Solutions of Kepler's Equation

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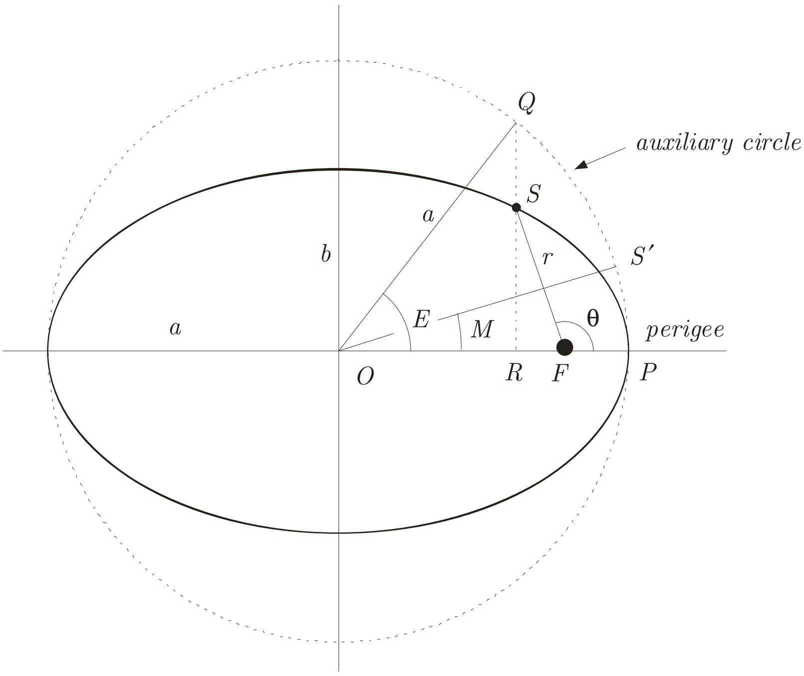

Figure 1. Satellite S in elliptical orbit about the earth F

Figure 1 shows a satellite S is in an elliptical orbit of period T about the earth F where T is the time between two successive passages through *perigee* P. The orbital ellipse has semi-axes a and b $\left(a > b\right)$ and eccentricity $e = \sqrt{\left(a^2 - b^2\right)/a^2}$ and the earth is at one of the focal points with $OF = ae$. The orbital ellipse has an auxiliary circle of radius a and S is located on the ellipse by the orbital radius r and the *true anomaly* θ and Q is located on the auxiliary circle by the radius a and the *eccentric anomaly* E. The line QR is perpendicular to the major axis $(2a)$ and passes through S. As S moves in its elliptical orbit, a fictitious satellite S', located on the auxiliary circle by the radius a and the *mean anomaly* M, moves around the auxiliary circle with constant angular velocity and with a period of revolution identical to the orbital period T. When the satellite S is at *perigee* P, anomalies θ, E and M all equal zero.

The following relationships are fundamental in orbital mechanics

M, E and e are related by Kepler's equation which is an outcome of Kepler's 2nd law (Deakin 2007)

$$M = E - e\sin E \tag{1}$$

E, θ and e are related by

$$\tan E = \frac{\sqrt{1-e^2}\sin\theta}{\cos\theta + e} \quad \text{or} \quad \tan\theta = \frac{\sqrt{1-e^2}\sin E}{\cos E - e} \tag{2}$$

e, r and θ are related by

$$r = \frac{a\left(1-e^2\right)}{1+e\cos\theta} \tag{3}$$

The orbital period T (in seconds) of the satellite is known to be (Deakin, 2007, p. 27)

$$T = 2\pi\sqrt{\frac{a^3}{GM}} = \frac{2\pi}{n} \tag{4}$$

where $GM = 3986005 \times 10^8 \ \ \mathrm{m}^3/\mathrm{s}^2$ is the geocentric gravitational constant that is the product of the universal constant of gravitation G and the mass of the earth M (GRS80) and n is the *mean motion* of the satellite, defined as

$$n = \sqrt{\frac{GM}{a^3}} \tag{5}$$

If t_0 is the time (in seconds) when the satellite passes through perigee and t is some time after t_0 then $t - t_0$ is the *time of flight* and the mean anomaly M can be expressed as

$$M = n\left(t - t_0\right) \tag{6}$$

and Kepler's equation written as

$$M = n\left(t - t_0\right) = E - e\sin E \tag{7}$$

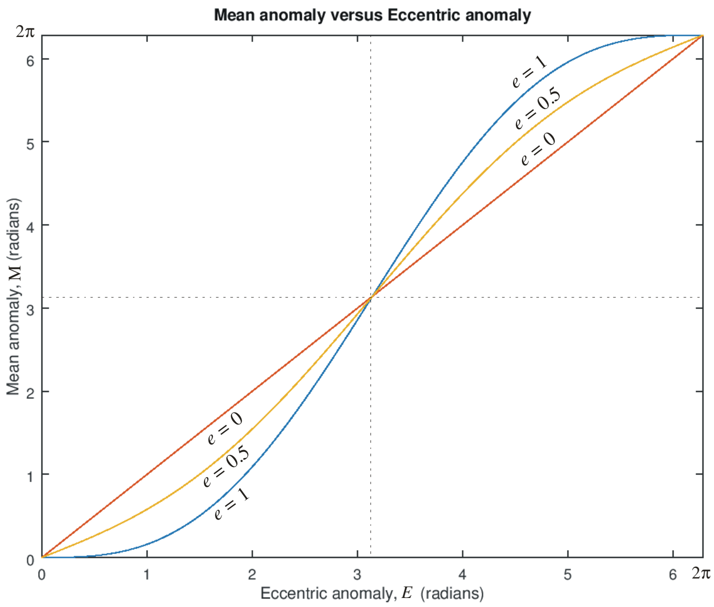

Figure 1. Mean anomaly M versus Eccentric anomaly E for $e = 0,\ 0.5,\ 1$

In orbital mechanics it is often required to compute the position of a satellite at time t when the mean motion n and t_0 are known quantities as well as orbital constants a and e. To achieve this, Kepler's equation must be solved for E and there are a number of different methods to solve Kepler's equation.

Of particular interest are three methods, (i) a series expansion, (ii) Newton-Raphson iteration and (iii) an iterative scheme using the bisection method.

Solution of Kepler's equation by trigonometric series

To solve Kepler's equation for the eccentric anomaly E equation (1) is recast as

$$E = M + e \sin E$$

and Lagrange's theorem[1] used to derive an expression for E as a trigonometric series in M

$$\begin{aligned} E = M &+ \left(e - \frac{1}{8}e^3 + \frac{1}{192}e^5 + \cdots \right) \sin M + \left(\frac{1}{2}e^2 - \frac{1}{6}e^4 + \frac{1}{48}e^6 + \cdots \right) \sin 2M \\ &+ \left(\frac{3}{8}e^3 - \frac{27}{128}e^5 + \cdots \right) \sin 3M + \left(\frac{1}{3}e^4 - \frac{4}{15}e^6 + \cdots \right) \sin 4M \\ &+ \left(\frac{125}{384}e^5 + \cdots \right) \sin 5M + \left(\frac{27}{80}e^6 + \cdots \right) \sin 6M + \cdots \end{aligned} \tag{8}$$

This equation, given by Battin (1987, eq. 5.17), ignores terms with coefficients e^7 and greater. It is a recasting of an earlier equation (also derived using Lagrange's theorem) given by Moulton (1914, p. 169). An explanation of Lagrange's theorem and the derivation of (8) is given in the Appendix.

With the aid of the computer algebra package *Maxima*[2] this series can be extended to higher orders of the eccentricity e and multiples of the mean anomaly M (see Appendix).

$$\begin{aligned} E = M &+ \left(e - \frac{1}{8}e^3 + \frac{1}{192}e^5 - \frac{1}{9216}e^7 + \frac{1}{737280}e^9 - \cdots \right) \sin M \\ &+ \left(\frac{1}{2}e^2 - \frac{1}{6}e^4 + \frac{1}{48}e^6 - \frac{1}{720}e^8 + \frac{1}{17280}e^{10} - \cdots \right) \sin 2M \\ &+ \left(\frac{3}{8}e^3 - \frac{27}{128}e^5 + \frac{243}{5120}e^7 - \frac{243}{40960}e^9 + \cdots \right) \sin 3M \\ &+ \left(\frac{1}{3}e^4 - \frac{4}{15}e^6 + \frac{4}{45}e^8 - \frac{16}{945}e^{10} + \cdots \right) \sin 4M \\ &+ \left(\frac{125}{384}e^5 - \frac{3125}{9216}e^7 + \frac{78125}{516096}e^9 - \cdots \right) \sin 5M + \left(\frac{27}{80}e^6 - \frac{243}{560}e^8 + \frac{2187}{8960}e^{10} - \cdots \right) \sin 6M \\ &+ \left(\frac{16807}{46080}e^7 - \frac{823543}{1474560}e^9 + \cdots \right) \sin 7M + \left(\frac{128}{315}e^8 - \frac{2048}{2835}e^{10} + \cdots \right) \sin 8M \\ &+ \left(\frac{531441}{1146880}e^9 - \cdots \right) \sin 9M + \left(\frac{78125}{145152}e^{10} - \cdots \right) \sin 10M + \cdots \end{aligned} \tag{9}$$

Colwell (1993, Appendix D) uses Lagrange's Theorem and the computer algebra system *Mathematica* to obtain a series for E where the coefficients of $e, e^2, e^3, \ldots, e^{10}$ are functions of powers of sines and cosines of M.

For orbits with small eccentricities $\left(e < 0.2 \right)$ the series (8) and (9) are rapidly convergent, but it is known that this series will diverge for some values of M when the eccentricity $e > 0.6627434194\ldots$ (Battin 1987, p.205). This result, first shown by Laplace (1749–1827), is known as the Laplace Limit.

Efficient evaluation of the trigonometric series for E

The trigonometric series (8) and (9) can be expressed in the form

$$E = M + \sum_{k=1}^{N} c_k \sin kM \tag{10}$$

[1] Also known as Lagrange's reversion theorem. In 1770, Joseph Louis Lagrange (1736–1813) published his power series for reversion: 'Nouvelle méthode pour résoudre les équations littérales par le moyen des séries' in *Mémoires de l Académie Royale des Sciences at Belles-Lettres de Berlin,* Vol. 24, pp. 251-326. (http://gallica.bnf.fr/ark:/12148/bpt6k229222d). See Wikipedia

[2] Maxima is based on a 1982 version of MACSYMA, which was developed at MIT with funding from the United States Department of Energy and other government agencies. See Appendix.

and for $N = 6$ the six coefficients $c_1, c_2, \ldots, c_6$ from (8) are

$$\begin{aligned} c_1 &= e - \frac{1}{8}e^3 + \frac{1}{192}e^5 + \cdots \quad c_2 = \frac{1}{2}e^2 - \frac{1}{6}e^4 + \frac{1}{48}e^6 + \cdots \quad c_3 = \frac{3}{8}e^3 - \frac{27}{128}e^5 + \cdots \\ c_4 &= \frac{1}{3}e^4 - \frac{4}{15}e^6 + \cdots \qquad c_5 = \frac{125}{384}e^5 + \cdots \qquad c_6 = \frac{27}{80}e^6 + \cdots \end{aligned} \tag{11}$$

The summation $S = \sum_{k=1}^{N} c_k \sin kM$ in (10) can be evaluated using Clenshaw summation[3] (see Appendix) that avoids multiple evaluations if the sine function and (10) can be written as

$$E = M + \sum_{k=1}^{N} c_k \sin kM = M + S = M + y_1 \sin M \tag{12}$$

where y_1 is obtained from the backward recurrence formula

$$y_k = \begin{cases} 0, & \text{for } k > N \\ 2\cos M\, y_{k+1} - y_{k+2} + c_k, & \text{for } k = N, N-1, N-2, \ldots, 3, 2, 1 \end{cases} \tag{13}$$

Using Clenshaw's summation means that there is one evaluation of $\sin M$ and one evaluation of $\cos M$ in determining the eccentric anomaly E.

Example 1

Use Clenshaw summation to evaluate E from (8) with $e = 0.100$ and $M = 5^\circ = 0.087266463$ radians.

The coefficients c_k from (11) are

e	k	e^k	c_k
0.1	1	0.100000000000	9.987505208E-02
	2	0.010000000000	4.983354167E-03
	3	0.001000000000	3.728906250E-04
	4	0.000100000000	3.306666667E-05
	5	0.000010000000	3.255208333E-06
	6	0.000001000000	3.375000000E-07

Clenshaw recurrence

k	y_k		
		$2\cos M =$	1.992389396
8	0	$\sin M =$	0.087155743
7	0		c_k
6	0.000000337500		3.375000000E-07
5	0.000003927640		3.255208333E-06
4	0.000040554554		3.306666667E-05
3	0.000449763450		3.728906250E-04
2	0.005838903540		4.983354167E-03
1	0.111058658132		9.987505208E-02

$$E = M + \sum_{k=1}^{N=6} c_k \sin kM = M + S = M + y_1 \sin M = \begin{cases} 0.096945862438 \text{ radians} \\ 5.554588758940 \text{ degrees} \end{cases}$$

A Matlab function *Kepler_series.m* is shown in the Appendix.

[3] Clenshaw summation evaluates a sum of products of indexed coefficients by functions which obey a recurrence relation.

Solution of Kepler's equation by Newton-Raphson iteration

As an alternative to the trigonometric series method, a value for E can be computed using the Newton-Raphson method for the real roots of the equation $f\left(E\right) = 0$ given in the form of an iterative equation

$$E_{n+1} = E_n - \frac{f\left(E_n\right)}{f'\left(E_n\right)} \tag{14}$$

where n denotes the n^{th} iteration and $f\left(E\right)$ is found from Kepler's equation (1) as

$$f\left(E\right) = E - e\sin E - M \tag{15}$$

The derivative $f'\left(E\right) = \dfrac{d}{dE}\left\{f\left(E\right)\right\}$ is given by

$$f'\left(E\right) = 1 - e\cos E \tag{16}$$

Substituting (15) and (16) into (14) gives

$$E_{n+1} = E_n - \frac{M_n - M}{1 - e\cos E_n} \tag{17}$$

where the second term is a correction to E_n and $M_n = E_n - e\sin E_n$

With an initial value E_0, the correction $\left(M_0 - M\right)\big/\left(1 - e\cos E_0\right)$ is evaluated and subtracted from E_0 giving an updated value E_1 and the process is repeated to obtain $E_2, E_3, \ldots$. This iterative process can be concluded when the difference between E_{n+1} and E_n reaches an acceptably small value.

Initial value E_0

It is common practice to use $E_0 = M$ as the initial (or starting value) in the iterative scheme, but it should be noted that this will not always lead to convergence.

Example 2

For values $M = 7^\circ$, $e = 0.999$ and with $E_0 = M$ as the initial value, the following results are obtained for the first 14 iterations

n	E_n (degrees)
0	7.000000000
1	832.869123399
2	275.954960202
3	-87.610599131
4	-48.562394340
5	-11.225112021
6	340.962526137
7	-5996.812219845
8	-2084.497865298
9	778.410987047
10	-737.535684055
11	14598.350404127
12	7099.442370278
13	1056.785610878
14	-12039.362753148

This example of non-convergence is shown in Meeus (1991, p. 189) with the comment that convergence to the true value $\left(E = 52.270261528 \text{ degrees}\right)$ did not occur until after the 47th iteration.

A faster convergence can be achieved with a better initial value E_0. The selection of initial values for iteration schemes has been the subject of many papers and Odell & Gooding (1986), Meeus (1991), Colwell (1993) and Esmaelzadeh & Ghadiri (2014) have summaries of modern iterative solutions and initial values. Here we use an equation for E_0 from Smith (1979), who in his development, included the following points relating to the solution for the root of $f(E) = E - e\sin E - M$:

(i) Due to symmetry (see Figure 1) it is only necessary to consider cases where $0 \le M \le \pi$

(ii) $f(M) = -e\sin M$ is negative or zero; $f(M+e) = e(1-\sin(M+e))$ is positive or zero and the derivative $f'(E) = 1 - e\cos E$ is positive which means that $f(E)$ must vanish somewhere within the interval $(M, M+e)$. Thus the solution for $f(E) = 0$ is bounded and

$$M \le E \le M + e \tag{18}$$

Smith (1979) then used these bounds in the equation for a straight line $\dfrac{y - y_1}{x - x_1} = \dfrac{y_2 - y_1}{x_2 - x_1}$ to obtain

$$\frac{0 - f(M)}{E_0 - M} = \frac{f(M+e) - f(M)}{(M+e) - M}$$

which is simplified and re-arranged as (Smith 1979, eq. 5)

$$E_0 = M + \frac{e\sin M}{1 - \sin(M+e) + \sin M} \tag{19}$$

Smith (1979) tested this initial value in solutions for E in Regions I and II that are known to be problematic in iterative solutions (Meeus 1991, Chapter 29, Figures 4, 5 and 6)

Region I:	$0.05 \le M \le \pi$	and	$0.01 \le e \le 0.99$
Region II:	$0.005 \le M \le 0.4$	and	$0.95 \le e \le 0.999$

Example 3

Using (19) with values $M = 7°$, $e = 0.999$ gives $E_0 = 38.527006574$ degrees and the correct result of $E = 52.270261528$ degrees is obtained after 5 iterations.

n	E_n (degrees)
0	38.527006574
1	57.412628477
2	52.682423402
3	52.273242571
4	52.270261686
5	52.270261528
6	52.270261528

Odell & Gooding (1986) note that (19) will not produce Newton-Raphson convergence for values of e greater than about 0.9995. The source of the problem, they say, is that if $0 < E_0 < E$ with $0 < M < E < \pi$ then E_1 (the estimate of E after one iteration) will exceed E (the true value) and sometimes will be much bigger than π. This is clearly the case with the example above (even with $e = 0.999$ being less than 0.9995) and this does not satisfy their requirements for convergence where every iteration must produce a value closer to the true value. But the iterative scheme does converge on the true solution after 5 iterations. Odell & Gooding (1986) are focussed on schemes of one or two iterations only to maximise computer efficiency.

Example 4

Interestingly using (19) with values $M = 7^\circ$, $e = 1$ gives $E_0 = 38.620614337$ degrees and the correct result of $E = 52.386793829$ degrees is obtained after 5 iterations.

n	E_n (degrees)
0	38.620614337
1	57.555617286
2	52.802877860
3	52.389831537
4	52.386793993
5	52.386793829
6	52.386793829

A Matlab function *Kepler_ Newton.m* is shown in the Appendix.

Solution of Kepler's equation using the Bisection Method

The Bisection Method is a numerical method for the real roots of $f(x) = 0$. The method is always convergent and is simple to implement, but it is relatively slow as its rate of convergence is linear.

It is based on a result from calculus known as the *Intermediate Value Theorem* or *Balzano's Theorem*[4] that can be expressed as (Apostol 1967):

> Let $f(x)$ be a continuous function at each point of a closed interval $[a,b]$ and assume that $f(a)$ and $f(b)$ have opposite signs, then there is at least one c in the open interval (a,b) such that $f(c) = 0$.

And c is a root of $f(x) = 0$ bounded by the open interval (a,b).

The Bisection Method can be understood by following Dahlquist & Björck (1974):

Suppose that $f(x)$ is continuous and two points a_0, b_0 are found at which the function values $f(a_0)$ and $f(b_0)$ have opposite signs, i.e. $f(a_0)f(b_0) < 0$.

A sequence of intervals $I_1 = (a_1, b_1) > I_2(a_2, b_2) > I_3(a_3, b_3) \ldots$, can be determined which all contain a root of $f(x) = 0$ and each interval is half the size of the previous interval. This sequence can be terminated when the interval bounds (a_k, b_k) become sufficiently close.

The intervals $I_k = (a_k, b_k)$, $k = 1, 2, 3, \ldots$ are determined recursively as follows:

The mid-point of the interval I_{k-1} is

$$c_k = \tfrac{1}{2}\left(a_{k-1} + b_{k-1}\right) \tag{20}$$

Evaluate $f(c_k)$ and determine the bounds of the next interval I_k according to the rule

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[4] Bernard Bolzano (1781–1848), a Catholic priest who made many important contributions to mathematics in the first half of the 19th century, was one of the first to recognize that many 'obvious' statements about continuous functions require proof. His observations concerning continuity were published posthumously in 1850 in *Paradoxien des Unendlichen* [Paradoxes of the infinite]. An English translation of Bolzano's paper on the Intermediate Value Theorem (Bolzano 1817) is given by Russ (1980).

$$\left(a_k, b_k\right) = \begin{cases} \left(c_k, b_{k-1}\right) & \text{if } f\left(c_k\right) < 0 \\ \left(a_{k-1}, c_k\right) & \text{if } f\left(c_k\right) > 0 \end{cases} \tag{21}$$

From the construction of $\left(a_k, b_k\right)$ it follows immediately that $f\left(a_k\right) < 0$ and $f\left(b_k\right) > 0$ and that each interval I_k of length $d_k = \left|b_k - a_k\right|$ contains a root of $f\left(x\right) = 0$. After n steps the root is contained in the interval of length d_n where

$$d_n = \tfrac{1}{2} d_{n-1} = \left(\tfrac{1}{2}\right)^2 d_{n-2} = \left(\tfrac{1}{2}\right)^3 d_{n-3} = \cdots = \left(\tfrac{1}{2}\right)^n d_0 \tag{22}$$

and $d_0 = \left|b_0 - a_0\right|$ is the initial interval length. A re-arrangement of (22) gives

$$2^n = \frac{d_0}{d_n} \tag{23}$$

If ε is some desired tolerance then the integer number of iterations n to achieve this tolerance is

$$n = \text{ceil}\left(\log_2 \frac{d_0}{\varepsilon}\right) = \text{ceil}\left(\frac{1}{\ln 2} \ln \frac{d_0}{\varepsilon}\right) = \text{ceil}\left(1.443 \ln \frac{d_0}{\varepsilon}\right) \tag{24}$$

where $\text{ceil}\left(x\right)$ is the ceiling function and $\log_2\left(x\right)$ is the binary logarithm (logarithm to the base 2). The ceiling function rounds a fractional number to the next highest integer and binary logarithms can be evaluated from the natural logarithm $\ln x \equiv \log_{\mathrm{e}} x$ using the rule: $\log_2 x = \dfrac{\ln x}{\ln 2} \approx 1.443 \ln x$. $\mathrm{e} \approx 2.7182818284 59$ is the base of the natural logarithms.

For example, if $d_0 = \tfrac{1}{2}\pi$ and the tolerance $\varepsilon = 1.0\ \text{E} - 15$ then $n = \text{ceil}\left(50.491\right) = 51$.

The Bisection method must succeed. If the interval happens to contain two or more roots, bisection will find one of them. If the interval contains no roots and merely straddles a singularity, it will converge on the singularity (Press et al. 1992).

The Appendix contains a Matlab function *Kepler_ Bisection.m* that is based on a BASIC program shown in *Sky & Telescope* August 1985 by Roger W Sinnott (1985). Sinnott calls his method 'Binary Search' and Meeus (1991) features Sinnott's binary search as a foolproof method of solving Kepler's equation for $0 < e \leq 1$.

Appendix

Reversion of a series

If we have an expression for a variable z as a series of powers or functions of another variable y then we may, by a reversion of the series, find an expression for y as series of functions of z. Reversion of a series can be done using Lagrange's theorem, a proof of which can be found in Battin (1987).

Suppose that

$$y = z + xF\left(y\right) \tag{25}$$

then Lagrange's theorem states that for any f

$$\begin{aligned} f\left(y\right) = f\left(z\right) &+ \frac{x}{1!}F\left(z\right)f'\left(z\right) \\ &+\frac{x^2}{2!}\frac{d}{dz}\left[\left\{F\left(z\right)\right\}^2 f'\left(z\right)\right] \\ &+\frac{x^3}{3!}\frac{d^2}{dz^2}\left[\left\{F\left(z\right)\right\}^3 f'\left(z\right)\right] \\ &+\cdots \\ &+\frac{x^n}{n!}\frac{d^{n-1}}{dz^{n-1}}\left[\left\{F\left(z\right)\right\}^n f'\left(z\right)\right] \\ &+\cdots \end{aligned} \tag{26}$$

As an example, consider Kepler's equation expressed in the form

$$E = M + e\sin E \tag{27}$$

And we wish to find an expression for E as a function of M.

Comparing the variables in equations (27) and (25), $z = M$, $y = E$, $x = e$ and $F\left(z\right) = F\left(M\right) = \sin M$.

And if we choose $f\left(y\right) = y$ then $f\left(z\right) = z$ and $f'\left(z\right) = 1$ then Lagrange's theorem gives

$$\begin{aligned} E = M + e\sin M &+ \frac{e^2}{2!}\frac{d}{dM}\left[\sin^2 M\right] + \frac{e^3}{3!}\frac{d^2}{dM^2}\left[\sin^3 M\right] + \frac{e^4}{4!}\frac{d^3}{dM^3}\left[\sin^4 M\right] + \cdots \\ &+ \frac{e^n}{n!}\frac{d^{n-1}}{dM^{n-1}}\left[\sin^n M\right] + \cdots \end{aligned} \tag{28}$$

This can also be expressed as (Colwell 1993)

$$E = M + \sum_{n=1}^{\infty} a_n\left(M\right)e^n \ , \quad a_n\left(M\right) = \frac{1}{n!}\frac{d^{n-1}}{dM^{n-1}}\left(\sin^n M\right) \tag{29}$$

Neglecting terms of order e^7 and higher, the derivatives up to the 5th order are

$$\begin{aligned} \frac{d}{dM}\sin^2 M &= 2\cos M\sin M = \sin 2M \\ \frac{d^2}{dM^2}\sin^3 M &= 6\cos^2 M\sin M - 3\sin^3 M = \frac{1}{4}\left(9\sin 3M - 3\sin M\right) \\ &= \frac{1}{2^2}\left(3^3\sin 3M - 3\sin M\right) \\ \frac{d^3}{dM^3}\sin^4 M &= 24\cos^3 M\sin M - 40\cos M\sin^3 M = 8\sin 4M - 4\sin 2M \\ &= \frac{1}{2^3}\left(4^3\sin 4M - 2^3\cdot 4\sin 2M\right) \end{aligned}$$

$$\begin{aligned}
\frac{d^4}{dM^4}\sin^5 M &= 120\cos^4 M\sin M - 440\cos^2 M\sin^3 M + 65\sin^5 M \\
&= \frac{1}{16}\Big(625\sin 5M - 405\sin 3M + 10\sin M\Big) \\
&= \frac{1}{2^4}\Big(5^4\sin 5M - 3^4\cdot 5\sin 3M + 10\sin M\Big) \\
\frac{d^5}{dM^5}\sin^6 M &= 720\cos^5 M\sin M - 4800\cos^3 M\sin^3 M + 2256\cos M\sin^5 M \\
&= 243\sin 6M - 192\sin 4M + 15\sin 2M \\
&= \frac{1}{2^5}\Big(6^5\sin 6M - 4^5\cdot 6\sin 4M + 2^5\cdot 15\sin 2M\Big)
\end{aligned}$$

Inserting the derivatives into (28) gives a series expression for E (Moulton 1914, p. 169)

$$\begin{aligned}
E = M + e\sin M + \frac{e^2}{2!}\sin 2M + \frac{e^3}{3!2^2}\Big(3^2\sin 3M - 3\sin M\Big) + \frac{e^4}{4!2^3}\Big(4^3\sin 4M - 2^3\cdot 4\sin 2M\Big) \\
+\frac{e^5}{5!2^4}\Big(5^4\sin 5M - 3^4\cdot 5\sin 3M + 10\sin M\Big) + \frac{e^6}{6!2^5}\Big(6^5\sin 6M - 4^5\cdot 6\sin 4M + 2^5\cdot 15\sin 2M\Big) + \cdots
\end{aligned}$$

that simplifies to

$$\begin{aligned}
E = M + e\sin M + \frac{e^2}{2}\sin 2M + \frac{e^3}{8}\Big(3\sin 3M - \sin M\Big) + \frac{e^4}{6}\Big(2\sin 4M - \sin 2M\Big) \\
+\frac{e^5}{384}\Big(125\sin 5M - 81\sin 3M + 2\sin M\Big) + \frac{e^6}{720}\Big(243\sin 6M - 192\sin 4M + 15\sin 2M\Big) + \cdots
\end{aligned} \tag{30}$$

This can be re-arranged as (Battin 1987, eq. 5.17)

$$\begin{aligned}
E = M &+ \left(e - \frac{1}{8}e^3 + \frac{1}{192}e^5 + \cdots\right)\sin M + \left(\frac{1}{2}e^2 - \frac{1}{6}e^4 + \frac{1}{48}e^6 + \cdots\right)\sin 2M \\
&+\left(\frac{3}{8}e^3 - \frac{27}{128}e^5 + \cdots\right)\sin 3M + \left(\frac{1}{3}e^4 - \frac{4}{15}e^6 + \cdots\right)\sin 4M \\
&+\left(\frac{125}{384}e^5 + \cdots\right)\sin 5M + \left(\frac{27}{80}e^6 + \cdots\right)\sin 6M + \cdots
\end{aligned} \tag{31}$$

Series reversion using Lagrange's Theorem and Maxima

Maxima is a fully functioned Computer Algebra System (CAS) and is a derivative of MACSYMA which had its origins in the 1960s at Massachusetts Institute of Technology (MIT). MACSYMA (Project MAC's SYmbolic MAnipulator and Project MAC was the Project on Mathematics And Computation) was the first of the 'modern' computer algebra systems and the forerunner of programs such as Maple and Mathematica. Its development grew out of research funded by the U.S. Department of Energy (DOE) and the source code (DOE MACSYMA) was maintained by William Schelter from 1982 until his death in 2001. In 1998 he obtained permission to release the Maxima source code under GNU[5] General Public License (GPL).

Maxima can be used in two modes; (i) typing simple input commands into the console screen that are acted on with the result as output printed to the console; or (ii) as a 'batch' file of instructions that are executed sequentially with output printed to the console. Batch files are the more useful way to use Maxima and the results shown in this paper have been generated from a Maxima text file 'Lagrange.txt' shown below, followed by a copy of the output screen.

[5] GNU is a recursive acronym for 'GNU's not Unix' chosen because GNU's design is Unix-like, but differs from Unix by being free software and containing no Unix code. GNU is a computer operating system developed by the GNU project aiming to be a complete Unix-compatible software system composed wholly of free software.

Maxima Batch File (Lagrange.txt) for trigonometric series for *E* to order e^{10}

```
/*****************************************************************/
/* Maxima program for Lagrange's Theorem                        */
/*                                                               */
/* path and file name:                                           */
/* D:\Projects\Geospatial\Geodesy\Satellite Orbits\Lagrange.txt  */
/*****************************************************************/

/* set the order to compute */
pow:10$
/* use Lagrange’s theorem to obtain a trigonometric series in M  */
E:M + e*sin(M)$
for k: 2 thru pow do
       (E: E + e^k/k!*diff(sin(M)^k,M,k-1))$
/* reduce the expression for E to a trigonometric series in multiple  */
/* values of M                                                        */
E : trigreduce(E)$
E : expand(E)$
/* group coefficients of of M and sin(kM)     */
coeff(E,M)*M + sum(coeff(E,sin(k*M))*sin(k*M),k,1,pow);
```

Maxima Console output of trigonometric series for *E* to order e^{10}

```
Maxima 5.24.0 http://maxima.sourceforge.net
using Lisp Clozure Common Lisp Version 1.7-dev-r14645M-trunk  (WindowsX8632)
Distributed under the GNU Public License. See the file COPYING.
Dedicated to the memory of William Schelter.
The function bug_report() provides bug reporting information.
(%i1)
read and interpret file: D:/Projects/Geospatial/Geodesy/Satellites Orbits/Lagrange.txt
(%i2)                                  pow : 10
(%i3)                              E : e sin(M) + M
                                             k         k
                                            e  diff(sin (M), M, k - 1)
(%i4)      for k from 2 thru pow do E : ------------------------- + E
                                                       k!
(%i5)                              E : trigreduce(E)
(%i6)                                E : expand(E)
(%i7)      sum(coeff(E, sin(k M)) sin(k M), k, 1, pow) + coeff(E, M) M
             10                      9                   8         10
      78125 e   sin(10 M)   531441 e  sin(9 M)    128 e    2048 e
(%o7) ------------------- + ------------------ + (------ - --------) sin(8 M)
            145152               1146880           315       2835
           7            9                      10         8        6
    16807 e    823543 e               2187 e     243 e    27 e
 + (-------- - ---------) sin(7 M) + (-------- - ------ + -----) sin(6 M)
     46080      1474560                 8960      560      80
           9          7         5
    78125 e    3125 e    125 e
 + (-------- - ------- + ------) sin(5 M)
     516096     9216      384
         10       8       6    4
      16 e     4 e    4 e    e
 + (- ------ + ---- - ---- + --) sin(4 M)
       945      45     15    3
           9         7        5       3
      243 e    243 e    27 e    3 e
 + (- ------ + ------ - ----- + ----) sin(3 M)
      40960     5120     128     8
      10     8     6    4    2
    e      e     e    e    e
 + (----- - --- + -- - -- + --) sin(2 M)
    17280   720   48   6    2
       9       7     5     3
      e       e     e     e
 + (------ - ---- + --- - -- + e) sin(M) + M
    737280   9216   192   8
(%i8)
```

Recurrence Relations

A recurrence relation is an equation that recursively defines a sequence. Once one or more initial terms are given each further term of the sequence is defined as a function of the preceding terms. As examples, consider the trigonometric functions

$$\sin k\phi = 2\cos\phi\sin\left(k-1\right)\phi - \sin\left(k-2\right)\phi \tag{32}$$

$$\cos k\phi = 2\cos\phi\cos\left(k-1\right)\phi - \cos\left(k-2\right)\phi \tag{33}$$

With initial values $\sin\left(0\right) = 0$, $\cos\left(0\right) = 1$ in (32) and (33) gives successively

$$\begin{aligned} \sin 2\phi &= 2\cos\phi\sin\phi, & \cos 2\phi &= 2\cos^2\phi - 1 \\ \sin 3\phi &= 2\cos\phi\sin 2\phi - \sin\phi, & \cos 3\phi &= 2\cos\phi\cos 2\phi - \cos\phi \\ \sin 4\phi &= 2\cos\phi\sin 3\phi - \sin 2\phi, & \cos 4\phi &= 2\cos\phi\cos 3\phi - \cos 2\phi \\ \sin 5\phi &= \cdots & \cos 5\phi &= \cdots \end{aligned}$$

Recurrence relations for even multiples are obtained by replacing ϕ with 2ϕ in (32) and (33) to give

$$\sin 2k\phi = 2\cos 2\phi\sin 2\left(k-1\right)\phi - \sin 2\left(k-2\right)\phi \tag{34}$$

$$\cos 2k\phi = 2\cos 2\phi\cos 2\left(k-1\right)\phi - \cos 2\left(k-2\right)\phi \tag{35}$$

Clenshaw summation

Suppose that a (truncated) sum S is denoted by

$$S = u_0F_0\left(x\right) + u_1F_1\left(x\right) + u_2F_2\left(x\right) + \cdots + u_NF_N\left(x\right) = \sum_{k=0}^{N} u_kF_k\left(x\right) \tag{36}$$

u_k are coefficients independent of *x,* and $F\left(x\right)$ obey the recurrence relation

$$F_{k+1}\left(x\right) = a_kF_k\left(x\right) + b_kF_{k-1}\left(x\right) \tag{37}$$

where the coefficients a_k, b_k may be functions of x as well as k. Note that in many applications a does not depend on k, and b is a constant independent of x or k.

The sum S can be evaluated from

$$S = b_1F_0\left(x\right)y_2 + F_1\left(x\right)y_1 + F_0\left(x\right)u_0 \tag{38}$$

where the quantities y_k are obtained from the backward (or reverse) recurrence formula

$$y_k = \begin{cases} 0, & \text{for } k > N \\ a_ky_{k+1} + b_{k+1}y_{k+2} + u_k, & \text{for } k = N, N-1, N-2, \ldots, 3, 2, 1 \end{cases} \tag{39}$$

Equation (39) is Clenshaw's recurrence formula and (38) is the associated sum; equations (38) and (39) combined are called *Clenshaw's summation* (Clenshaw 1955, Deakin & Hunter 2011).

Clenshaw's summation can be explained by writing out (36) as

$$\begin{aligned} S = {} & u_NF_N\left(x\right) + u_{N-1}F_{N-1}\left(x\right) + u_{N-2}F_{N-2}\left(x\right) + \cdots + u_8F_8\left(x\right) + u_7F_7\left(x\right) + u_6F_6\left(x\right) + \cdots \\ & + u_2F_2\left(x\right) + u_1F_1\left(x\right) + u_0F_0\left(x\right) \end{aligned} \tag{40}$$

and re-arranging (39) as

$$u_k = y_k - a_ky_{k+1} - b_ky_{k+2} \tag{41}$$

Then substituting (41) into (40) gives

$$
\begin{aligned}
S = & \left[y_N - a_N y_{N+1} - b_{N+1} y_{N+2}\right] F_N(x) + \left[y_{N-1} - a_{N-1} y_N - b_N y_{N+1}\right] F_{N-1}(x) \\
& + \left[y_{N-2} - a_{N-2} y_{N-1} - b_{N-1} y_N\right] F_{N-2}(x) + \cdots \\
& + \left[y_8 - a_8 y_9 - b_9 y_{10}\right] F_8(x) + \left[y_7 - a_7 y_8 - b_8 y_9\right] F_7(x) + \left[y_6 - a_6 y_7 - b_7 y_8\right] F_6(x) + \cdots \\
& + \left[y_2 - a_2 y_3 - b_3 y_4\right] F_2(x) + \left[y_1 - a_1 y_2 - b_2 y_3\right] F_1(x) + \left[u_0 + b_1 y_2 - b_1 y_2\right] F_0(x)
\end{aligned} \tag{42}
$$

Noting that in the last line $b_1 y_2$ has been added and subtracted. Examining the terms containing a factor of y_8 in (42) involves

$$\left[F_8(x) - a_7 F_7(x) - b_7 F_6(x)\right] y_8 \tag{43}$$

And as a consequence of the recurrence relation (37) the term in $[\ \ ]$ will equal zero and similarly for all other y_k down through and including y_2. The only surviving terms in (42) are u_0, y_1 and $b_1 y_2$; and so the sum S is given by (38).

Summation $S = \sum_{k=1}^{N} c_k \sin k\phi$

Consider the (truncated) trigonometric series

$$S = c_1 \sin\phi + c_2 \sin 2\phi + c_3 \sin 3\phi + \cdots + c_N \sin N\phi = \sum_{k=1}^{N} c_k \sin k\phi \tag{44}$$

The trigonometric functions $\sin\phi, \sin 2\phi, \sin 3\phi, \ldots$ obey the recurrence relation (32) so S can be evaluated using Clenshaw summation. Write the recurrence relation (32) in another form replacing k with $k+1$ giving

$$\sin(k+1)\phi = 2\cos\phi \sin k\phi - \sin(k-1)\phi \tag{45}$$

Equation (45) has the same form as (37) where $F_k(x) = \sin k\phi$, $a_k = 2\cos\phi$ and $b_k = -1$. Clenshaw's backward recurrence formula (39) becomes

$$y_k = \begin{cases} 0, & \text{for } k > N \\ 2\cos\phi\, y_{k+1} - y_{k+2} + c_k, & \text{for } k = N, N-1, N-2, \ldots, 3, 2, 1 \end{cases} \tag{46}$$

The associated sum (see equation (38) with $F_0(x) = \sin(0) = 0$ and $F_1(x) = \sin\phi$) is

$$S = \sum_{k=1}^{N} c_k \sin k\phi = y_1 \sin\phi \tag{47}$$

Matlab function Kepler_ series.m

```
function E = Kepler_series(e,M)
%
% Kepler_series is a function that solves Kepler's equation
% M = E - e*sin(E) for E using a trigonometric series to the order e^10 derived
% from Lagrange's Theorem.  M = mean anomaly, E = eccentric anomaly and
% e = orbit eccentricity.  Clenshaw summation is used to evaluate the series.
%
% example:  >> format long g
%           >> d2r = 180/pi;
%           >> M = 5/d2r;
%           >> e = 0.1;
%           >> E = Kepler_series(e,M);
%           >> E
%           E = 0.0969458710753345
%           >>

%------------------------------------------------------------------------------
% Function:  Kepler_series
%
% Usage:     E = Kepler_series(e,M);
%
% Author:    R.E.Deakin,
%            1/443 Station Street,
%            BONBEACH, VIC 3196, AUSTRALIA.
%            email: randm.deakin@gmail.com
%            Version  1.0   22 December 2017
%
% Functions required:
%    none
%
% Purpose:
%    Solution of Kepler's equation: M = E - e*sin(E) for E
%    where M = mean anomaly, E = eccentric anomaly and e = orbit eccentricity
%
% Variables:
%    A       - A = 2*cos(M) is a constant in Clenshaw recurrence
%    c()     - array of coefficients in the trigonometric series
%    e       - orbital eccentricity
%    e2,e3,...  - powers of eccentricity
%    E       - eccentric anomaly (radians)
%    k       - integer counter
%    M       - mean anomaly (radians)
%    N       - order of series
%    y1,y2   - Clenshaw recurrence variables
%
% Remarks:
%    The solution of Kepler's equation for the eccentric anomaly E is given in
%    the form of a trigonometric series:
%    E = M + c(1)*sin(M) + c(2)*sin(2*M) + ... + c(N)*sin(N*M)
%    where the coefficients c(1), c(2), ..., c(N) are functions of powers of
%    the eccentricity e.
%    Clenshaw summation is used to evaluate the series.
%    The series is accurate for values of eccentricity e < 0.2.
%    For values of e > 0.2, Newton-Raphson, or another iterative scheme should
%    be used.
%
% References:
%    [1] Battin, R.H., (1987), 'An Introduction to the Mathematics and Methods
%           of Astrodynamics', American Institute of Aeronautics and
%           Astronautics (AIAA), Inc., New York.
%    [2] Deakin, R.E., (2017), 'Solutions of Kepler's Equation', Private Notes,
%           December, 2017.
%------------------------------------------------------------------------------
```

```
% calculate the value of the eccentric anomaly E using the trigonometric
% series expansion given in Ref[1], eq.(5.16), p.202 and Ref[2].

% set powers of eccentricity
e2 = e*e;
e3 = e2*e;
e4 = e3*e;
e5 = e4*e;
e6 = e5*e;
e7 = e6*e;
e8 = e7*e;
e9 = e8*e;
e10 = e9*e;

% set an array c() for the coefficients c(1),c(2),...,c(10)
N = 10;
c = zeros(N,1);

% set the values of the coefficients c1, c2, ... c10  Ref[2], eq.(9)
c(1) = (e-1/8*e3+1/192*e5-1/9216*e7+1/737280*e9);
c(2) = (1/2*e2-1/6*e4+1/48*e6-1/720*e8+1/17280*e10);
c(3) = (3/8*e3-27/128*e5+243/5120*e7-243/40960*e9);
c(4) = (1/3*e4-4/15*e6+4/45*e8-16/945*e10);
c(5) = (125/384*e5-3125/9216*e7+78125/516096*e9);
c(6) = (27/80*e6-243/560*e8+2187/8960*e10);
c(7) = (16807/46080*e7-823543/1474560*e9);
c(8) = (128/315*e8-2048/2835*e10);
c(9) = (531441/1146880*e9);
c(10) = (78125/145152*e10);

% set up y1 and y2 for Clenshaw's backward recurrence
y2 = 0;
y1 = 0;

% calculate y1 from Clenshaw's backward recurrence
A = 2*cos(M);
for k = N:-2:1
    y2 = A*y1-y2+c(k);
    y1 = A*y2-y1+c(k-1);
end

% calculate the eccentric anomaly
E = M + y1*sin(M);
return
```

Matlab function Kepler_ Newton.m

```
function [E,count] = Kepler_Newton(e,M)
%
% Kepler_Newton is a function that solves Kepler's equation
% M = E - e*sin(E) for E
% where M = mean anomaly, E = eccentric anomaly and e = orbit eccentricity
% count is the n umber of iterations
%
% example:  >> format long g
%           >> d2r = 180/pi;
%           >> M = 5/d2r;
%           >> e = 0.1;
%           >> [E,count] = Kepler_Newton(e,M);
%           >> E
%           E = 0.0969458710759671     % radians
%           >> count
%           count = 3                  % number of iterations

%--------------------------------------------------------------------------
% Function:  Kepler_Newton
%
% Usage:     [E,count] = Kepler_Newton(e,M);
%
% Author:    R.E.Deakin,
%            1/443 Station Street,
%            BONBEACH, VIC 3196, AUSTRALIA.
%            email: randm.deakin@gmail.com
%            Version  1.0   01 December 2017
%
% Functions required:
%    None
%
% Purpose:
%    Solution of Kepler's equation M = E - e*sin(E) for E
%    where M = mean anomaly, E = eccentric anomaly and e = orbit eccentricity
%
% Variables:
%    corrn   - corrn = F/df
%    count   - number of iterations
%    dF      - derivative of F, dF = 1 - e*cos(Psi)
%    e       - orbital eccentricity
%    E       - eccentric anomaly (radians)
%    F       - function F = E - e*sin(E) - M
%    M       - mean anomaly (radians)
%    tol     - tolerance (a small value)
%
% Remarks:
%    This function uses Newton-Raphson Iteration to solve Kepler's equation for
%    the eccentric anomaly E.  A starting value for the iterative process is
%    obtained from eq. (5) of Ref. [1]
%
% References:
%    [1] Smith, G.R., (1979), 'A simple efficient starting value for the
%           iterative solution of kepler's equation', Celestial Mechanics, Vol.
%           19, pp. 163-166.
%    [2] Odell, A.W. & Gooding, R.H., (1986), 'Procedures for solving Kepler's
%           equation', Celestial Mechanics, Vol. 38, pp. 307-334.
%    [3] Deakin, R.E., (2017), 'Solutions of Kepler's Equation', Private Notes,
%           December, 2017.
%--------------------------------------------------------------------------

% Compute the initial value of E
E = M + e*(sin(M)/(1-sin(M+e)+sin(M)));  % eq. (5) of ref [1]

% set starting values for corrn and count and the tolerance
corrn = 1;
```

```
count = 0;
tol = 1e-15;

% Newton-Raphson iteration
while (abs(corrn) > tol)
  F  = E - e*sin(E) - M;        % function F(M) = E - e*sin(E) - M = 0
  dF = 1.0 - e*cos(E);          % derivative of F(M)
  if(abs(dF) < tol)
    fprintf('\n*** derivative 1 - e*cos(E) = 0 *** \n\n');
    break;
  endif
  corrn = F/dF;
  E = E - corrn;
  count = count+1;
  if(count>50)
    fprintf('\n*** no convergence after 50 iterations ***\n\n');
    break;
  endif
end
```

Matlab function Kepler_Bisection.m

```
function [E,count] = Kepler_Bisection(e,M)
%
% Kepler_Bisection is a function that solves Kepler's equation
% M = E - e*sin(E) for E
% where M = mean anomaly, E = eccentric anomaly and e = orbit eccentricity
% count is the n umber of iterations
%
% example:  >> format long g
%           >> d2r = 180/pi;
%           >> M = 5/d2r;
%           >> e = 0.1;
%           >> [E,count] = Kepler_Bisection(e,M);
%           >> E
%           E = 0.0969458710759658      % radians
%           >> count
%           count = 46                  % number of iterations

%--------------------------------------------------------------------------
% Function:  Kepler_Bisection
%
% Usage:     [E,count] = Kepler_Bisection(e,M);
%
% Author:    R.E.Deakin,
%            1/443 Station Street,
%            BONBEACH, VIC 3196, AUSTRALIA.
%            email: randm.deakin@gmail.com
%            Version  1.0   01 December 2017
%
% Functions required:
%    None
%
% Purpose:
%    Solution of Kepler's equation M = E - e*sin(E) for E
%    where M = mean anomaly, E = eccentric anomaly and e = orbit eccentricity
%
% Variables:
%    count   - number of iterations
%    d       - length of interval
%    F       - F = sign(M)  F = -1 if M < 0; F = 1 if M > 0; F = 0 if M = 0
%    e       - orbital eccentricity
%    E       - eccentric anomaly (radians)
%    M       - mean anomaly (radians)
```

```
%    M_new   - new value of M
%    tol     - tolerance (a small value)
%    twopi   - 2*pi
%
% Remarks:
%    This function uses a Bisection iterative scheme to solve Kepler's equation
%    for the eccentric anomaly E.
%    The (unknown) eccentric anomaly is bounded M <= E <= (M+e) and the initial
%    value for the iteration scheme is M+(e/2) and the initial interval width is
%    d = e.
%
% References:
%    [1] Meeus, J., (1991), 'Astronomical Algorithms', Willmann-Bell, Inc.,
%           Richmond, Virginia, USA.
%    [2] Deakin, R.E., (2017), 'Solutions of Kepler's Equation', Private Notes,
%           December, 2017.
%--------------------------------------------------------------------------

twopi = 2.0*pi;

% Reduce M to a value in the range -twopi < M < twopi
F = sign(M);
M = abs(M)/twopi;
M = (M-fix(M))*twopi*F;
% Make M a value in the range  0 < M < twopi
if M < 0
  M = M + twopi;
endif
% determine the sign of reduced M
F = 1;
if M > pi
  M = twopi - M;
  F = -1;
endif
% set starting values for the iterative process
% E is bounded such that M <= E <= (M+e) so the mid-point is M+(e/2).  This is
% the initial value for E.
E = M+(e/2);
d = e/2;
% set the tolerance and iteration count
tol = 1.0E-15;
count = 0;
while (d > tol)
  M_new = E - e*sin(E);
  E     = E + sign(M-M_new)*d;
  d = d/2;
  count = count+1;
  if(count > 60)
   fprintf('\n*** no convergence after 60 iterations ***');
   break;
  endif
end
E = F*E;
```

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