A REVIEW OF LEAST SQUARES THEORY APPLIED TO TRAVERSE ADJUSTMENT

R E DEAKIN
Department of Land Information
RMIT University
GPO Box 2476V MELBOURNE VIC 3001
AUSTRALIA

Phone: +61 3 9925 2213
Fax: +61 3 9663 2517
e-mail: deakin@rmit.edu.au

This paper was originally published in two parts in *The Australian Surveyor,* Vol. 36, No. 3, September 1991, pp. 245-53 and Vol. 36, No. 4, December 1991, pp. 281-90. Minor changes have been made to notation.

R.E.Deakin,
January 1999.

ABSTRACT

This paper presents the general outline for a least squares adjustment of a traverse network by *Variation of Coordinates*. The standard methods set out lend themselves to computer solution and thetechnique is adaptable to Resections and Intersections, Triangulation and Trilateration schemes and combinations thereof. A method of constraining bearings, distances and angles in the traverse network is outlined and necessary matrix equations developed for solution. Equations are also developed which enable precisions of adjusted bearings and distances to be estimated.

INTRODUCTION

A traverse is the basic element of many surveys. Good survey practice demands that traverses be closed so that miscloses may be used to assess the precision of the traverse measurements. If the misclose is within acceptable limits, it is usual for the surveyor to adjust the measurements so as to eliminate the misclose. This is sound practice as the traverse is now mathematically correct and hence all figures derived from this traverse should also be mathematically correct.

Shepherd (1968, pp.317-347), lists some of the traverse adjustments that may be used:

> Bowditch, Transit (Wilson's method), Smirnoff, Crandall, Scale Factor axis method, and the xy (Ormsby) method.

The common virtue of these adjustment methods is their mathematical simplicity. Unfortunately these techniques are **non-rigorous** and often based on assumptions that are not valid for todays survey equipment or techniques. Traverse measurements derived from these adjustments may suffer accordingly.

The method of *least squares* is a **rigorous** technique that can be applied to the adjustment of single traverses as well as networks of connected traverses to yield the most likely values of the survey measurements. Leahy (1974) has an excellent summary of the theoretical basis and development of the least squares technique.

This paper will attempt to outline the basic principles of least squares as applied to the adjustment of traverse networks.

A worked example of a traverse adjustment is provided in Appendix B as a means of understanding the principles and methods used.

WHAT IS A LEAST SQUARES TRAVERSE ADJUSTMENT?

A traverse is a combination of two basic survey measurements, distances and directions, which are indirect measurements of the coordinates of traverse points. Assuming that mistakes and systematic errors are eliminated, traverse directions and distances are affected by small random errors which manifest themselves as miscloses in closed traverses. A residual has the same magnitude as an error but opposite sign and since distances and directions are indirect measurements of coordinates, it could be said that the residuals are functions of the coordinates of the traverse points.

A least squares traverse adjustment is the determination of a set of traverse coordinates which makes *the sum of the squares of the residuals a minimum*.

This set of coordinates will be the most likely and the underlined section above is often referred to as the *Least Squares Principle*.

HOW IS THE LEAST SQUARES PRINCIPLE APPLIED?

The diagram below shows the bearing and distance between traverse points

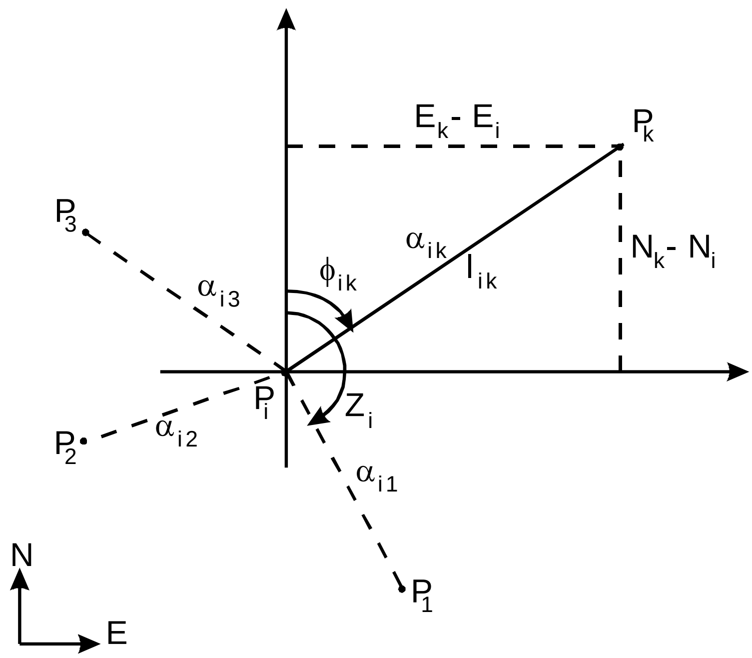

Figure 1.

where

$P_1, P_2, P_3, \cdots P_i, P_k$	traverse points (P_i is the instrument point),
$\alpha_{i1}, \alpha_{i2}, \alpha_{i3}, \cdots \alpha_{ik}$	observed directions from P_i,
$l_{i1}, l_{i2}, l_{i3}, \cdots l_{ik}$	observed distances from P_i,
E_i, N_i, E_k, N_k	coordinates of points P_i and P_k respectively,
ϕ_{ik} and s_{ik}	bearing and distance of line P_i to P_k,
Z_i	orientation constant.

Referring to Figure 1, *Observation Equations* for the direction and distance of the line line P_i to P_k, are;

$$\alpha_{ik} + v_{ik} + Z_i = \phi_{ik} = \tan^{-1}\left\{\frac{E_k - E_i}{N_k - N_i}\right\} \tag{1}$$

$$l_{ik} + v_{ik} = s_{ik} = \sqrt{(E_k - E_i)^2 + (N_k - N_i)^2} \tag{2}$$

where v_{ik} is the *residual.*

In equations (1) and (2) the observed directions and distances are expressed as non-linear functions of the coordinates E_i, N_i and E_k, N_k. Mikhail and Gracie (1981, pp.266-272), show how these equations can be expressed as linearized approximations using Taylor's Theorem as;

$$\alpha_{ik} + v_{ik} + Z_i = \phi'_{ik} - a_{ik}\Delta N_i - b_{ik}\Delta E_i + a_{ik}\Delta N_k + b_{ik}\Delta E_k$$

$$l_{ik} + v_{ik} = s'_{ik} - c_{ik}\Delta N_i - d_{ik}\Delta E_i + c_{ik}\Delta N_k + d_{ik}\Delta E_k$$

These equations can be re-arranged as *residual equations* for *directions* and *distances* respectively,

$$v_{ik} = -a_{ik}\Delta N_i - b_{ik}\Delta E_i + a_{ik}\Delta N_k + b_{ik}\Delta E_k - f_{ik} \tag{3}$$

$$v_{ik} = -c_{ik}\Delta N_i - d_{ik}\Delta E_i + c_{ik}\Delta N_k + d_{ik}\Delta E_k - f_{ik} \tag{4}$$

where

ΔN and ΔE are small corrections to approximate coordinates E' and N' such that $E = E' + \Delta E$ and $N = N' + \Delta N$.
ΔZ is a small correction to the approximate orientation constant Z' such that $Z = Z' + \Delta Z$.

$a_{ik} = -\dfrac{\sin\phi'_{ik}}{s'_{ik}}$ and $b_{ik} = \dfrac{\cos\phi'_{ik}}{s'_{ik}}$ are *direction* coefficients.

$c_{ik} = \cos\phi'_{ik}$ and $d_{ik} = \sin\phi'_{ik}$ are *distance* coefficients.

ϕ' and s' are the computed bearing and distance using the approximate coordinates.

f_{ik} takes the general form:	$f = \text{observed - computed}$
in equation (3)	$f_{ik} = (\alpha_{ik} + Z'_i) - \phi'_{ik}$
in equation (4)	$f_{ik} = l_{ik} - s'_{ik}$

Inspection of equations (3) and (4) shows that a residual for each measurement can be expressed as a linear function of the unknown corrections to approximate coordinates of the end points of the traverse line. In the case of direction measurements, an unknown correction to the approximate orientation constant at the instrument point is also included. The "f-term" is a function of the measurement and the approximate coordinates.

Equations of the form of (3) and (4) can be written for each observed direction and distance and in a closed traverse or traverse network there will be more equations than unknowns. These residual equations can be represented in matrix form as;

$$\mathbf{v} = \mathbf{B}\mathbf{x} - \mathbf{f} \tag{5}$$

where

u	number of unknowns,
n	number of equations,
$\mathbf{v}_{(n,1)}$	is an n by 1 vector of residuals,
$\mathbf{B}_{(n,u)}$	is an n by u coefficient matrix containing various combinations of direction and distance coefficients,
$\mathbf{x}_{(u,1)}$	is a u by 1 vector of unknown corrections to the approximate coordinates and orientation constants,
$\mathbf{f}_{(n,1)}$	is an n by 1 vector of numeric terms derived from the measurements and computed values of bearings and distances.

The residual equations given in matrix form by equation (5) are partitioned in the following manner;

$$\begin{bmatrix} \mathbf{v}_1 \\ \hline \mathbf{v}_2 \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \hline \mathbf{B}_{21} & \mathbf{B}_{22} \end{array}\right] \begin{bmatrix} \mathbf{x}_1 \\ \hline \mathbf{x}_2 \end{bmatrix} - \begin{bmatrix} \mathbf{f}_1 \\ \hline \mathbf{f}_2 \end{bmatrix} \tag{6}$$

where the elements

$\mathbf{v}_1$ and $\mathbf{v}_2$	are the residuals associated with the directions and distances respectively
$\mathbf{x}_1$ and $\mathbf{x}_2$	are the corrections to the approximate coordinates and orientations respectively
$\mathbf{f}_1$ and $\mathbf{f}_2$	are the numerical terms associated with directions and distances respectively
$\mathbf{B}_{11}$	is a sub-matrix of direction coefficients a and b of the corrections to the approximate coordinates
$\mathbf{B}_{12}$	is a sub-matrix of coefficients of the corrections to the approximate orientation constants
$\mathbf{B}_{21}$	is a sub-matrix of distance coefficients c and d of the corrections to the approximate coordinates
$\mathbf{B}_{22}$	is a sub-matrix of zeroes

Associated with every set of measurements is a square matrix Σ_{mm} of order n and known as the variance matrix. The diagonal elements of Σ_{mm} contain *variances* of measurements whilst the off diagonal elements contain *covariances* between measurements. The variance of a measurement is its *standard deviation* squared and hence a measure of precision, whilst the covariance is a measure of the dependence between two measurements. If two measurements are independent then their covariance will be zero.

In any least squares adjustment it is assumed that apriori estimates of the variances and covariances are available and the variance matrix Σ_{mm} is estimated by $\mathbf{Q}_{mm}$

In many least squares adjustments the measurements are independent and hence the apriori variance matrix $\mathbf{Q}$ will be diagonal and its inverse is commonly known as the *weight matrix* $\mathbf{W} = \mathbf{Q}^{-1}$ whose diagonal elements are the weights of the particular measurements.

The sum of the squares of the residuals can be represented in matrix form as:

$$\mathbf{v}^T \mathbf{W} \mathbf{v} = (\mathbf{B}\mathbf{x} - \mathbf{f})^T \mathbf{W} (\mathbf{B}\mathbf{x} - \mathbf{f}) = \varphi$$

where $\mathbf{v}^T$ is the transpose of $\mathbf{v}$. Mikhail and Gracie (1981, pp.68-73), show how calculus is used to minimize φ and obtain a set of normal equations of the form;

$$(\mathbf{B}^T \mathbf{W} \mathbf{B})\mathbf{x} = \mathbf{B}^T \mathbf{W} \mathbf{f} \tag{7a}$$

or

$$\mathbf{N}\mathbf{x} = \mathbf{t} \tag{7b}$$

The solutions of the normal equations are;

$$\mathbf{x} = \mathbf{N}^{-1}\mathbf{t} \tag{8}$$

where

$\mathbf{N}_{(u,u)}$ is a u by u matrix of coefficients of the normal equations (7b),
$\mathbf{t}_{(u,1)}$ is a u by 1 column vector of numeric terms.

and the vector $\mathbf{x}$ contains the corrections to the approximate coordinates and orientation constants which makes the sum of the squares of the residuals a minimum. This technique of traverse adjustment is known as *Variation of Coordinates* and is essentially an iterative process of solving for corrections to approximate values which is terminated when the corrections reach some predetermined value.

CONSTRAINTS

The least squares technique set out above requires at least two points in the traverse network to have fixed coordinate values before a solution for the corrections to the approximate coordinates of the other points can be determined. This is illustrated by pinning the network diagram to the wall through two traverse points. Remove one pin and the network will rotate around the other thereby making the coordinates of the other traverse points indeterminate. This constraining of the network can also be achieved by holding one point fixed and the bearing of a traverse line fixed.

As well as these minimal constraints it is also possible to have additional fixed points and constrained bearings in the network as well as distances and angles constrained to certain fixed values.

Constraining bearings, distances and angles in a traverse network to specified values means the following equations must be satisfied.

$$
\begin{aligned}
\tan\phi_{ik} &= \frac{E_k - E_i}{N_k - N_i} \\
s_{ik} &= \sqrt{(E_k - E_i)^2 + (N_k - N_i)^2} \\
\theta_{iLR} &= \tan^{-1}\left\{\frac{E_R - E_i}{N_R - N_i}\right\} - \tan^{-1}\left\{\frac{E_L - E_i}{N_L - N_i}\right\}
\end{aligned}
$$

where ϕ, s and θ are constrained or fixed traverse bearings, distances and angles respectively and the subscripts L and R refer to the left and right traverse stations defining an angle at the instrument point.

Using the techniques outlined previously leads to the following *constraint equations* for bearings, distances and angles respectively.

$$-\tan\phi_{ik}\Delta N_i + \Delta E_i + \tan\phi_{ik}\Delta N_k - \Delta E_k - g_{ik} = 0 \tag{9}$$

$$-c_{ik}\Delta N_i - d_{ik}\Delta E_i + c_{ik}\Delta N_k + d_{ik}\Delta E_k - g_{ik} = 0 \tag{10}$$

$$-(a_{iR} - a_{iL})\Delta N_i - (b_{iR} - b_{iL})\Delta E_i + a_{iR}\Delta N_R + b_{iR}\Delta E_R - a_{iL}\Delta N_L - b_{iL}\Delta E_L - g_{iLR} = 0 \tag{11}$$

where

$$g_{ik} = (E'_k - E'_i) - (N'_k - N'_i)\tan\theta_{ik} \qquad \text{in equation (9)}$$

$$g_{ik} = s_{ik} - s'_{ik} \qquad \text{in equation (10)}$$

$$g_{ik} = \theta_{iLR} - (\theta'_{iR} - \theta'_{iL}) \qquad \text{in equation (11)}$$

These constraint equations can be expressed in matrix form as;

$$\mathbf{Cx} - \mathbf{g} = \mathbf{0} \tag{12}$$

where

c	number of constraint equations,
u	number of unknowns,
$\mathbf{C}_{(c,u)}$	is a c by u coefficient matrix,
$\mathbf{x}_{(u,1)}$	is a u by 1 vector of unknown corrections to the approximate coordinates and orientation constants,
$\mathbf{g}_{(c,1)}$	is a c by 1 vector of elements derived from the constrained values of bearings, distances and angles and the approximate traverse coordinates.

Mikhail (1976, pp.214-33), shows how the least squares principle can be applied (using constrained minima by Lagrange multipliers) to the combined constraint and residual equations (5) and (12) to give the solution for the vector of unknowns **x**, and the Lagrange multipliers **k** as a partitioned matrix equation of the form;

$$\begin{bmatrix} \mathbf{x} \\ \hline \mathbf{k} \end{bmatrix} = \left[\begin{array}{c:c} -\mathbf{N} & \mathbf{C}^T \\ \hdashline \mathbf{C} & \mathbf{0} \end{array}\right]^{-1} \begin{bmatrix} -\mathbf{t} \\ \hline \mathbf{g} \end{bmatrix} \tag{13}$$

where

the vector to the left of the equal sign contains $(u+c)$ elements, the first u elements are the corrections to the approximate coordinates and orientation constants and the remaining c elements are Lagrange multipliers, the partitioned square matrix to the right of the equal sign is of order *(u+c)* whose upper left sub-matrix of order *u* by *u* contains the coefficients of the normal equations multiplied by -1, the lower left sub-matrix of order *c* by *u* contains the coefficients of the constraint equations, upper right sub-matrix of order *u* by *c* is the transpose of the lower left sub-matrix, and the lower right sub-matrix of order *c* by *c* contains zeros,

the vector to the right of the equal sign contains *(u+c)* elements, the first *u* elements are the numeric terms multiplied by -1 and the remaining *c* elements are the numerical terms of the constraint equations.

Equation (13) may be written as;

$$\left[\begin{array}{c}\mathbf{x}\\ \hline \mathbf{k}\end{array}\right] = \left[\begin{array}{c:c}\alpha & \beta^T\\ \hdashline \beta & \gamma\end{array}\right]\left[\begin{array}{c}-\mathbf{t}\\ \hline \mathbf{g}\end{array}\right] \tag{14}$$

where the partitioned matrix to the right of the equal sign is the inverse of the matrix in equation (13).

Constraint equations add flexibility to the adjustment process. If desired, any lines in the traverse network can be constrained to particular values of bearings and distances. The two extreme cases being;

(a) when all traverse bearings are fixed, in which case only the traverse distances will be adjusted and

(b) when all traverse distances are fixed, in which case only the traverse bearings will be adjusted.

It may also be desirable to maintain certain angles in the traverse network as they were "set out" in the field, such as an angle of 180° at a traverse point on line between two other points, or perhaps an angle of 90° between two traverse lines at a particular point.

These constraints can be accommodated in the adjustment by combining the particular constraint equations with the normal equations and solving equation (13).

PRECISION ESTIMATION

Traverse adjustment by the method of Least Squares allows precision estimation of the adjusted coordinates of traverse points as well as derived bearings and distances.

The variance matrix of the traverse measurements $\mathbf{Q}_{mm}$ is a necessary apriori estimate of the "true" measurement variance matrix Σ_{mm}. Mikhail (1976, pp.285-88) shows that the relationship between the true variance matrix and the apriori estimates is:

$$\Sigma_{mm} = \sigma_0^2 \mathbf{Q}_{mm} \tag{15}$$

where σ_0^2 is the *variance factor*.

The variance factor is defined as "the sum of the squares of the residuals divided by the degrees of freedom" and can be computed from;

$$\sigma_0^2 = \frac{\mathbf{v}^T \mathbf{W} \mathbf{v}}{r} = \frac{\mathbf{f}^T \mathbf{W} \mathbf{f} - \mathbf{x}^T \mathbf{t}}{r} \tag{16}$$

where $r = n - u + c$ is the degrees of freedom.

The degrees of freedom in a least squares traverse adjustment is a combination of the number of redundant measurements and the constraints in the network and when this figure reaches a statistically significant level then the expected value of the variance factor is unity if the apriori estimates of the measurement variances are correct. The difference between unity and the variance factor is often used as a measure of confidence in the adjustment results but is dependent upon a sound knowledge of the apriori measurement variances as well as the degrees of freedom in the network.

Mikhail (1976, pp.77-81, 159-61, 229-30) shows that the Law of Propagation of Variances applied to the least squares adjustment gives the variance matrix of the adjusted quantities as:

$$\Sigma_{xx} = \sigma_0^2 \mathbf{N}^{-1} \tag{17}$$

when there are no constraints in the network and

$$\Sigma_{xx} = -\sigma_0^2 \alpha \tag{18}$$

when there are constraints in the network (α is obtained from (14)).

The variance matrix Σ_{xx} contains the variances and covariances of the "adjusted quantities". The diagonal elements will be the variances of coordinates or orientation constants in the same sequence as the elements of the vector **x**. The off-diagonal elements of Σ_{xx} are the covariances between particular coordinates and/or orientation constants. The variance matrix Σ_{xx} may be partitioned in the following manner;

$$\Sigma_{xx} = \left[\begin{array}{c|c} \Sigma_{11} & \Sigma_{12} \\ \hline \Sigma_{21} & \Sigma_{22} \end{array}\right] \tag{19}$$

where

Σ_{11} is a symmetric sub-matrix containing the variances and covariances of the adjusted coordinates

Σ_{22} is a symmetric sub-matrix containing the variances and covariances of the adjusted orientation constants

Σ_{12} is a sub-matrix containing the covariances between coordinates and orientation constants

$\Sigma_{21} = \Sigma_{12}^T$

PRECISION OF ADJUSTED TRAVERSE COORDINATES

An upper-triangular portion of the symmetric variance sub-matrix Σ_{11} is shown below

$$\Sigma_{11} = \begin{bmatrix}
\sigma_{N_1}^2 & \sigma_{N_1E_1} & \sigma_{N_1N_2} & \sigma_{N_1E_2} & \cdots & \sigma_{N_1N_i} & \sigma_{N_1E_i} & \sigma_{N_1N_k} & \sigma_{N_1E_k} & \cdots \\
 & \sigma_{E_1}^2 & \sigma_{E_1N_2} & \sigma_{E_1E_2} & \cdots & & & & \vdots & \\
 & & \sigma_{N_2}^2 & \sigma_{N_2E_2} & \cdots & & & & & \\
 & & & \sigma_{E_2}^2 & \cdots & & & & & \\
 & & & & \ddots & \vdots & & & \vdots & \\
 & & & & & \sigma_{N_i}^2 & \sigma_{N_iE_i} & \sigma_{N_iN_k} & \sigma_{N_iE_k} & \cdots \\
 & & & & & & \sigma_{E_i}^2 & \sigma_{E_iN_k} & \sigma_{E_iE_k} & \cdots \\
 & & & & & & & \sigma_{N_k}^2 & \sigma_{N_kE_k} & \cdots \\
 & & & & & & & & \sigma_{E_k}^2 & \cdots \\
 & & & & & & & & & \ddots
\end{bmatrix}$$

where

$\sigma_{N_i}^2, \sigma_{E_i}^2, \sigma_{N_k}^2, \sigma_{E_k}^2$ are the variances of the adjusted coordinates at points P_i and P_k

and

$$\begin{array}{ccc} \sigma_{N_iE_i} & \sigma_{N_iN_k} & \sigma_{N_iE_k} \\ & \sigma_{E_iN_k} & \sigma_{E_iE_k} \\ & & \sigma_{N_kE_k} \end{array} \quad \text{are the covariances}$$

The standard deviation of an adjusted coordinate is simply the square root of the appropriate variance.

PRECISION OF COMPUTED TRAVERSE BEARINGS AND DISTANCES

Bearings and distances computed from the adjusted coordinates of the traverse network are non-linear functions of the coordinates and can be represented in a linearized matrix form as;

$$\begin{bmatrix} \boldsymbol{\phi} \\ \hline \mathbf{s} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{11} \\ \hline \mathbf{B}_{21} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \end{bmatrix} \tag{20}$$

where

$\boldsymbol{\phi}$	are the computed bearings
$\mathbf{s}$	are the computed distances
$\mathbf{B}_{11}, \mathbf{B}_{21}$ and $\mathbf{x}_1$	are defined in equation (6)

[Note that there are the same number of computed bearings and distances as there are observed directions and distances respectively and in the same order as equation (6)].

Applying the *Law of Propagation of Variances* gives;

$$\begin{bmatrix} \sigma_\phi^2 \\ \hline \sigma_s^2 \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{11} \\ \hline \mathbf{B}_{21} \end{bmatrix} \boldsymbol{\Sigma}_{11} \begin{bmatrix} \mathbf{B}_{11} \\ \hline \mathbf{B}_{21} \end{bmatrix}^T \tag{21}$$

where

σ_ϕ^2 and σ_s^2	are the variances of the adjusted bearings and distances respectively.

The matrix equation (21) gives the following formulae for the variance of an adjusted bearing and distance as;

$$\begin{aligned} \sigma_{\phi_{ik}}^2 &= a_{ik}^2\left(\sigma_{N_i}^2 + \sigma_{N_k}^2 - 2\sigma_{N_iN_k}\right) + b_{ik}^2\left(\sigma_{E_i}^2 + \sigma_{E_k}^2 - 2\sigma_{E_iE_k}\right) \\ &\quad + 2\,a_{ik}\,b_{ik}\left(\sigma_{N_iE_i} + \sigma_{N_kE_k} - \sigma_{N_iE_k} - \sigma_{N_kE_i}\right) \end{aligned} \tag{21a}$$

and

$$\begin{aligned} \sigma_{s_{ik}}^2 &= c_{ik}^2\left(\sigma_{N_i}^2 + \sigma_{N_k}^2 - 2\sigma_{N_iN_k}\right) + d_{ik}^2\left(\sigma_{E_i}^2 + \sigma_{E_k}^2 - 2\sigma_{E_iE_k}\right) \\ &\quad + 2c_{ik}\,d_{ik}\left(\sigma_{N_iE_i} + \sigma_{N_kE_k} - \sigma_{N_iE_k} - \sigma_{N_kE_i}\right) \end{aligned} \tag{21b}$$

and the standard deviations of the adjusted bearings and distances are the square roots of the variances.

ERROR ELLIPSES

Error ellipses are a graphical representation of the precision of the adjusted coordinates of a traverse point with respect to the fixed points in the network. They can be used to gauge the "strengths and weaknesses" in a network. Circular ellipses indicate a strong position fix and elongated ellipses indicate a weak position fix.

Mikhail (1976, pp.30-31), gives the equations for the lengths of the semi-axes of the error ellipse as;

$$a^2 = \frac{1}{2}\left(\sigma_N^2 + \sigma_E^2 + \sqrt{\left(\sigma_N^2 - \sigma_E^2\right)^2 + \left(2\sigma_{NE}\right)^2} \right) \tag{22a}$$

$$b^2 = \frac{1}{2}\left(\sigma_N^2 + \sigma_E^2 - \sqrt{\left(\sigma_N^2 - \sigma_E^2\right)^2 + \left(2\sigma_{NE}\right)^2} \right) \tag{22b}$$

and the angle θ between north and the major axis of the error ellipse as;

$$\tan 2\theta = \frac{2\sigma_{NE}}{\sigma_N^2 - \sigma_E^2} \tag{23}$$

The correct quadrant of 2θ is determined from the signs of the numerator and denominator in equation (23).

It is interesting to note that precision estimation can take place before any measurements are taken. A scale diagram of the network is sufficient to determine approximate bearings and distances from which the coefficient matrix B may be deduced. Solving the system in the manner set out will enable the variance matrix $\mathbf{\Sigma}_{xx}$ to be "estimated" and error ellipse parameters calculated. The strength of the network can then be assessed and additional measurements taken if desired.

An important point to bear in mind when assessing precision is the number of constraints in the network. Holding <u>one</u> point fixed and constraining <u>one</u> bearing to a particular value are the minimum constraints that can be placed on a network using this method of solution and will yield a particular number for the sum of the squares of the residuals $\mathbf{v}^T\mathbf{W}\mathbf{v}$. Applying additional constraints will cause $\mathbf{v}^T\mathbf{W}\mathbf{v}$ to increase with a commensurate increase in the variances and covariances in $\mathbf{\Sigma}_{xx}$ and an apparent loss of precision in the adjusted coordinates. Constraints should be carefully chosen so as not to distort the network or concentrate residuals at particular traverse lines.

CONCLUSION

Least Squares is an adjustment technique founded on well accepted principles of measurements and their errors and is regarded as superior to all other methods of adjustment.

The Least Squares method of adjustment, *Variation of Coordinates,* outlined above is a systematic method of determining the most likely values of traverse coordinates when the number of measurements exceeds the number of unknowns, as happens in all closed traverses. The technique is adaptable to many surveying applications such as Resections and Intersections, Triangulation and Trilateration schemes as well as combinations of these and lends itself to computer solution.

The inclusion of *constraints* in the form of bearings, distances and angles adds a degree of flexibility to this well proven adjustment process.

BIBLIOGRAPHY

Leahy, F.J., 1974. 'Two hundred years of adjustment of survey measurements', *Two Centuries of Surveying (1874-1974-2074): Proceedings of The Institution of Surveyors, Australia, 17th Australian Survey Congress,* Melbourne, 23 February -1 March 1974, pp.19-29.

Mikhail, E.M., 1976. *Observations and Least Squares,* IEP–A Dun-Donnelley, New York.

Mikhail, E.M. and G. Gracie, 1981. *Analysis and Adjustment of Survey Measurements,* Van Nostrand Reinhold, New York.

Shepherd, F.A., 1968. *Surveying Problems and Solutions,* Edward Arnold Ltd., London.