Figure 1 shows a satellite $S$ is in an elliptical orbit of period $T$ about the earth $F$ where $T$ is the time between two successive passages through perigee $P$. The orbital ellipse has semi-axes $a$ and $b$ ($a > b$) and eccentricity $e = \sqrt{(a^2 - b^2)/a^2}$ and the earth is at one of the focal points with $OF = ae$. The orbital ellipse has an auxiliary circle of radius $a$ and $S$ is located on the ellipse by the orbital radius $r$ and the \textit{true anomaly} $\theta$ and $Q$ is located on the auxiliary circle by the radius $a$ and the \textit{eccentric anomaly} $E$. The line $QR$ is perpendicular to the major axis ($2a$) and passes through $S$. As $S$ moves in its elliptical orbit, a fictitious satellite $S'$, located on the auxiliary circle by the radius $a$ and the \textit{mean anomaly} $M$, moves around the auxiliary circle with constant angular velocity and with a period of revolution identical to the orbital period $T$. When the satellite $S$ is at \textit{perigee} $P$, anomalies $\theta$, $E$ and $M$ all equal zero.

The following relationships are fundamental in orbital mechanics

$M$, $E$ and $e$ are related by Kepler’s equation which is an outcome of Kepler’s 2nd law (Deakin 2007)

$$M = E - e \sin E$$  \hspace{1cm} (1)

$E$, $\theta$ and $e$ are related by

$$\tan E = \frac{\sqrt{1 - e^2} \sin \theta}{\cos \theta + e} \quad \text{or} \quad \tan \theta = \frac{\sqrt{1 - e^2} \sin E}{\cos E - e}$$  \hspace{1cm} (2)

e, r and $\theta$ are related by
\[ r = \frac{a \left(1 - e^2\right)}{1 + e \cos \theta} \]  

(3)

The orbital period \( T \) (in seconds) of the satellite is known to be (Deakin, 2007, p. 27)

\[ T = 2\pi \sqrt{\frac{a^3}{GM}} = \frac{2\pi}{n} \]  

(4)

where \( GM = 3986005 \times 10^8 \text{ m}^3/\text{s}^2 \) is the geocentric gravitational constant that is the product of the universal constant of gravitation \( G \) and the mass of the earth \( M \) (GRS80) and \( n \) is the mean motion of the satellite, defined as

\[ n = \sqrt{\frac{GM}{a^3}} \]  

(5)

If \( t_0 \) is the time (in seconds) when the satellite passes through perigee and \( t \) is some time after \( t_0 \) then \( t - t_0 \) is the time of flight and the mean anomaly \( M \) can be expressed as

\[ M = n(t - t_0) \]  

(6)

and Kepler’s equation written as

\[ M = n(t - t_0) = E - e \sin E \]  

(7)

Figure 1. Mean anomaly \( M \) versus Eccentric anomaly \( E \) for \( e = 0, 0.5, 1 \)

In orbital mechanics it is often required to compute the position of a satellite at time \( t \) when the mean motion \( n \) and \( t_0 \) are known quantities as well as orbital constants \( a \) and \( e \). To achieve this, Kepler’s equation must be solved for \( E \) and there are a number of different methods to solve Kepler’s equation.

Of particular interest are three methods, (i) a series expansion, (ii) Newton-Raphson iteration and (iii) an iterative scheme using the bisection method.
Solution of Kepler’s equation by trigonometric series

To solve Kepler’s equation for the eccentric anomaly \( E \) equation (1) is recast as

\[
E = M + e \sin E
\]

and Lagrange’s theorem used to derive an expression for \( E \) as a trigonometric series in \( M \)

\[
E = M + \left( e - \frac{1}{8}e^3 + \frac{1}{192}e^5 + \cdots \right) \sin M + \left( \frac{1}{2}e^2 - \frac{1}{6}e^4 + \frac{1}{48}e^6 + \cdots \right) \sin 2M
\]

\[
+ \frac{3}{8}e^3 - \frac{27}{128}e^5 + \cdots \sin 3M + \left( \frac{1}{3}e^4 - \frac{4}{15}e^6 + \cdots \right) \sin 4M
\]

\[
+ \frac{125}{384}e^5 + \cdots \sin 5M + \left( \frac{27}{80}e^6 + \cdots \right) \sin 6M + \cdots
\]

(8)

This equation, given by Battin (1987, eq. 5.17), ignores terms with coefficients \( e^7 \) and greater. It is a recasting of an earlier equation (also derived using Lagrange’s theorem) given by Moulton (1914, p. 169). An explanation of Lagrange’s theorem and the derivation of (8) is given in the Appendix.

With the aid of the computer algebra package Maxima\(^2\) this series can be extended to higher orders of the eccentricity \( e \) and multiples of the mean anomaly \( M \) (see Appendix).

\[
E = M + \left( e - \frac{1}{8}e^3 + \frac{1}{192}e^5 - \frac{1}{9216}e^7 + \frac{1}{737280}e^9 - \cdots \right) \sin M
\]

\[
+ \frac{1}{2}e^2 - \frac{1}{6}e^4 + \frac{1}{48}e^6 - \frac{1}{720}e^8 + \frac{1}{17280}e^{10} - \cdots \sin 2M
\]

\[
+ \frac{3}{8}e^3 - \frac{27}{128}e^5 + \frac{243}{5120}e^7 - \frac{243}{40960}e^9 + \cdots \sin 3M
\]

\[
+ \frac{1}{3}e^4 - \frac{4}{15}e^6 + \frac{4}{45}e^8 - \frac{16}{945}e^{10} + \cdots \sin 4M
\]

\[
+ \frac{125}{384}e^5 - \frac{3125}{9216}e^7 + \frac{78125}{516096}e^9 - \cdots \sin 5M + \left( \frac{27}{80}e^6 - \frac{243}{560}e^8 + \frac{2187}{8960}e^{10} - \cdots \right) \sin 6M
\]

\[
+ \frac{16807}{40680}e^7 - \frac{823543}{2097152}e^9 + \cdots \sin 7M + \left( \frac{128}{315}e^8 - \frac{2048}{2835}e^{10} + \cdots \right) \sin 8M
\]

\[
+ \frac{531441}{1146880}e^9 - \cdots \sin 9M + \left( \frac{78125}{145152}e^{10} - \cdots \right) \sin 10M + \cdots
\]

(9)

Colwell (1993, Appendix D) uses Lagrange’s Theorem and the computer algebra system Mathematica to obtain a series for \( E \) where the coefficients of \( e, e^2, e^3, \ldots, e^{10} \) are functions of powers of sines and cosines of \( M \).

For orbits with small eccentricities \( e < 0.2 \) the series (8) and (9) are rapidly convergent, but it is known that this series will diverge for some values of \( M \) when the eccentricity \( e > 0.6627434194 \)… (Battin 1987, p.205). This result, first shown by Laplace (1749–1827), is known as the Laplace Limit.

Efficient evaluation of the trigonometric series for \( E \)

The trigonometric series (8) and (9) can be expressed in the form

\[
E = M + \sum_{k=1}^{N} c_k \sin kM
\]


\(^2\) Maxima is based on a 1982 version of MACSYMA, which was developed at MIT with funding from the United States Department of Energy and other government agencies. See Appendix.
and for $N = 6$ the six coefficients $c_1, c_2, \ldots, c_6$ from (8) are

$$
c_1 = e - \frac{1}{8} e^3 + \frac{1}{192} e^5 + \cdots,
\quad c_2 = \frac{1}{2} e^2 - \frac{1}{6} e^4 + \frac{1}{48} e^6 + \cdots,
\quad c_3 = \frac{3}{8} e^3 - \frac{27}{128} e^5 + \cdots,
\quad c_4 = \frac{1}{3} e^4 - \frac{4}{15} e^6 + \cdots,
\quad c_5 = \frac{125}{384} e^6 + \cdots,
\quad c_6 = \frac{27}{80} e^6 + \cdots
$$

(11)

The summation $S = \sum_{k=1}^{N} c_k \sin kM$ in (10) can be evaluated using Clenshaw summation (see Appendix) that avoids multiple evaluations if the sine function and (10) can be written as

$$
E = M + \sum_{k=1}^{N} c_k \sin kM = M + S = M + y_1 \sin M
$$

(12)

where $y_1$ is obtained from the backward recurrence formula

$$
y_k = \begin{cases} 
0, & \text{for } k > N \\
2 \cos M y_{k+1} - y_{k+2} + c_k, & \text{for } k = N, N - 1, N - 2, \ldots, 3, 2, 1 
\end{cases}
$$

(13)

Using Clenshaw’s summation means that there is one evaluation of $\sin M$ and one evaluation of $\cos M$ in determining the eccentric anomaly $E$.

**Example 1**

Use Clenshaw summation to evaluate $E$ from (8) with $e = 0.100$ and $M = 5^\circ = 0.087266463$ radians.

The coefficients $c_k$ from (11) are

<table>
<thead>
<tr>
<th>$e$</th>
<th>$k$</th>
<th>$e^k$</th>
<th>$c_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1</td>
<td>0.100000000000</td>
<td>9.987505208E-02</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0.010000000000</td>
<td>4.983354167E-03</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0.001000000000</td>
<td>3.728906250E-04</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>0.000100000000</td>
<td>3.306666667E-05</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>0.000010000000</td>
<td>3.255208333E-06</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>0.000001000000</td>
<td>3.375000000E-07</td>
</tr>
</tbody>
</table>

Clenshaw recurrence

<table>
<thead>
<tr>
<th>$k$</th>
<th>$y_k$</th>
<th>$2 \cos M = 1.992389396$</th>
<th>$\sin M = 0.087155743$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>0</td>
<td>$c_k$</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.000000337500</td>
<td>3.375000000E-07</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.00003927640</td>
<td>3.255208333E-06</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.00040554554</td>
<td>3.306666667E-05</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.000449763450</td>
<td>3.728906250E-04</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.005838903540</td>
<td>4.983354167E-03</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.111058658132</td>
<td>9.987505208E-02</td>
<td></td>
</tr>
</tbody>
</table>

$$
E = M + \sum_{k=1}^{N=6} c_k \sin kM = M + S = M + y_1 \sin M = \begin{cases} 
0.096945862438 \text{ radians} \\
5.545588758940 \text{ degrees} 
\end{cases}
$$

A Matlab function `Kepler_series.m` is shown in the Appendix.

---

3 Clenshaw summation evaluates a sum of products of indexed coefficients by functions which obey a recurrence relation.
Solution of Kepler’s equation by Newton-Raphson iteration

As an alternative to the trigonometric series method, a value for $E$ can be computed using the Newton-Raphson method for the real roots of the equation $f(E) = 0$ given in the form of an iterative equation

\[ E_{n+1} = E_n - \frac{f(E_n)}{f'(E_n)} \quad (14) \]

where $n$ denotes the $n^{th}$ iteration and $f(E)$ is found from Kepler’s equation (1) as

\[ f(E) = E - e \sin E - M \quad (15) \]

The derivative $f'(E) = \frac{d}{dE} \{ f(E) \} \) is given by

\[ f'(E) = 1 - e \cos E \quad (16) \]

Substituting (15) and (16) into (14) gives

\[ E_{n+1} = E_n - \frac{M_n - M}{1 - e \cos E_n} \quad (17) \]

where the second term is a correction to $E_n$ and $M_n = E_n - e \sin E_n$

With an initial value $E_0$, the correction $(M_0 - M)/(1 - e \cos E_0)$ is evaluated and subtracted from $E_0$ giving an updated value $E_1$ and the process is repeated to obtain $E_2, E_3, \ldots$. This iterative process can be concluded when the difference between $E_{n+1}$ and $E_n$ reaches an acceptably small value.

Initial value $E_0$

It is common practice to use $E_0 = M$ as the initial (or starting value) in the iterative scheme, but it should be noted that this will not always lead to convergence.

Example 2

For values $M = 7^\circ$, $e = 0.999$ and with $E_0 = M$ as the initial value, the following results are obtained for the first 14 iterations

<table>
<thead>
<tr>
<th>$n$</th>
<th>$E_n$ (degrees)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>7.0000000000</td>
</tr>
<tr>
<td>1</td>
<td>832.869123399</td>
</tr>
<tr>
<td>2</td>
<td>275.954960202</td>
</tr>
<tr>
<td>3</td>
<td>-87.610599131</td>
</tr>
<tr>
<td>4</td>
<td>-48.562394340</td>
</tr>
<tr>
<td>5</td>
<td>-11.225112021</td>
</tr>
<tr>
<td>6</td>
<td>340.962526137</td>
</tr>
<tr>
<td>7</td>
<td>340.962526137</td>
</tr>
<tr>
<td>8</td>
<td>-5996.81219845</td>
</tr>
<tr>
<td>9</td>
<td>-2084.497865298</td>
</tr>
<tr>
<td>10</td>
<td>-778.410987047</td>
</tr>
<tr>
<td>11</td>
<td>-737.535684055</td>
</tr>
<tr>
<td>12</td>
<td>14598.35040127</td>
</tr>
<tr>
<td>13</td>
<td>7099.442370278</td>
</tr>
<tr>
<td>14</td>
<td>1056.785610878</td>
</tr>
</tbody>
</table>

This example of non-convergence is shown in Meeus (1991, p. 189) with the comment that convergence to the true value $(E = 52.270261528$ degrees) did not occur until after the 47th iteration.
A faster convergence can be achieved with a better initial value $E_0$. The selection of initial values for iteration schemes has been the subject of many papers and Odell & Gooding (1986), Meeus (1991), Colwell (1993) and Esmaelzadeh & Ghadiri (2014) have summaries of modern iterative solutions and initial values. Here we use an equation for $E_0$ from Smith (1979), who in his development, included the following points relating to the solution for the root of $f(E) = E - e \sin E - M$:

(i) Due to symmetry (see Figure 1) it is only necessary to consider cases where $0 \leq M \leq \pi$

(ii) $f(M) = -e \sin M$ is negative or zero; $f(M + e) = e(1 - \sin(M + e))$ is positive or zero and the derivative $f'(E) = 1 - e \cos E$ is positive which means that $f(E)$ must vanish somewhere within the interval $(M, M + e)$. Thus the solution for $f(E) = 0$ is bounded and $M \leq E \leq M + e$ (18)

Smith (1979) then used these bounds in the equation for a straight line $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$ to obtain

$$\frac{0 - f(M)}{E_0 - M} = f(M + e) - f(M)$$

$$(M + e) - M$$

which is simplified and re-arranged as (Smith 1979, eq. 5)

$$E_0 = M + \frac{e \sin M}{1 - \sin (M + e) + \sin M}$$

(19)

Smith (1979) tested this initial value in solutions for $E$ in Regions I and II that are known to be problematic in iterative solutions (Meeus 1991, Chapter 29, Figures 4, 5 and 6)

Region I: $0.05 \leq M \leq \pi$ and $0.01 \leq e \leq 0.99$

Region II: $0.005 \leq M \leq 0.4$ and $0.95 \leq e \leq 0.999$

**Example 3**

Using (19) with values $M = 7^\circ$, $e = 0.999$ gives $E_0 = 38.527006574$ degrees and the correct result of $E = 52.270261528$ degrees is obtained after 5 iterations.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$E_n$ (degrees)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>38.527006574</td>
</tr>
<tr>
<td>1</td>
<td>57.412628477</td>
</tr>
<tr>
<td>2</td>
<td>52.682423402</td>
</tr>
<tr>
<td>3</td>
<td>52.273242571</td>
</tr>
<tr>
<td>4</td>
<td>52.270261686</td>
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<tr>
<td>5</td>
<td>52.270261528</td>
</tr>
<tr>
<td>6</td>
<td>52.270261528</td>
</tr>
</tbody>
</table>

Odell & Gooding (1986) note that (19) will not produce Newton-Raphson convergence for values of $e$ greater than about 0.9995. The source of the problem, they say, is that if $0 < E_0 < E$ with $0 < M < E < \pi$ then $E_1$ (the estimate of $E$ after one iteration) will exceed $E$ (the true value) and sometimes will be much bigger than $\pi$. This is clearly the case with the example above (even with $e = 0.999$ being less than 0.9995) and this does not satisfy their requirements for convergence where every iteration must produce a value closer to the true value. But the iterative scheme does converge on the true solution after 5 iterations. Odell & Gooding (1986) are focussed on schemes of one or two iterations only to maximise computer efficiency.
Example 4

Interestingly using (19) with values $M = 7^\circ$, $e = 1$ gives $E_0 = 38.620614337$ degrees and the correct result of $E = 52.386793829$ degrees is obtained after 5 iterations.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$E_n$ (degrees)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>38.620614337</td>
</tr>
<tr>
<td>1</td>
<td>57.55617286</td>
</tr>
<tr>
<td>2</td>
<td>52.802877860</td>
</tr>
<tr>
<td>3</td>
<td>52.38931537</td>
</tr>
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<td>4</td>
<td>52.386793993</td>
</tr>
<tr>
<td>5</td>
<td>52.386793829</td>
</tr>
<tr>
<td>6</td>
<td>52.386793829</td>
</tr>
</tbody>
</table>

A Matlab function *Kepler_Newton.m* is shown in the Appendix.

**Solution of Kepler’s equation using the Bisection Method**

The Bisection Method is a numerical method for the real roots of $f(x) = 0$. The method is always convergent and is simple to implement, but it is relatively slow as its rate of convergence is linear.

It is based on a result from calculus known as the Intermediate Value Theorem or Bolzano’s Theorem⁴ that can be expressed as (Apostol 1967):

Let $f(x)$ be a continuous function at each point of a closed interval $[a,b]$ and assume that $f(a)$ and $f(b)$ have opposite signs, then there is at least one $c$ in the open interval $(a,b)$ such that $f(c) = 0$.

And $c$ is a root of $f(x) = 0$ bounded by the open interval $(a,b)$.

The Bisection Method can be understood by following Dahlquist & Björck (1974):

Suppose that $f(x)$ is continuous and two points $a_0, b_0$ are found at which the function values $f(a_0)$ and $f(b_0)$ have opposite signs, i.e. $f(a_0)f(b_0) < 0$.

A sequence of intervals $I_1 = (a_1, b_1) > I_2 (a_2, b_2) > I_3 (a_3, b_3) ...$, can be determined which all contain a root of $f(x) = 0$ and each interval is half the size of the previous interval. This sequence can be terminated when the interval bounds $(a_k, b_k)$ become sufficiently close.

The intervals $I_k = (a_k, b_k)$, $k = 1, 2, 3, ...$ are determined recursively as follows:

The mid-point of the interval $I_{k-1}$ is

$$c_k = \frac{1}{2}(a_{k-1} + b_{k-1})$$

Evaluate $f(c_k)$ and determine the bounds of the next interval $I_k$ according to the rule

---

⁴ Bernard Bolzano (1781–1848), a Catholic priest who made many important contributions to mathematics in the first half of the 19th century, was one of the first to recognize that many ‘obvious’ statements about continuous functions require proof. His observations concerning continuity were published posthumously in 1850 in *Paradoxien des Unendlichen* [Paradoxes of the infinite]. An English translation of Bolzano’s paper on the Intermediate Value Theorem (Bolzano 1817) is given by Russ (1980).
\[(a_k, b_k) = \begin{cases} (c_{k-1}, b_{k-1}) & \text{if } f(c_k) < 0 \\ (a_{k-1}, c_k) & \text{if } f(c_k) > 0 \end{cases} \] (21)

From the construction of \((a_k, b_k)\) it follows immediately that \(f(a_k) < 0\) and \(f(b_k) > 0\) and that each interval \(I_k\) of length \(d_k = |b_k - a_k|\) contains a root of \(f(x) = 0\). After \(n\) steps the root is contained in the interval of length \(d_n\) where

\[d_n = \frac{1}{2} d_{n-1} = \left(\frac{1}{2}\right)^2 d_{n-2} = \left(\frac{1}{2}\right)^3 d_{n-3} = \cdots = \left(\frac{1}{2}\right)^n d_0\] (22)

and \(d_0 = |b_0 - a_0|\) is the initial interval length. A re-arrangement of (22) gives

\[2^n = \frac{d_0}{d_n}\] (23)

If \(\varepsilon\) is some desired tolerance then the integer number of iterations \(n\) to achieve this tolerance is

\[n = \text{ceil} \left[ \log_2 \frac{d_0}{\varepsilon} \right] = \text{ceil} \left[ \frac{1}{\ln 2} \ln \frac{d_0}{\varepsilon} \right] = \text{ceil} \left[ 1.443 \ln \frac{d_0}{\varepsilon} \right]\] (24)

where \(\text{ceil}(x)\) is the ceiling function and \(\log_2(x)\) is the binary logarithm (logarithm to the base 2). The ceiling function rounds a fractional number to the next highest integer and binary logarithms can be evaluated from the natural logarithm \(\ln x \equiv \log_e x\) using the rule: \(\log_2 x = \frac{\ln x}{\ln 2} \approx 1.443 \ln x\).

\(e \approx 2.718281828459\) is the base of the natural logarithms.

For example, if \(d_0 = \frac{1}{2} \pi\) and the tolerance \(\varepsilon = 1.0 \times 10^{-15}\) then \(n = \text{ceil}(50.491) = 51\).

The Bisection method must succeed. If the interval happens to contain two or more roots, bisection will find one of them. If the interval contains no roots and merely straddles a singularity, it will converge on the singularity (Press et al. 1992).

The Appendix contains a Matlab function \texttt{Kepler\_Bisection.m} that is based on a BASIC program shown in \textit{Sky & Telescope} August 1985 by Roger W Sinnott (1985). Sinnott calls his method ‘Binary Search’ and Meeus (1991) features Sinnott’s binary search as a foolproof method of solving Kepler’s equation for \(0 < e \leq 1\).
Appendix

Reversion of a series

If we have an expression for a variable $z$ as a series of powers or functions of another variable $y$ then we may, by a reversion of the series, find an expression for $y$ as series of functions of $z$. Reversion of a series can be done using Lagrange’s theorem, a proof of which can be found in Battin (1987).

Suppose that

$$ y = z + xF(y) $$

then Lagrange’s theorem states that for any $f$

$$ f(y) = f(z) + \frac{x}{1!} F(z) f'(z) $$

$$ + \frac{x^2}{2!} d \left[ \left( F(z) \right)^2 f'(z) \right] $$

$$ + \frac{x^3}{3!} d^2 \left[ (F(z))^3 f'(z) \right] $$

$$ + \cdots $$

$$ + \frac{x^n}{n!} d^{n-1} \left[ (F(z))^n f'(z) \right] $$

As an example, consider Kepler’s equation expressed in the form

$$ \sin E = M + e \sin E $$

and we wish to find an expression for $E$ as a function of $M$.

Comparing the variables in equations (27) and (25), $z = M, y = E, x = e$ and $F(z) = F(M) = \sin M$.

And if we choose $f(y) = y$ then $f(z) = z$ and $f'(z) = 1$ then Lagrange’s theorem gives

$$ E = M + e \sin M + \frac{e^2}{2! dM} \left[ \sin^2 M \right] + \frac{e^3}{3! dM^2} \left[ \sin^3 M \right] + \frac{e^4}{4! dM^3} \left[ \sin^4 M \right] + \cdots $$

$$ + \frac{e^n}{n! dM^{n-1}} \left[ \sin^n M \right] $$

This can also be expressed as (Colwell 1993)

$$ E = M + \sum_{n=1}^{\infty} a_n(M) e^n, \quad a_n(M) = \frac{1}{n!} \frac{d^{n-1}}{dM^{n-1}} \left( \sin^n M \right) $$

Neglecting terms of order $e^7$ and higher, the derivatives up to the 5th order are

$$ \frac{d}{dM} \sin^2 M = 2 \cos M \sin M = \sin 2M $$

$$ \frac{d^2}{dM^2} \sin^3 M = 6 \cos^2 M \sin M - 3 \sin^3 M = \frac{1}{4} (9 \sin 3M - 3 \sin M) $$

$$ = \frac{1}{2^2} \left( 3^3 \sin 3M - 3 \sin M \right) $$

$$ \frac{d^3}{dM^3} \sin^4 M = 24 \cos^3 M \sin M - 40 \cos M \sin^3 M = 8 \sin 4M - 4 \sin 2M $$

$$ = \frac{1}{2^3} \left( 4^3 \sin 4M - 2^3 \cdot 4 \sin 2M \right) $$
\[
\begin{align*}
\frac{d^4}{dM^4} \sin^5 M &= 120 \cos^4 M \sin M - 440 \cos^2 M \sin^3 M + 65 \sin^5 M \\
&= \frac{1}{16} \left( 625 \sin 5M - 405 \sin 3M + 10 \sin M \right) \\
&= \frac{1}{24} \left( 5^4 \sin 5M - 3^4 \cdot 5 \sin 3M + 10 \sin M \right) \\
\frac{d^5}{dM^5} \sin^6 M &= 720 \cos^5 M \sin M - 4800 \cos^3 M \sin^3 M + 2256 \cos M \sin^5 M \\
&= 243 \sin 6M - 192 \sin 4M + 15 \sin 2M \\
&= \frac{1}{25} \left( 6^5 \sin 6M - 4^5 \cdot 6 \sin 4M + 2^5 \cdot 15 \sin 2M \right)
\end{align*}
\]

Inserting the derivatives into (28) gives a series expression for \( E \) (Moulton 1914, p. 169)

\[
E = M + e \sin M + \frac{e^2}{2!} \sin 2M + \frac{e^3}{3!} \left( 3^2 \sin 3M - 3 \sin M \right) + \frac{e^4}{4!} \left( 4^2 \sin 4M - 2^2 \cdot 4 \sin 2M \right) \\
+ \frac{e^5}{5!} \left( 5^4 \sin 5M - 3^4 \cdot 5 \sin 3M + 10 \sin M \right) + \frac{e^6}{6!} \left( 6^5 \sin 6M - 4^5 \cdot 6 \sin 4M + 2^5 \cdot 15 \sin 2M \right) + \cdots
\]

that simplifies to

\[
E = M + e \sin M + \frac{e^2}{2} \sin 2M + \frac{e^3}{8} \left( 3 \sin 3M - \sin M \right) + \frac{e^4}{6} \left( 2 \sin 4M - \sin 2M \right) \\
+ \frac{e^5}{384} \left( 125 \sin 5M - 81 \sin 3M + 2 \sin M \right) + \frac{e^6}{720} \left( 243 \sin 6M - 192 \sin 4M + 15 \sin 2M \right) + \cdots \quad (30)
\]

This can be re-arranged as (Battin 1987, eq. 5.17)

\[
E = M + \left( e - \frac{1}{8} e^3 + \frac{1}{192} e^5 + \cdots \right) \sin M + \left( \frac{1}{2} e^2 - \frac{1}{6} e^4 + \frac{1}{48} e^6 + \cdots \right) \sin 2M \\
+ \left( \frac{3}{8} e^3 - \frac{27}{128} e^5 + \cdots \right) \sin 3M + \left( \frac{1}{15} e^4 - \frac{4}{15} e^6 + \cdots \right) \sin 4M \\
+ \left( \frac{125}{384} e^5 + \cdots \right) \sin 5M + \left( \frac{27}{80} e^6 + \cdots \right) \sin 6M + \cdots \quad (31)
\]

**Series reversion using Lagrange’s Theorem and Maxima**

Maxima is a fully functioned Computer Algebra System (CAS) and is a derivative of MACSYMA which had its origins in the 1960s at Massachusetts Institute of Technology (MIT). MACSYMA (Project MAC’s SYmbolic MANipulator and Project MAC was the Project on Mathematics And Computation) was the first of the ‘modern’ computer algebra systems and the forerunner of programs such as Maple and Mathematica. Its development grew out of research funded by the U.S. Department of Energy (DOE) and the source code (DOE MACSYMA) was maintained by William Schelter from 1982 until his death in 2001. In 1998 he obtained permission to release the Maxima source code under GNU General Public License (GPL).

Maxima can be used in two modes; (i) typing simple input commands into the console screen that are acted on with the result as output printed to the console; or (ii) as a ‘batch’ file of instructions that are executed sequentially with output printed to the console. Batch files are the more useful way to use Maxima and the results shown in this paper have been generated from a Maxima text file ‘Lagrange.txt’ shown below, followed by a copy of the output screen.

---

\(^5\) GNU is a recursive acronym for ‘GNU’s not Unix’ chosen because GNU’s design is Unix-like, but differs from Unix by being free software and containing no Unix code. GNU is a computer operating system developed by the GNU project aiming to be a complete Unix-compatible software system composed wholly of free software.
Maxima Batch File (Lagrange.txt) for trigonometric series for $E$ to order $e^{10}$

```maxima
/* set the order to compute */
pow:10$
/* use Lagrange's theorem to obtain a trigonometric series in M */
E:M + e*sin(M)$
for k: 2 thru pow do
  (E: E + e^k/k!*diff(sin(M)^k,M,k-1))$
/* reduce the expression for E to a trigonometric series in multiple */
E : trigreduce(E)$
E : expand(E)$
/* group coefficients of of M and sin(kM) */
coeff(E,M)*M + sum(coeff(E,sin(k*M))*sin(k*M),k,1,pow);
```

Maxima Console output of trigonometric series for $E$ to order $e^{10}$

Maxima 5.24.0 http://maxima.sourceforge.net
using Lisp Clozure Common Lisp Version 1.7-dev-r14645M-trunk (WindowsX8632)
Distributed under the GNU Public License. See the file COPYING.
Dedicated to the memory of William Schelter.
The function bug_report() provides bug reporting information.

(%i1) read and interpret file: D:/Projects/Geospatial/Geodesy/Satellites Orbits/Lagrange.txt
(%i2) pow : 10
(%i3) E : e sin(M) + M
  k       k
       e diff(sin(M), M, k - 1)
for k from 2 thru pow do E : ----------------------------- + E
  k!
(%i4) E : trigreduce(E)
(%i5) E : expand(E)
(%i6) sum(coeff(E, sin(k M)) sin(k M), k, 1, pow) + coeff(E, M) M
 10        9       8       10
78125 e   sin(10 M)  531441 e  sin(9 M) 128 e   2048 e
 10        9       8       10
145152 1146880 315 2835
7 9 10 8 6

16807 e 823543 e 2187 e 243 e 27 e
46080 1474560 8960 560 80
9 7 5

78125 e 3125 e 125 e
516096 5126 384
10 8 6 4

16 e 4 e 4 e
243 e 243 e 27 e 3 e
40960 5120 128 8
10 8 6 4

17280 720 48 6 2
9 7 5 3

737280 9216 192 8
(%o7)```

11
Recurrence Relations

A recurrence relation is an equation that recursively defines a sequence. Once one or more initial terms are given each further term of the sequence is defined as a function of the preceding terms. As examples, consider the trigonometric functions

\[
\sin k\phi = 2\cos\phi \sin(k-1)\phi - \sin(k-2)\phi \tag{32}
\]

\[
\cos k\phi = 2\cos\phi \cos(k-1)\phi - \cos(k-2)\phi \tag{33}
\]

With initial values \(\sin(0) = 0, \cos(0) = 1\) in (32) and (33) gives successively

\[
\sin 2\phi = 2\cos\phi \sin\phi, \quad \cos 2\phi = 2\cos^2\phi - 1
\]

\[
\sin 3\phi = 2\cos\phi \sin 2\phi - \sin\phi, \quad \cos 3\phi = 2\cos\phi \cos 2\phi - \sin 2\phi
\]

\[
\sin 4\phi = 2\cos\phi \sin 3\phi - \sin 2\phi, \quad \cos 4\phi = 2\cos\phi \cos 3\phi - \cos 2\phi
\]

\[
\sin 5\phi = \cdots, \quad \cos 5\phi = \cdots
\]

Recurrence relations for even multiples are obtained by replacing \(\phi\) with \(2\phi\) in (32) and (33) to give

\[
\sin 2k\phi = 2\cos 2\phi \sin 2(k-1)\phi - \sin 2(k-2)\phi \tag{34}
\]

\[
\cos 2k\phi = 2\cos 2\phi \cos 2(k-1)\phi - \cos 2(k-2)\phi \tag{35}
\]

Clenshaw summation

Suppose that a (truncated) sum \(S\) is denoted by

\[
S = u_0F_0(x) + u_1F_1(x) + u_2F_2(x) + \cdots + u_NF_N(x) = \sum_{k=0}^N u_kF_k(x) \tag{36}
\]

\(u_k\) are coefficients independent of \(x\), and \(F(x)\) obey the recurrence relation

\[
F_{k+1}(x) = a_kF_k(x) + b_kF_{k-1}(x) \tag{37}
\]

where the coefficients \(a_k, b_k\) may be functions of \(x\) as well as \(k\). Note that in many applications \(a\) does not depend on \(k\), and \(b\) is a constant independent of \(x\) or \(k\).

The sum \(S\) can be evaluated from

\[
S = b_0F_0(x)y_2 + F_1(x)y_1 + F_0(x)u_0 \tag{38}
\]

where the quantities \(y_k\) are obtained from the backward (or reverse) recurrence formula

\[
y_k = \begin{cases} 0, & \text{for } k > N \\ a_ky_{k+1} + b_{k+1}y_{k+2} + u_k, & \text{for } k = N, N-1, N-2, \ldots, 3, 2, 1 \end{cases} \tag{39}
\]

Equation (39) is Clenshaw’s recurrence formula and (38) is the associated sum; equations (38) and (39) combined are called Clenshaw’s summation (Clenshaw 1955, Deakin & Hunter 2011).

Clenshaw’s summation can be explained by writing out (36) as

\[
S = u_NF_N(x) + u_{N-1}F_{N-1}(x) + u_{N-2}F_{N-2}(x) + \cdots + u_0F_0(x) + u_1F_1(x) + \cdots
\]

and re-arranging (39) as

\[
u_k = y_k - a_ky_{k+1} - b_ky_{k+2} \tag{41}
\]
Then substituting (41) into (40) gives

\[
S = [y_N - a_N y_{N+1} - b_{N+1} y_{N+2}] F_N(x) + \cdots + [y_{N-1} - a_{N-1} y_N - b_N y_{N+1}] F_{N-1}(x) \\
+ [y_{N-2} - a_{N-2} y_{N-1} - b_{N-1} y_N] F_{N-2}(x) + \cdots \\
+ [y_0 - a_0 y_1 - b_0 y_2] F_0(x) + [y_1 - a_1 y_2 - b_1 y_3] F_1(x) + \cdots \\
+ [y_2 - a_2 y_3 - b_2 y_4] F_2(x) + [y_1 - a_1 y_2 - b_1 y_3] F_1(x) + [u_0 + b_0 y_2 - b_1 y_3] F_0(x) 
\]

(42)

Noting that in the last line \(b_1 y_2\) has been added and subtracted. Examining the terms containing a factor of \(y_8\) in (42) involves

\[
[F_8(x) - a_8 F_7(x) - b_7 F_6(x)] y_8 
\]

(43)

And as a consequence of the recurrence relation (37) the term in \([\ ]\) will equal zero and similarly for all other \(y_k\) down through and including \(y_2\). The only surviving terms in (42) are \(u_0, y_1\) and \(b_1 y_2\); and so the sum \(S\) is given by (38).

**Summation** \(S = \sum_{k=1}^{N} c_k \sin k \phi\)

Consider the (truncated) trigonometric series

\[
S = c_1 \sin \phi + c_2 \sin 2 \phi + c_3 \sin 3 \phi + \cdots + c_N \sin N \phi = \sum_{k=1}^{N} c_k \sin k \phi 
\]

(44)

The trigonometric functions \(\sin \phi, \sin 2 \phi, \sin 3 \phi, \ldots\) obey the recurrence relation (32) so \(S\) can be evaluated using Clenshaw summation. Write the recurrence relation (32) in another form replacing \(k\) with \((k + 1)\) giving

\[
\sin (k + 1) \phi = 2 \cos \phi \sin k \phi - \sin (k - 1) \phi 
\]

(45)

Equation (45) has the same form as (37) where \(F_k(x) = \sin k \phi\), \(a_k = 2 \cos \phi\) and \(b_k = -1\). Clenshaw’s backward recurrence formula (39) becomes

\[
y_k = \begin{cases} 
0, & \text{for } k > N \\
2 \cos \phi y_{k+1} - y_{k+2} + c_k, & \text{for } k = N, N-1, N-2, \ldots, 3, 2, 1 
\end{cases} 
\]

(46)

The associated sum (see equation (38) with \(F_0(x) = \sin (0) = 0\) and \(F_1(x) = \sin \phi\)) is

\[
S = \sum_{k=1}^{N} c_k \sin k \phi = y_1 \sin \phi
\]

(47)
function E = Kepler_series(e,M)

% Kepler_series is a function that solves Kepler's equation
% M = E - e*sin(E) for E using a trigonometric series to the order e^10 derived
% from Lagrange's Theorem.  M = mean anomaly, E = eccentric anomaly and
% e = orbit eccentricity.  Clenshaw summation is used to evaluate the series.
%
% example:  >> format long g
% >> d2r = 180/pi;
% >> M = 5/d2r;
% >> e = 0.1;
% >> E = Kepler_series(e,M);
% >>
% E = 0.0969458710753345
%

%--------------------------------------------------------------------------
% Function:  Kepler_series
% Usage:     E = Kepler_series(e,M);
% Author:    R.E.Deakin,  
% 1/443 Station Street, 
% BONBEACH, VIC 3196, AUSTRALIA.  
% email: randm.deakin@gmail.com
% Version   1.0   22 December 2017
%
% Functions required:  none
% Purpose:  Solution of Kepler's equation: M = E - e*sin(E) for E
% where M = mean anomaly, E = eccentric anomaly and e = orbit eccentricity
% Variables:
% A       - A = 2*cos(M) is a constant in Clenshaw recurrence
% c()     - array of coefficients in the trigonometric series
% e       - orbital eccentricity
% e2,e3,... - powers of eccentricity
% E       - eccentric anomaly (radians)
% k       - integer counter
% M       - mean anomaly (radians)
% N       - order of series
% y1,y2   - Clenshaw recurrence variables
%
% Remarks:  The solution of Kepler's equation for the eccentric anomaly E is given in
% the form of a trigonometric series:
% E = M + c(1)*sin(M) + c(2)*sin(2*M) + ... + c(N)*sin(N*M)
% where the coefficients c(1), c(2), ..., c(N) are functions of powers of
% the eccentricity e.
% Clenshaw summation is used to evaluate the series.
% The series is accurate for values of eccentricity e < 0.2.
% For values of e > 0.2, Newton-Raphson, or another iterative scheme should be used.
%
% References:
% of Astrodynamics', American Institute of Aeronautics and
% Astronautics (AIAA), Inc., New York.
% December, 2017.
%--------------------------------------------------------------------------
% calculate the value of the eccentric anomaly \( E \) using the trigonometric
% series expansion given in Ref[1], eq.(5.16), p.202 and Ref[2].

% set powers of eccentricity
\( e^2 = e^2 \times e; \)
\( e^3 = e^2 \times e; \)
\( e^4 = e^3 \times e; \)
\( e^5 = e^4 \times e; \)
\( e^6 = e^5 \times e; \)
\( e^7 = e^6 \times e; \)
\( e^8 = e^7 \times e; \)
\( e^9 = e^8 \times e; \)
\( e^{10} = e^9 \times e; \)

% set an array \( c() \) for the coefficients \( c(1), c(2), \ldots, c(10) \)
\( N = 10; \)
c = zeros(N,1);

% set the values of the coefficients \( c_1, c_2, \ldots, c_{10} \) Ref[2], eq.(9)
c(1) = (e-1/8*\( e^3 \)+1/192*\( e^5 \)-1/9216*\( e^7 \)+1/737280*\( e^9 \));
c(2) = (1/2*\( e^2 \)-1/6*\( e^4 \)+1/48*\( e^6 \)-1/720*\( e^8 \)+1/17280*\( e^{10} \));
c(3) = (3/8*\( e^3 \)-27/128*\( e^5 \)+243/5120*\( e^7 \)-243/40960*\( e^9 \));
c(4) = (1/3*\( e^4 \)-4/15*\( e^6 \)+4/45*\( e^8 \)-16/945*\( e^{10} \));
c(5) = (125/384*\( e^5 \)-3125/9216*\( e^7 \)+78125/516096*\( e^9 \));
c(6) = (27/80*\( e^6 \)-243/560*\( e^8 \)+2187/8960*\( e^{10} \));
c(7) = (16807/46080*\( e^7 \)-823543/1474560*\( e^9 \));
c(8) = (128/315*\( e^8 \)-2048/2835*\( e^{10} \));
c(9) = (531441/1146880*\( e^9 \));
c(10) = (78125/145152*\( e^{10} \));

% set up \( y_1 \) and \( y_2 \) for Clenshaw's backward recurrence
\( y_2 = 0; \)
\( y_1 = 0; \)

% calculate \( y_1 \) from Clenshaw's backward recurrence
\( A = 2*cos(M); \)
for \( k = N:-2:1 \)
\( y_2 = A*y_1-y_2+c(k); \)
\( y_1 = A*y_2-y_1+c(k-1); \)
end

% calculate the eccentric anomaly
\( E = M + y_1*\sin(M); \)
return
Matlab function Kepler_Newton.m

function [E,count] = Kepler_Newton(e,M)
%
% Kepler_Newton is a function that solves Kepler's equation
% M = E - e*sin(E) for E
% where M = mean anomaly, E = eccentric anomaly and e = orbit eccentricity
% count is the number of iterations
%
% example: >> format long g
% >> d2r = 180/pi;
% >> M = 5/d2r;
% >> e = 0.1;
% >> [E,count] = Kepler_Newton(e,M);
% >> E
% E = 0.0969458710759671 % radians
% >> count
% count = 3 % number of iterations
%
%------------------------------------------------------------------------
% Function:  Kepler_Newton
%
% Usage:     [E,count] = Kepler_Newton(e,M);
%
% Author:    R.E.Deakin,
% 1/443 Station Street,
% BONBEACH, VIC 3196, AUSTRALIA.
% email:  randm.deakin@gmail.com
% Version   1.0   01 December 2017
%
% Functions required:
% None
%
% Purpose:
%  Solution of Kepler's equation M = E - e*sin(E) for E
%  where M = mean anomaly, E = eccentric anomaly and e = orbit eccentricity
%
% Variables:
%  corrn   - corrn = F/df
%  count   - number of iterations
%  dF      - derivative of F, dF = 1 - e*cos(Psi)
%  e       - orbital eccentricity
%  E       - eccentric anomaly (radians)
%  F       - function F = E - e*sin(E) - M
%  M       - mean anomaly (radians)
%  tol     - tolerance (a small value)
%
% Remarks:
%  This function uses Newton-Raphson Iteration to solve Kepler's equation for
%  the eccentric anomaly E. A starting value for the iterative process is
%  obtained from eq. (5) of Ref. [1]
%
% References:
%  [1] Smith, G.R., (1979), 'A simple efficient starting value for the
%      iterative solution of kepler's equation', Celestial Mechanics, Vol.
%      19, pp. 163-166.
%      December, 2017.
%------------------------------------------------------------------------
%
% Compute the initial value of E
E = M + e*(sin(M)/(1-sin(M+e)+sin(M)));  % eq. (5) of ref [1]
%
% set starting values for corrn and count and the tolerance
% corrn = 1;
count = 0;
tol = 1e-15;

% Newton-Raphson iteration
while (abs(corrn) > tol)
    F  = E - e*sin(E) - M;        % function F(M) = E - e*sin(E) - M = 0
    dF = 1.0 - e*cos(E);          % derivative of F(M)
    if(abs(dF) < tol)
        fprintf('
*** derivative 1 - e*cos(E) = 0 *** \n\n');
        break;
    endif
    corrn = F/dF;
    E = E - corrn;
    count = count+1;
    if(count>50)
        fprintf('
*** no convergence after 50 iterations ***\n\n');
        break;
    endif
end

Matlab function Kepler_Bisection.m

function [E,count] = Kepler_Bisection(e,M)

% Kepler_Bisection is a function that solves Kepler's equation
% M = E - e*sin(E) for E
% where M = mean anomaly, E = eccentric anomaly and e = orbit eccentricity
% count is the number of iterations
%
% example: >> format long g
% >> d2r = 180/pi;
% >> M = 5/d2r;
% >> e = 0.1;
% >> [E,count] = Kepler_Bisection(e,M);
% >> E
% E = 0.0969458710759658 % radians
% >> count
% count = 46 % number of iterations

%-----------------------------------------------------------------------
% Function:  Kepler_Bisection
% %
% Usage:     [E,count] = Kepler_Bisection(e,M);
% %
% Author:    R.E.Deakin,
% 1/443 Station Street,
% BONBEACH, VIC 3196, AUSTRALIA.
% email: randm.deakin@gmail.com
% Version  1.0   01 December 2017
%
% Functions required:
% None
%
% Purpose:
% Solution of Kepler's equation M = E - e*sin(E) for E
% where M = mean anomaly, E = eccentric anomaly and e = orbit eccentricity
%
% Variables:
% count   - number of iterations
% d       - length of interval
% F       - F = sign(M)  F = -1 if M < 0; F = 1 if M > 0; F = 0 if M = 0
% e       - orbital eccentricity
% E       - eccentric anomaly (radians)
% M       - mean anomaly (radians)
% M_new   - new value of M
% tol     - tolerance (a small value)
% twopi   - 2*pi
%
% Remarks:
% This function uses a Bisection iterative scheme to solve Kepler's equation
% for the eccentric anomaly E.
% The (unknown) eccentric anomaly is bounded M <= E <= (M+e) and the initial
% value for the iteration scheme is M+(e/2) and the initial interval width is
% d = e.
% %
% References:
% Richmond, Virginia, USA.
% December, 2017.
%------------------------------------------------------------------------

% twopi = 2.0*pi;

% Reduce M to a value in the range -twopi < M < twopi
F = sign(M);
M = abs(M)/twopi;
M = (M-fix(M))*twopi*F;

% Make M a value in the range 0 < M < twopi
if M < 0
   M = M + twopi;
endif

% determine the sign of reduced M
F = 1;
if M > pi
   M = twopi - M;
   F = -1;
endif

% set starting values for the iterative process
% E is bounded such that M <= E <= (M+e) so the mid-point is M+(e/2). This is
% the initial value for E.
E = M+(e/2);
d = e/2;

% set the tolerance and iteration count
tol = 1.0E-15;
count = 0;
while (d > tol)
   M_new = E - e*sin(E);
   E = E + sign(M-M_new)*d;
   d = d/2;
   count = count+1;
   if(count > 60)
      fprintf('
*** no convergence after 60 iterations ***
');
      break;
   endif
end
E = F*E;
References


Deakin, R.E., (2007), *Satellite Orbits*, Lecture Notes, School of Mathematical & Geospatial Sciences, RMIT University, 51 pages, August 2007


URL: http://www.skyandtelescope.com/astronomy-resources/basic-programs-from-sky-telescope/


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