{45}$$

Molodensky–Badekas $$\mathbf{l}_2 = (1 + ds)\mathbf{R}_s \overline{\mathbf{l}}_1 + \mathbf{t}_M \tag{46}$$

where the $\mathbf{t}_B, \mathbf{t}_M$ are Bursa–Wolf and Molodensky–Badekas translations respectively.

Equating equations (45) and (46) gives

$$(1 + ds)\mathbf{R}_s \mathbf{l}_1 + \mathbf{t}_B = (1 + ds)\mathbf{R}_s \overline{\mathbf{l}}_1 + \mathbf{t}_M \tag{47}$$

Letting $s = 1 + ds$ and $\overline{\mathbf{l}}_1 = \mathbf{l}_1 - \mathbf{g}_1$ then equation (47) becomes

$$\begin{aligned} s\,\mathbf{R}_s \mathbf{l}_1 + \mathbf{t}_B &= s\,\mathbf{R}_s \left(\mathbf{l}_1 - \mathbf{g}_1\right) + \mathbf{t}_M \\ &= s\,\mathbf{R}_s \mathbf{l}_1 - s\,\mathbf{R}_s \mathbf{g}_1 + \mathbf{t}_M \end{aligned}$$

giving

$$\mathbf{t}_M = \mathbf{t}_B + s\,\mathbf{R}_s \mathbf{g}_1 \tag{48}$$

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