ENGINEERING SURVEYING 1

VERTICAL CURVES

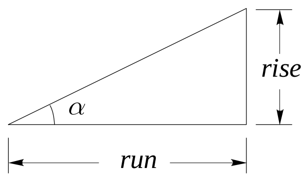
Whenever two gradients intersect on a road or railway, it is necessary to connect them with a vertical curve to improve visibility (summit curves), prevent shock impacts or passenger discomfort (both summit and sag curves) and to improve visual appearance. Vertical curves allow vehicles to pass smoothly from one gradient to another.

In practice, road and railway gradients are comparatively flat and it is often unimportant what type of vertical curve is used; the usual curves are circular or parabolic. However, it is best to use a vertical curve having a constant rate of change of gradient, i.e., a *parabola* and as it turns out, *parabolic vertical curves* are very easy to calculate and use.

For some freeways (high-speed roads) a vertical curve whose rate of change of gradient increases or decreases with the length of the curve is sometimes used, e.g., a cubic parabola. However, since freeways are generally made up of relatively flat gradients, curves of this type are sometimes regarded as an unnecessary refinement. The cubic parabola is sometimes used as a sag vertical curve, where its properties allow a uniform rate of increase of centrifugal force (greater passenger comfort) and less filling in the valley is required. Unless indicated otherwise, in these notes, a vertical curve is assumed to be a simple parabola.

1. GRADIENTS

A **gradient** is a dimensionless number, $gradient = \frac{\text{rise}}{\text{run}}$ or $gradient = \tan\alpha$.



In road and railway design, **gradients** are usually expressed in percentages; e.g., a road of +4% gradient rises 4 units *vertically* in 100 units *horizontally*. Thus, a gradient of *p*% is equal to *p*/100. Gradients rising from left to right are positive and gradients falling left to right are negative. Representing gradients as percentages has a useful connection with calculus: if vertical distances are measured along the *y*-axis and horizontal distances along the *x*-axis then a 4% gradient is a mathematical gradient of 0.04 or $dy/dx = 0.04$. Thus the gradient (in %) divided by 100 is the derivative.

Vertical curves connect two gradients and in sectional view, the gradient to the left of the vertical curve will be denoted by p% and the gradient to the right will be denoted by q%. Alternative notations are g_1% or G_1% for the left-hand gradient and g_2% or G_2% for the right-hand gradient.

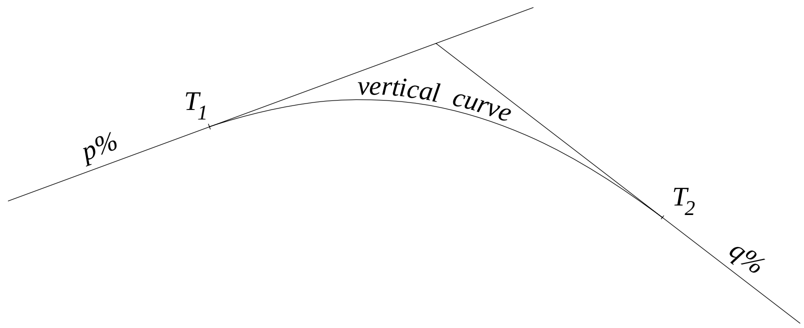


Figure 1.1

Gradients can also be expressed as 1 in x, i.e., 1 vertical in x horizontal. A gradient of +4% is equivalent to a gradient of 1 in 25. If the gradient is expressed as p%, then it is equivalent to a gradient of 1 in x where $x = 100/p$.

2. TYPES OF VERTICAL CURVES

(i) *Summit curves:* Vertical curves where the total change in gradient is negative.

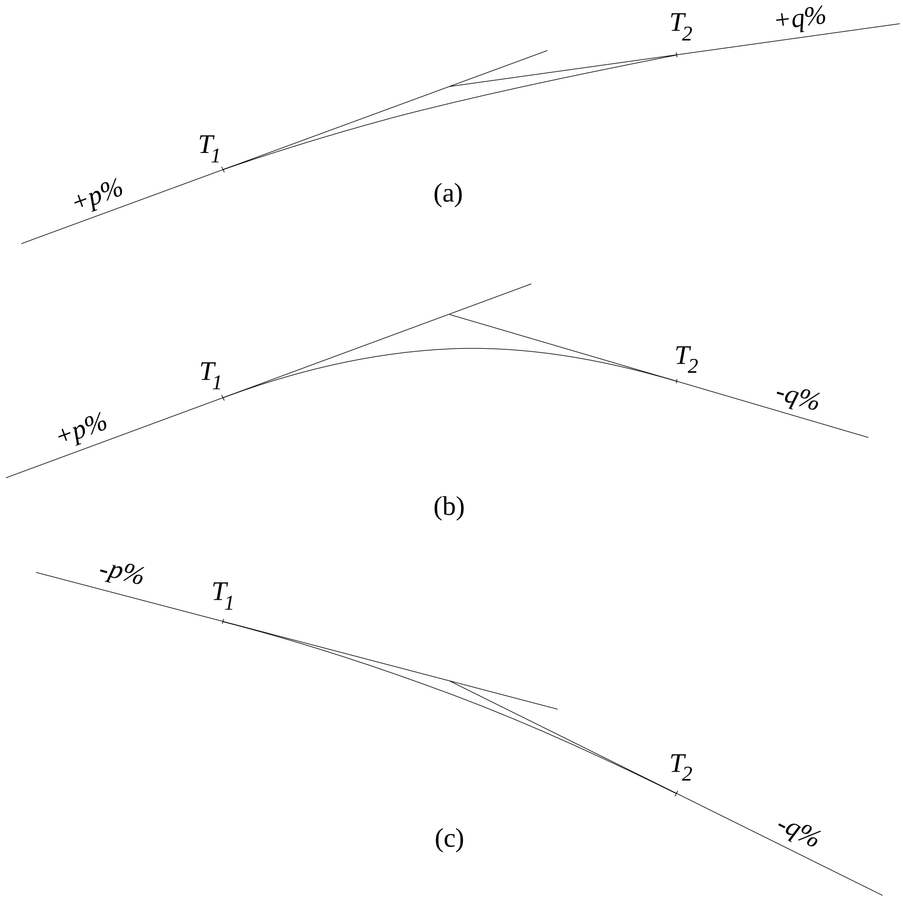


Figure 2.1

Let A be the total change in gradient, then in percent

$$A\% = q\% - p\% \tag{1.1}$$

In Figure 2.1(a) the gradients are $p\%$ = +3.5% and $q\%$ = +1.4%, in Figure 2.1(b) the gradients are $p\%$ = +3.5% and $q\%$ = -3.0% and in Figure 2.1(c) they are $p\%$ = -2.5% and $q\%$ = -5.0%.

The total change in gradient for each curve is

Figure 2.1(a) $A\% = +1.4 - (+3.5) = -2.1\%$,

Figure 2.1(b) $A\% = -3.0 - (+3.5) = -6.5\%$

Figure 2.1(c) $A\% = -5.0 - (-2.5) = -2.5\%$

In each case, A is a negative quantity and the vertical curves are summit curves.

(ii) *Sag curves:* Vertical curves where the total change in gradient is positive.

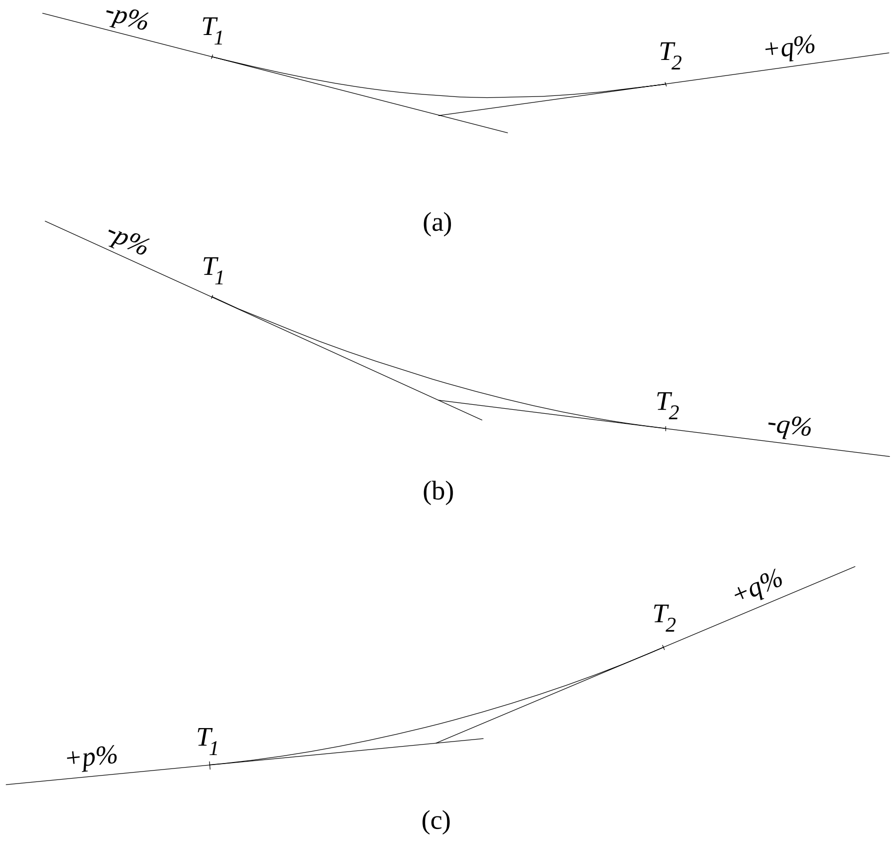


Figure 2.2

In Figure 2.2(a) the gradients are $p\%$ = -2.5% and $q\%$ = +1.5%, in Figure 2.2(b) the gradients are $p\%$ = -4.5% and $q\%$ = -1.0% and in Figure 2.2(c) they are $p\%$ = +1.0% and $q\%$ = +4.0%.

The total change in gradient for each curve is

Figure 2.2(a) $A\% = +1.5 - (-2.5) = +4.0\%$,

Figure 2.2(b) $A\% = -1.0 - (-4.5) = +3.5\%$

Figure 2.2(c) $A\% = +4.0 - (+1.0) = +3.0\%$

In each case, A is a positive quantity and the vertical curves are sag curves. Sag vertical curves are also known as valley curves.

It should be noted that a summit curve will only have a "true" high point and a sag curve will only have true low point when there is a change of sign between the grades.

3. EQUATION OF A VERTICAL CURVE

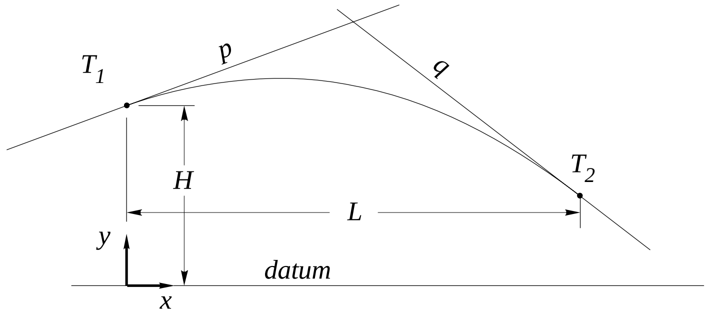


Figure 3.1

In Figure 3.1 two gradients, p and q are joined by a vertical curve of length L. T_1, T_2 are tangent points and the x-y coordinate origin is vertically below T_1 with the x-axis being the datum for reduced levels y. H is the reduced level of T_1.

The basic requirement for the vertical curve is that the rate of change of gradient (with respect to horizontal distance) shall be constant. This requirement can be expressed in two ways.

(i) Since the gradient can also be the derivative dy/dx then the rate of change of gradient is a second derivative and we may write.

$$\frac{d^2y}{dx^2} = K \tag{3.1}$$

(ii) Since the rate of change of gradient is a constant *K*, then it is also equal to the total change in gradient *A* divided by the length *L*.

$$\frac{q-p}{L}=\frac{A}{L}=K \tag{3.2}$$

The gradient at any point on the vertical curve can be found by integrating equation (3.1)

$$\frac{dy}{dx}=\int\left(\frac{d^2y}{dx^2}\right)dx=\int K\,dx=Kx+C_1$$

C_1 is a constant of integration and when $x = 0$ (the tangent point T_1) the gradient is *p*. Thus $C_1 = p$ and equation for the gradient becomes

$$\frac{dy}{dx}=Kx+p \tag{3.3}$$

or

$$\frac{dy}{dx}=\left(\frac{q-p}{L}\right)x+p \tag{3.4}$$

The reduced level *y* (height above datum) of any point on the vertical curve can be found by integrating (3.3)

$$y=\int\left(\frac{dy}{dx}\right)dx=\int\left(Kx+p\right)dx=\frac{Kx^2}{2}+px+C_2$$

C_2 is a constant of integration and when $x = 0$ (the tangent point T_1) the reduced level is *H*. Thus $C_2 = H$ and substituting the expression for *K*, the equation for the reduced level (*y*-value) becomes

$$y=\left(\frac{q-p}{2L}\right)x^2+px+H \tag{3.5}$$

This is an equation of the form $y=ax^2+bx+c$ (a parabola) where *a* is half the rate of change of gradient $\left(\text{ie, } a=\frac{q-p}{2L}=\frac{K}{2}\right)$, *b* is the initial gradient at T_1 $(\text{ie, } b=p)$ and *c* is the of reduced level of T_1 $(\text{ie, } c=H)$.

If the gradients are given in percentages then equation (3.5) becomes

$$y=\left(\frac{q\%-p\%}{200L}\right)x^2+\left(\frac{p\%}{100}\right)x+H \tag{3.6}$$

When the sign of the coefficient of x^2 is negative, the curve is a summit curve and when the coefficient is positive it is a sag curve. This will be demonstrated by numerical examples for each of the curves in Figures 2.1 and 2.2.

Summit vertical curves in Figure 2.1:

Figure 2.1(a) $p\% = +3.5\%$ $q\% = +1.4\%$ then $K = \dfrac{+1.4-(+3.5)}{100L} = \dfrac{-2.1}{100L}$

$$y = \left(\frac{-2.1}{200L}\right)x^2 + \left(\frac{3.5}{100}\right)x + H$$

Figure 2.1(b) $p\% = +3.5\%$ $q\% = -3.0\%$ then $K = \dfrac{-3.0-(+3.5)}{100L} = \dfrac{-6.5}{100L}$

$$y = \left(\frac{-6.5}{200L}\right)x^2 + \left(\frac{3.5}{100}\right)x + H$$

Figure 2.1(c) $p\% = -2.5\%$ $q\% = -5.0\%$ then $K = \dfrac{-5.0-(-2.5)}{100L} = \dfrac{-2.5}{100L}$

$$y = \left(\frac{-2.5}{200L}\right)x^2 + \left(\frac{-2.5}{100}\right)x + H$$

Sag vertical curves in Figure 2.2:

Figure 2.2(a) $p\% = -2.5\%$ $q\% = +1.5\%$ then $K = \dfrac{+1.5-(-2.5)}{100L} = \dfrac{4.0}{100L}$

$$y = \left(\frac{4.0}{200L}\right)x^2 + \left(\frac{-2.5}{100}\right)x + H$$

Figure 2.2(b) $p\% = -4.5\%$ $q\% = -1.0\%$ then $K = \dfrac{-1.0-(-4.5)}{100L} = \dfrac{3.5}{100L}$

$$y = \left(\frac{3.5}{200L}\right)x^2 + \left(\frac{-4.5}{100}\right)x + H$$

Figure 2.2(c) $p\% = +1.0\%$ $q\% = +4.0\%$ then $K = \dfrac{+4.0-(+1.0)}{100L} = \dfrac{3.0}{100L}$

$$y = \left(\frac{3.0}{200L}\right)x^2 + \left(\frac{1.0}{100}\right)x + H$$

It should be clear from the derivation of the equation of the vertical curve and from the evaluations above, that once the gradients have been fixed then only one other parameter needs to be fixed to define the curve uniquely. That is, either the rate of change of gradient *K*, or the length of the vertical curve *L*. The determination of these parameters is dealt with in later sections.

4. PROPERTIES OF THE PARABOLIC VERTICAL CURVE

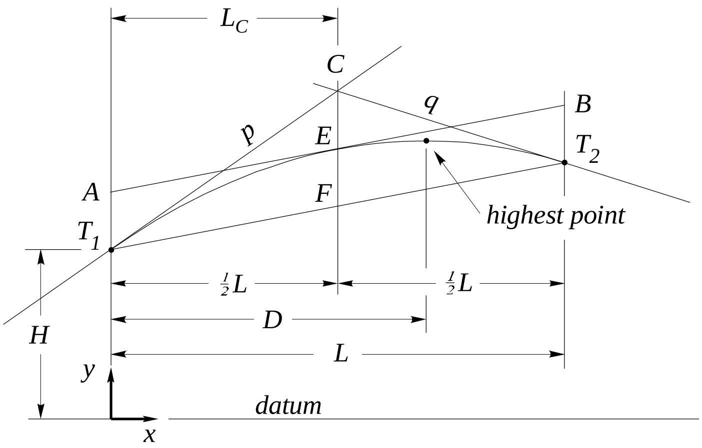


Figure 4.1

In Figure 4.1, $T_1 E T_2$ is a vertical parabolic curve between two grades *p* and *q* which intersect at *C*. T_1, T_2 are tangent points and the *x*-*y* coordinate origin is vertically below T_1 with the *x*-axis being the datum for reduced levels *y*. *H* is the reduced level of T_1. The horizontal length of the vertical curve is *L* and the highest point of the curve is a distance *D* from T_1.

[1] The equation of the curve is

$$y = \left(\frac{q-p}{2L}\right)x^2 + px + H$$

Substituting *L* for *x*, the reduced level of T_2 becomes

$$\left(\frac{q-p}{2L}\right)L^2 + pL + H = \left(q+p\right)\frac{L}{2} + H$$

From Figure 4.1, the horizontal distance from T_1 to *C* is L_C and we can use this distance to calculate the reduced level of T_2 as

$$H + pL_C + q\left(L - L_C\right)$$

Equating these two expressions for the reduced level of T_2 gives

$$\begin{aligned}\left(q+p\right)\frac{L}{2} + H &= H + pL_C + q\left(L - L_C\right)\\ &= H + \left(p-q\right)L_C + 2q\frac{L}{2}\end{aligned}$$

Cancelling and re-arranging gives

$$(p-q)\frac{L}{2} = (p-q)L_C$$

i.e.,
$$L_C = \frac{L}{2} \tag{4.1}$$

This very important relationship: <u>the horizontal distances from the tangent points to the intersection point are equal</u> is of considerable use in solving vertical curve problems.

[2] In Figure 4.1, *F* is the mid-point of the line T_1T_2 and the reduced level (RL) of *F* is the mean of the reduced levels of T_1 and T_2.

$$\mathrm{RL_F} = \frac{(q+p)L}{4} + H$$

The reduced level of *E*, a point on the curve at a horizontal distance *L/2* from T_1 is

$$\mathrm{RL_E} = \left(\frac{q-p}{2L}\right)\left(\frac{L}{2}\right)^2 + p\frac{L}{2} + H = \frac{(q+3p)L}{8} + H$$

The reduced level of *C* is

$$\mathrm{RL_C} = \frac{pL}{2} + H$$

The vertical distances *CE* and *CF* are the differences between the RL's of *C* and *E*, and *C* and *F*

$$CE = \frac{(p-q)L}{8}, \quad CF = \frac{(p-q)L}{4}$$

i.e.,
$$CE = \frac{CF}{2} \tag{4.2}$$

Hence, <u>the parabola bisects the vertical from the intersection point *C* to the mid-point of the line joining the tangent points</u>.

[3] The gradient of the curve at *E* is given by equation (3.4) as

$$\left(\frac{dy}{dx}\right)_E = \left(\frac{q-p}{L}\right)\left(\frac{L}{2}\right) + p = \frac{q+p}{2}$$

This is also the gradient of the line joining the tangent points, hence in Figure 4.1, <u>*AB* is parallel to T_1T_2 and is tangential to the curve at *E*, a point midway between the tangent points</u>.

[4] In Figure 4.1, D is the horizontal distance to the highest point of the curve (or the lowest point if the curve was a true sag curve). The gradient will be zero at the highest point (or lowest) and since the gradient is also the derivative, then we can set equation (3.4) equal to zero and solve x

$$\frac{dy}{dx} = \left(\frac{q-p}{L}\right)x + p = 0$$

giving the <u>horizontal distance to the high (or low) point of a parabolic vertical curve</u> as

$$D = \left(\frac{-p}{q-p}\right)L \tag{4.3}$$

As mentioned previously, a true summit curve (having a highest point) and a true sag curve (having a lowest point) will only occur when there is a change of sign between q and p. Note that the term $q-p$ in equation (4.3) is the numerical sum of the two gradients and that D will always be a positive quantity.

e.g., *Summit* $p\% = +4.0\%$ $q\% = -3.0\%$

$$D = \left(\frac{-4}{-3-4}\right)L = \left(\frac{4}{7}\right)L$$

e.g., *Sag* $p\% = -4.0\%$ $q\% = +3.0\%$

$$D = \left(\frac{4}{3-(-4)}\right)L = \left(\frac{4}{7}\right)L$$

5. EXAMPLES OF PARABOLIC VERTICAL CURVE COMPUTATIONS

In these example computations, it is assumed that vertical curves are parabolic and that the length of the curve has been fixed according to certain design principles. These design principles (for determining the length of vertical curves) are covered in subsequent sections of these notes.

5.1 Locating tangent points of vertical curves

Example 1

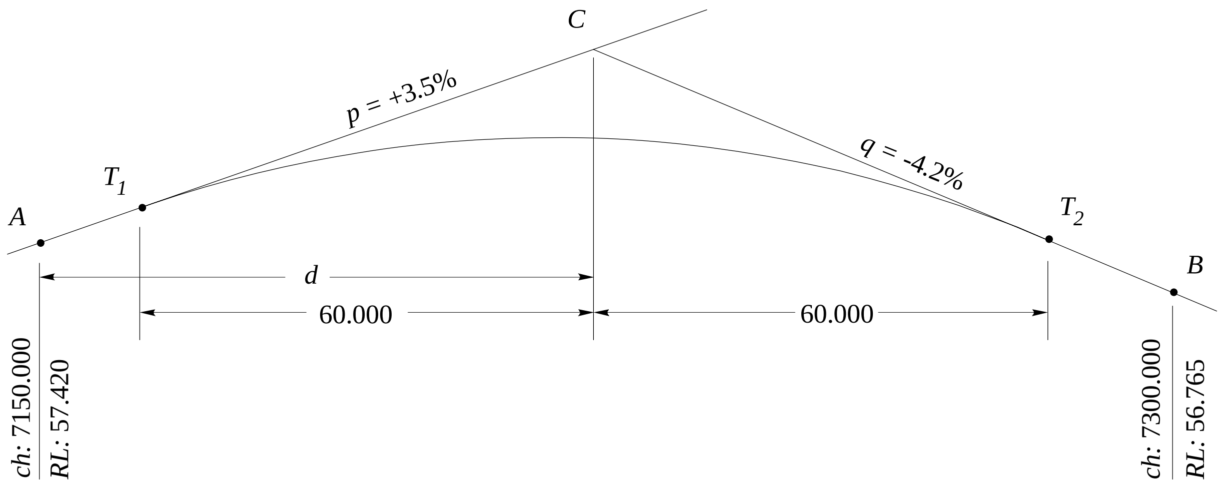


Figure 5.1

Figure 5.1 shows a rising gradient of +3.5% followed by a falling gradient of –4.2% connected by a vertical parabolic curve of horizontal length 120 m. *A*, a point on the rising grade, has a chainage of 7150.000 m and a reduced level (RL) of 57.420 m and *B*, a point on the falling grade, has a chainage of 7300.000 m and RL 56.765 m.
Calculate the chainage and RL of the tangents T_1 and T_2 and the intersection point *C*.

Let the horizontal distance from *A* to the intersection point *C* be *d*, then since *AB* = 150.000 m

$$\mathrm{RL}_C = \mathrm{RL}_A + \left(\frac{3.5}{100}\right)d$$

$$\mathrm{RL}_B = \mathrm{RL}_C + \left(\frac{-4.2}{100}\right)(150 - d) \quad \text{or} \quad \mathrm{RL}_C = \mathrm{RL}_B + \left(\frac{4.2}{100}\right)(150 - d)$$

Equating the expressions for RL_C gives

$$\mathrm{RL}_A + \left(\frac{3.5}{100}\right)d = \mathrm{RL}_B + \left(\frac{4.2}{100}\right)(150 - d)$$

and $$d = 73.312 \text{ m}$$

The chainage of the intersection point is 7150.000 + 73.312 = 7223.312 m and the chainages of the tangent points are 60 m either side of the intersection (due to the symmetry of the curve). Having determined the chainages, the RL's follow from the grades, giving

Point	Chainage	RL
A	7150.000	57.420
T_1	7163.312	57.886
C	7223.312	59.986
T_2	7283.312	57.466
B	7300.000	56.765

Table 5.1

Example 2

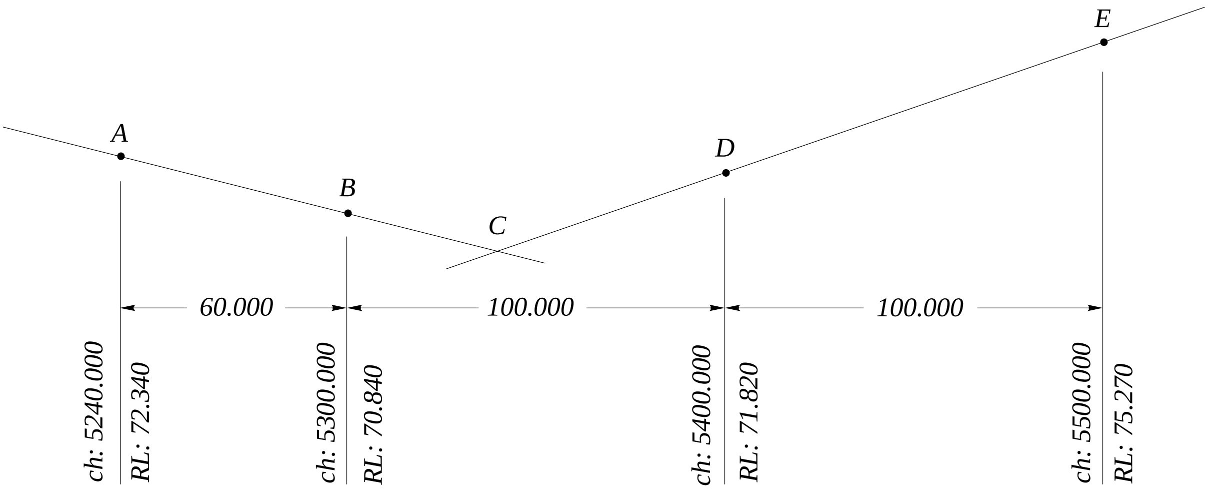


Figure 5.2

In Figure 5.2 points *A*, *B*, *D* and *E* lie on intersecting grades which are to be connected by a parabolic vertical curve of horizontal length 150 m. The chainages and RL's are shown on the diagram and the grades intersect at *C*.
Calculate the chainage and RL of the tangents T_1 and T_2 and the intersection point *C*.

$$\text{Gradient } AB = \left(\frac{70.840 - 72.340}{60.000}\right) = \frac{-1.500}{60} = -0.0250 = -2.50\%$$

$$\text{Gradient } DE = \left(\frac{75.270 - 71.820}{100.000}\right) = \frac{3.450}{100} = 0.0345 = +3.45\%$$

Let the horizontal distance from *B* to the intersection point *C* be *d*, then

$$\mathrm{RL}_C = \mathrm{RL}_B + \left(\frac{-2.50}{100}\right)d$$

$$\mathrm{RL}_D = \mathrm{RL}_C + \left(\frac{+3.45}{100}\right)(100-d) \quad \text{or} \quad \mathrm{RL}_C = \mathrm{RL}_D - \left(\frac{3.45}{100}\right)(100-d)$$

Equating the expressions for RL_C gives

$$\mathrm{RL}_B - \left(\frac{2.50}{100}\right)d = \mathrm{RL}_D - \left(\frac{3.45}{100}\right)(100-d)$$

and $$d = 41.513 \text{ m}$$

The chainage of the intersection point is 5300.000 + 41.513 = 5341.513 m and the chainages of the tangent points are 75 m either side of the intersection (due to the symmetry of the curve). Having determined the chainages, the RL's follow from the grades, giving

Point	Chainage	RL
A	5240.000	72.340
T_1	5266.513	71.677
B	5300.000	70.840
C	5341.513	69.802
D	5400.000	71.820
T_2	5416.513	72.390
E	5500.000	75.270

Table 5.2

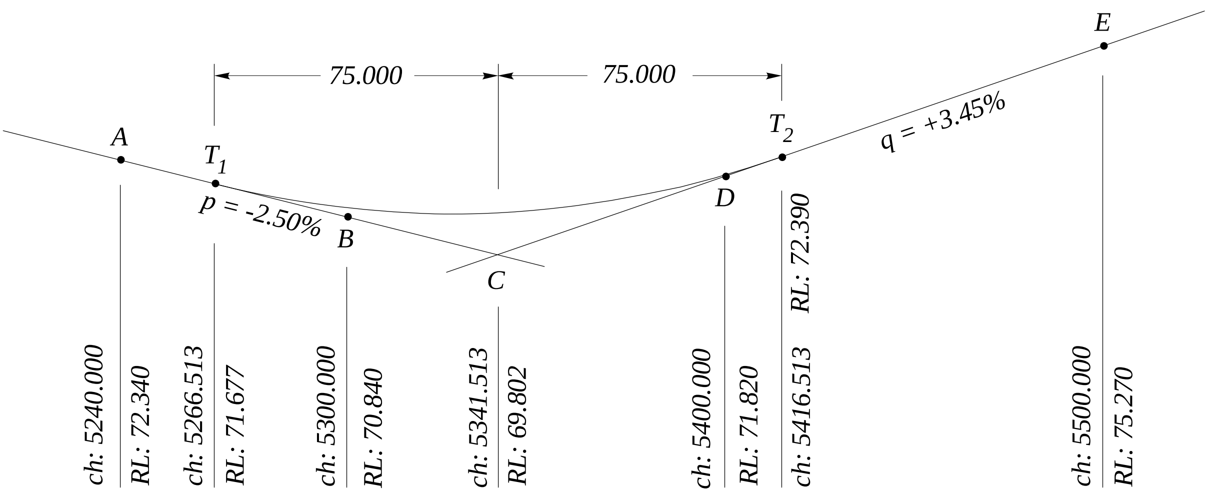


Figure 5.3

5.2 Computation of Reduced Levels of points on vertical curves

Once the chainage and reduced level (RL) of the tangent points have been determined, it remains to compute the RL's of points along the vertical curve. This can be achieved by using equation (3.6)

$$y = \left(\frac{q\% - p\%}{200L} \right) x^2 + \left(\frac{p\%}{100} \right) x + H \tag{3.6}$$

where the gradients p and q are given in percentages, x is the horizontal distance from the tangent point T_1 to the point on the curve, H is the RL of T_1 and y is the RL of the point on the curve.

Example 3

Consider the parabolic vertical curve determined in Example 1:

L = 120 m, p = +3.50%, q = –4.20%
ch T_1 = 7163.312 m, RL T_1 = 57.886 m
ch T_2 = 7283.312 m, RL T_2 = 57.466 m

Compute the RL's of points on the vertical curve at even 20 m chainages.

The equation, see (3.6) above is

$$\text{RL} = \left(\frac{q\% - p\%}{200L} \right) x^2 + \left(\frac{p\%}{100} \right) x + H = \left(\frac{-0.0385}{120} \right) x^2 + 0.0350x + 57.886$$

The tabulated results are

Point	**Chainage**	x	**RL**
T_1	7163.312	0	57.886
1	7180	16.688	58.381
2	7200	36.688	58.738
3	7220	56.688	58.839
4	7240	76.688	58.683
5	7260	96.688	58.271
6	7280	116.688	57.602
T_2	7283.312	120.000	57.466

Table 5.3

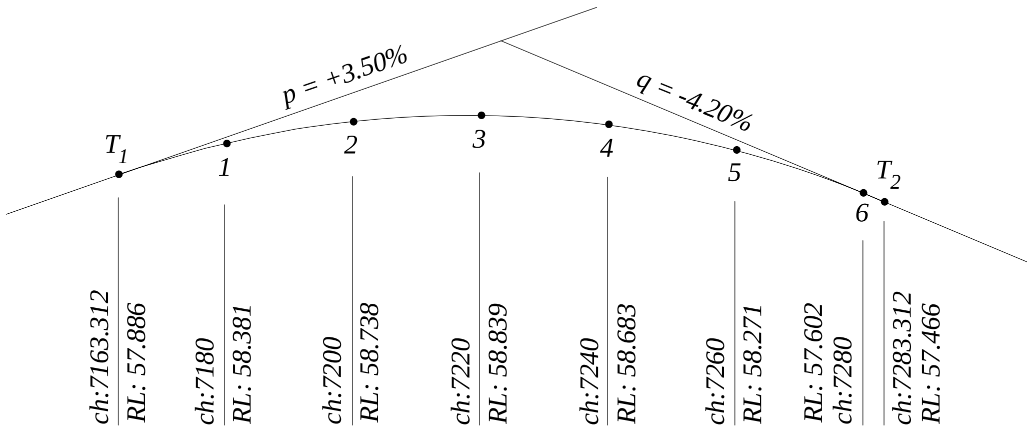


Figure 5.4

Example 4

Consider the parabolic vertical curve determined in Example 2:

L = 150 m, p = –2.50%, q = +3.45%

ch T_1 = 5266.513 m, RL T_1 = 71.677 m

ch T_2 = 5416.513 m, RL T_2 = 72.390 m

Compute the RL's of points on the vertical curve at even 20 m chainages.

The equation, see (3.6) above is

$$\text{RL} = \left(\frac{q\% - p\%}{200L}\right)x^2 + \left(\frac{p\%}{100}\right)x + H = \left(\frac{0.029750}{150}\right)x^2 - 0.0250x + 71.677$$

The tabulated results are

Point	**Chainage**	***x***	**RL**
T_1	5266.513	0	71.677
1	5280	13.487	71.376
2	5300	33.487	71.062
3	5320	53.487	70.907
4	5340	73.487	70.911
5	5360	93.487	71.073
6	5380	113.487	71.394
7	5400	133.487	71.874
T_2	5416.513	150.000	72.390

Table 5.4

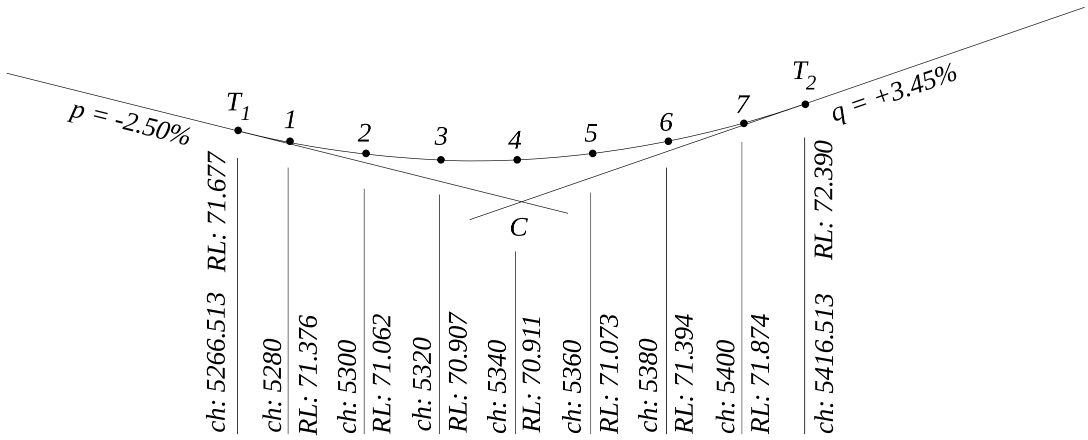


Figure 5.5

Exercise 1

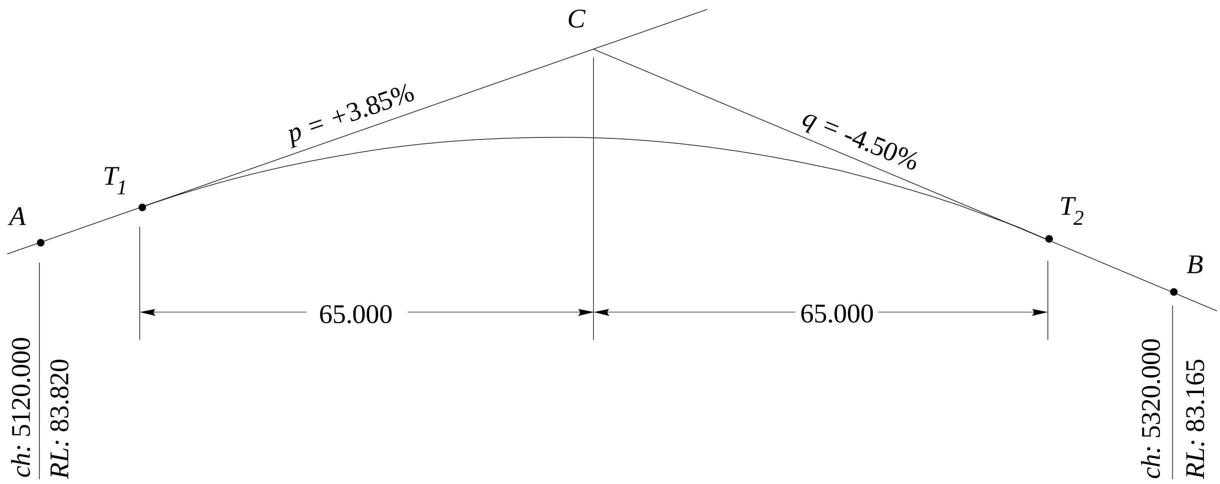


Figure 5.6

Figure 5.6 shows a rising gradient of +3.85% followed by a falling gradient of –4.50% connected by a vertical parabolic curve of horizontal length 130 m. *A*, a point on the rising grade, has a chainage of 5120.000 m and a reduced level (RL) of 83.820 m and *B*, a point on the falling grade, has a chainage of 5320.000 m and RL 83.165 m.

Calculate the following:

(i) The chainages and RL's of the tangents T_1 and T_2 and the intersection point *C*.
(ii) The RL's of points on the curve at even 20 m chainages.
(iii) The chainage and RL of the mid-point of the curve.
(iv) The chainage and RL of the high-point of the curve.

Exercise 2

A falling grade of 4% meets a rising grade of 5% at chainage 1500.000 m and RL 64.750 m. At chainage 1460.000 m, the underside of a bridge has a RL of 71.250 m. The two gradients are to be joined by a parabolic vertical curve of maximum length (rounded down to nearest 20 m) to give at least 4 metres clearance under the bridge.

Calculate the following:

(i) The length of the vertical curve. This length should then be rounded down to the nearest 20 m for use in the following calculations.
(ii) The chainages and RL's of the tangent points.
(iii) The clearance between the curve and the underside of the bridge. Note that this should be at least 4 m.
(iv) The RL's of points on the curve at even 20 m chainages.
(v) The chainage and RL of the mid-point of the curve.
(vi) The chainage and RL of the low-point of the curve.

6. DESIGN CONSIDERATIONS FOR PARABOLIC VERTICAL CURVES

In previous sections, it has been shown that a vertical parabolic curve is completely defined if the following elements are known.

(i) Intersecting gradients p and q,
(ii) Chainage and RL of the intersection point (or the tangent point) and
(iii) Horizontal length L.

The determination of a suitable length L is normally the responsibility of the traffic engineer or designer. For any design speed, the <u>minimum</u> length of a vertical curve will depend on one of two factors, namely the limitation of vertical acceleration or the allowable minimum sight distance.

6.1 Length of Vertical Parabolic Curve determined by Limitation of Vertical Acceleration.

This is a method of computing L taking into account the design speed of the road v (m/s) or more commonly V (kph) and the allowable vertical acceleration a. The design speed is usually taken to be the 85 percentile speed, i.e., the speed that is not exceeded by 85% of the vehicles using the road. The vertical acceleration a, is the result of the vehicle traversing the curve at a constant speed; its vertical velocity component changing as the grade changes from p at T_1 to q at T_2. This change in the vertical component of velocity means that the vehicle is subject to an acceleration a, (centripetal acceleration) given by

$$a = \frac{v^2}{r} \tag{6.1}$$

r is the radius of circular curve approximating the vertical parabolic curve and $1/r$ is the curvature κ. The general equation for the curvature of a curve $y = f(x)$ is given by

$$\kappa = \pm \frac{\dfrac{d^2y}{dx^2}}{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{3/2}} \tag{6.2}$$

For a vertical curve, $\dfrac{dy}{dx}$ is the gradient, which is usually small and $\left(\dfrac{dy}{dx}\right)^2$ will be exceedingly small and may be neglected, giving the curvature κ (as an approximation)

$$\kappa \simeq \frac{d^2y}{dx^2} \tag{6.3}$$

For parabolic vertical curves, $\dfrac{d^2y}{dx^2}$ (the rate of change of gradient) is a constant *K*, and from equations (3.1) and (3.2)

$$\frac{d^2y}{dx^2} = K = \frac{q-p}{L} \tag{6.4}$$

Now, since $1/r = \kappa \simeq d^2y/dx^2$ and using equations (6.1) and (6.4) the vertical acceleration is

$$a = \frac{(q-p)v^2}{L} \tag{6.5}$$

Using the relationship $v\ \text{(m/s)} = \dfrac{V\ \text{(kph)}}{3.6}$ and re-arranging equation (6.5) gives the length of the vertical curve as

$$L = \frac{(q-p)V^2}{12.96a} \tag{6.6}$$

or for grades in percentages

$$L = \frac{(q\% - p\%)V^2}{1296a} \tag{6.7}$$

For design purposes, the maximum value of vertical acceleration *a*, should not exceed 0.1*g* (*g* is the acceleration due to gravity $\simeq$ 9.8 m/s) and otherwise come within the range 0.10*g* to 0.05*g*, depending on the importance of the road. On major highways, a maximum value of *a* = 0.05*g* is considered satisfactory.

6.2 Minimum Sight Distance

Sight distance is defined as the extent of a drivers clear view a road, sufficient to enable the drive to react to an emergency or to permit safe overtaking of a vehicle travelling at less than design speed.

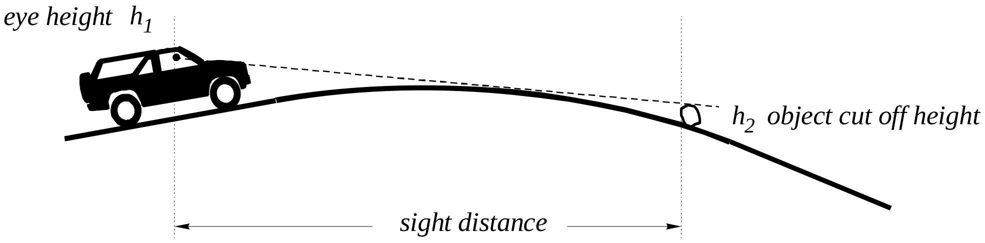


Figure 6.1 Driver's view of an object on a summit vertical curve.

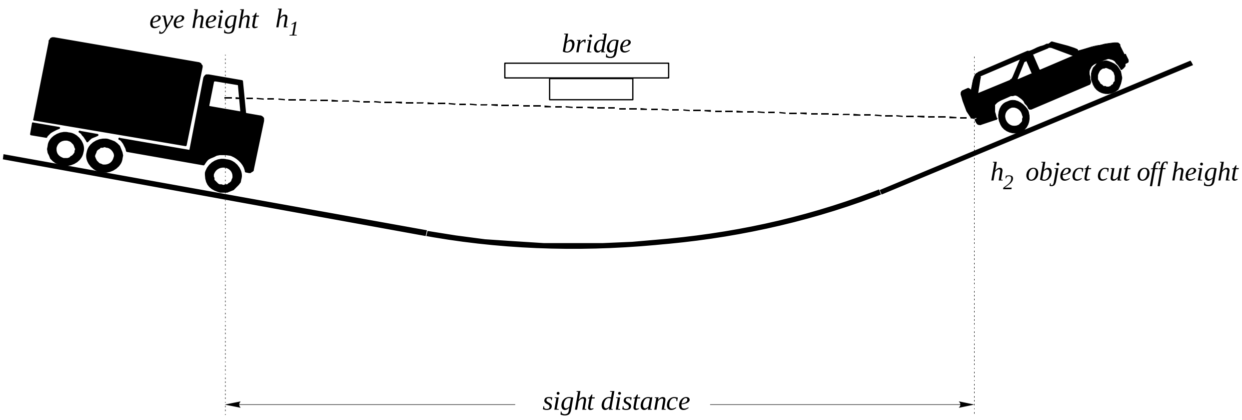


Figure 6.2 Driver's view of an object on a sag vertical curve.

Constraints assumed for computation of sight distances:

Height of eye of driver (h_1)

- passenger vehicle 1.15 m
- commercial vehicle 1.80 m

Object cut-off height (h_2). The driver is assumed to react to their view of that portion of the object over this height

- approaching vehicle 1.15 m
- stationary object on road 0.20 m
- vehicle tail or stop light 0.60 m

6.3 Stopping Distance D_S

A theoretical stopping distance D_S can be derived from the equation

$$\text{Stopping Distance} = \text{Reaction Distance} + \text{Braking Distance} \tag{6.8}$$

6.3.1 Reaction Distance D_R

A driver confronted with an emergency has firstly to <u>perceive</u> and secondly to <u>react</u> to a situation before they apply foot to brake. The time lapse between initial perception and the instant when the vehicle brakes act is called the <u>total reaction time</u> (R_T). It can vary between 0.5 and 3 seconds depending upon circumstances (drivers perception, atmospheric conditions, etc). For design purposes a figure of $R_T = 2.5$ sec is usually adopted

$$\text{Reaction Distance } D_R = R_T\, v = R_T \frac{V}{3.6} \tag{6.9}$$

where R_T is reaction time in seconds
v is velocity in metres per second (m/s)
V is velocity in kilometres per hour (kph)

6.3.2 Braking Distance D_B

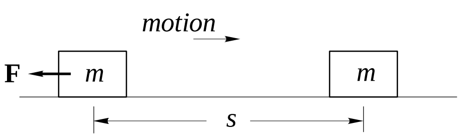
Having applied the brakes, the vehicle still has to stop. Braking distance is defined as the length of roadway travelled from the time the brakes start to act until the vehicle is brought to a halt. To establish a formula, consider the *kinetic energy* of the vehicle:

> Kinetic energy W_K is the energy possessed by a body because of its motion; it is measured by the work done by the body as it is brought to rest.

Consider a body of mass *m* decelerating to rest. From the 3rd equation of motion

$$v^2 = u^2 + 2as$$

where v is final velocity
u is initial velocity
a is acceleration
s is displacement



Since the final velocity will be zero $(v = 0)$ and the acceleration will be negative then

$$0 = u^2 + 2as$$

$$s = \frac{u^2}{2a}$$

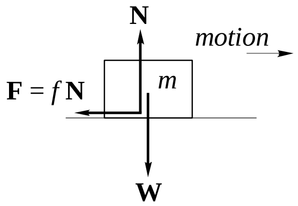
Now Work = Force $\times$ Distance $= F \times s = F\left(\dfrac{u^2}{2a}\right)$ and Force = mass $\times$ acceleration $= ma$

therefore

Work $= ma\left(\dfrac{u^2}{2a}\right) = \dfrac{1}{2}mu^2$ and since kinetic energy is a measure of the work done then

$$W_k = \frac{1}{2}\ \text{mass} \times (\text{velocity})^2 = \frac{1}{2}mv^2 \qquad (6.10)$$

For a body of mass *m*, decelerating to rest on a horizontal surface, the force **F** $= f$ **N** causing the deceleration is a function of the normal reaction force **N**. The normal force is equal to the weight **W**, a force whose magnitude is $W = mg$ and f is the *Coefficient of Longitudinal Friction*. The magnitude of the decelerating force is



$$F = f\,N = f\,W = fmg$$

Since the kinetic energy W_k is the work done (Force × Distance) as the body is brought to rest, then

$$\text{Force} \times \text{Distance} = W_k$$
$$fmg \times d = \frac{1}{2}mv^2$$

giving the *Braking Distance* D_B as

$$\text{Braking Distance } D_B = \frac{v^2}{2gf} \qquad (6.11)$$

Using a value of the acceleration due to gravity of g = 9.8 m/s and with V in kph ($V = \dfrac{v}{3.6}$) then

$$D_B = \frac{V^2}{254\,f} \qquad (6.12)$$

Substituting equations (6.9) and (6.12) into equation (6.8) gives

$$\text{Stopping Distance } D_S = D_R + D_B = R_T\frac{V}{3.6} + \frac{V^2}{254\,f} \qquad (6.13)$$

Using a value for the reaction time $R_T = 2.5\,\text{sec}$ gives a common formula for the stopping distances as

$$D_S = 0.7V + \frac{V^2}{254\,f} \qquad (6.14)$$

Recent tests by the Australian Road Research Board (ARRB)[1] have found that, on good dry pavements modern passenger cars can consistently achieve deceleration rates in excess of 1.0*g*. However, the values used for design purposes should allow for degradation of pavements skid resistance when wet and for a reasonable amount of surface polishing. The values for the coefficient of longitudinal friction given in Table 6.1 below are taken from *Rural Road Design – Guide to the Geometric Design of Rural Roads,* AUSTROADS, Sydney, 1993, p.28. The lower values assumed for the higher speeds reflect the reduction in wet pavement skid resistance with increasing speed and the need for lateral vehicle control over the longer braking distances.

Initial Speed V (kph)	Coefficient of Longitudinal Friction f
50	0.52
80	0.43
100	0.39
130	0.33

Table 6.1 Values of Coefficient of Longitudinal Friction f

6.4 Length of Summit Vertical Curve for Stopping Sight Distance D

Stopping Sight Distance D (Non-Overtaking Sight Distance) is used to determine the minimum length of a vertical curve. D shall be equal to the Stopping Distance D_S of a vehicle travelling at design speed V when an unobstructed view is provided between a point h_1 (eye height) above road pavement and a stationary object of height h_2 (object cut off height) in the lane of travel.

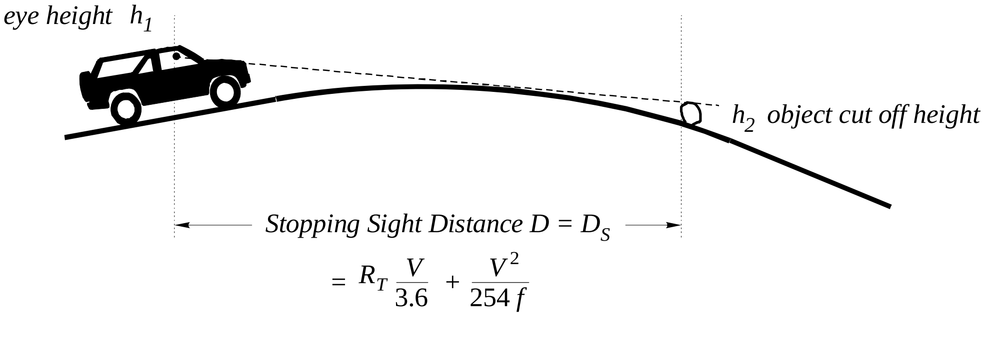


Figure 6.3

[1] Samuels, S.E. and Jarvis, J.R., 1978, *Acceleration and Deceleration of Modern Vehicles,* ARRB Research Report, ARR No.86 – from *Rural Road Design – Guide to the Geometric Design of Rural Roads,* AUSTROADS, Sydney, 1993, p.28.

To determine a minimum vertical curve length *L*, using Stopping Sight Distance *D*, three cases arise

(i) $L > D$: length of vertical curve greater than the stopping sight distance. In this case, the vehicle and the object are both on the vertical curve.
(ii) $L = D$: length of vertical curve equal to the stopping sight distance. In this case, the vehicle is at the beginning of the vertical curve and the object is at the end of the curve.
(iii) $L < D$: length of vertical curve shorter than the stopping sight distance. In this case, the vehicle and the object are both on the grades joined by the vertical curve.

6.4.1 $L > D$: Length of Summit Vertical Curve Greater than Stopping Sight Distance

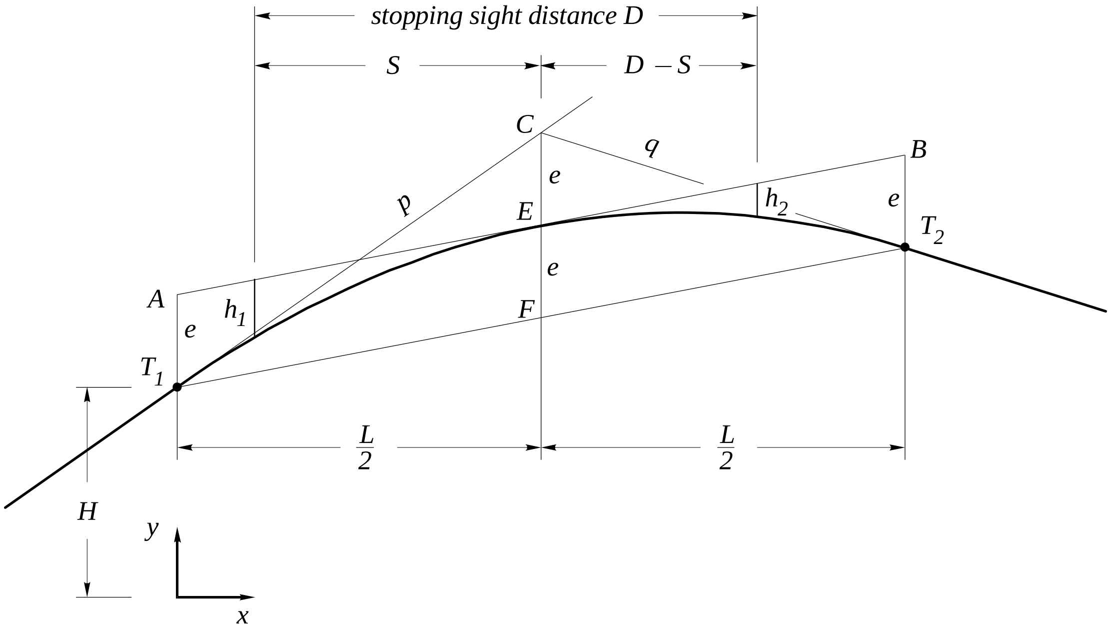


Figure 6.4

In Figure 6.4 the following is known (see Section 4 – Properties of the Parabolic Vertical Curve). The line *AB* is tangential to the curve at *E* (the mid-point of the curve) and is parallel to the line between the tangent points. The distance $e = T_1A = CE = T_2B$ is

$$e = CE = \frac{CF}{2} = \frac{(q-p)}{8}L \qquad (6.15)$$

and the sign of *e* is given by the sign of $(q-p)$.

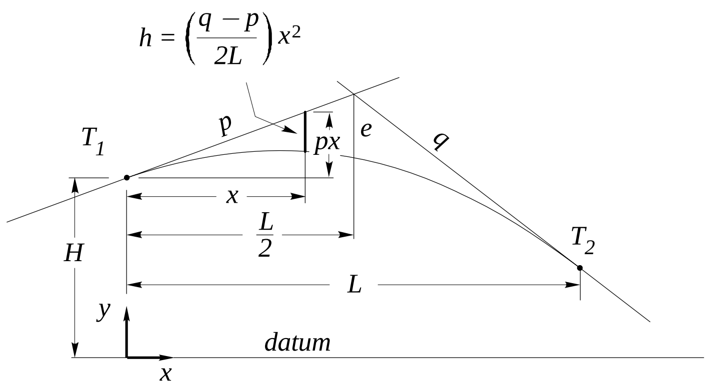


Figure 6.5

Equation (6.15) can be verified by considering Figure 6.5 and the equation of the parabolic vertical curve

$$y = \left(\frac{q-p}{2L}\right)x^2 + px + H \tag{6.16}$$

For a point on the curve at a horizontal distance x from the tangent point T_1 the following three components of equation (6.16) are shown on Figure 6.5.

(i) the vertical distance from the grade to the curve is $h = \left(\frac{q-p}{2L}\right)x^2$,

(ii) the vertical distance from the grade to the horizontal line passing through T_1 is px and

(iii) the vertical distance from T_1 to the datum is H.

The sum of the three components is the y-coordinate of the point on the curve.

For the mid-point of the curve, the vertical distance e is given by the first component of equation (6.16)

$$e = \left(\frac{q-p}{2L}\right)\left(\frac{L}{2}\right)^2 = \frac{(q-p)}{8}L$$

This demonstrates a useful property of a parabolic vertical curve, i.e.,; the vertical distance h from a tangent line to a point on the curve is given by

$$h = \left(\frac{q-p}{2L}\right)x^2 \tag{6.17}$$

where x is the horizontal distance from the tangent point. Noting, in Figure 6.4, that AB is a tangent and E is a tangent point, equation (6.17) can be used to derive expressions for the distances h_1 and h_2, noting that $q-p$ in equation (6.17) has been reversed to make h_1 and h_2 positive quantities.

$$h_1 = \left(\frac{p-q}{2L}\right)S^2 \tag{6.18}$$

$$h_2 = \left(\frac{p-q}{2L}\right)\left(D-S\right)^2 \tag{6.19}$$

Using equations (6.15), (6.18) and (6.19) the following manipulations yield an expression for the length of a vertical curve.

By taking square roots of both sides of equations (6.18) and (6.19) we have

$$\sqrt{h_1} = S\sqrt{\frac{p-q}{2L}}$$

$$\sqrt{h_2} = \left(D-S\right)\sqrt{\frac{p-q}{2L}}$$

Adding the equations eliminates S

$$\sqrt{h_1} + \sqrt{h_2} = D\sqrt{\frac{p-q}{2L}}$$

Squaring both sides and re-arranging gives

$$L = \frac{D^2\left(p-q\right)}{2\left(\sqrt{h_1}+\sqrt{h_2}\right)^2} \tag{6.20}$$

If the gradients are given in percentages then equation (6.20) becomes

$$L = \frac{D^2\left(p\%-q\%\right)}{200\left(\sqrt{h_1}+\sqrt{h_2}\right)^2} \tag{6.21}$$

6.4.2 $L = D$: Length of Summit Vertical Curve equal to the Stopping Sight Distance

In this case, letting $D = L$ in equations (6.20) and (6.21) gives

$$L = \frac{2\left(\sqrt{h_1}+\sqrt{h_2}\right)^2}{p-q} \tag{6.22}$$

or, if gradients are given in percentages

$$L = \frac{200\left(\sqrt{h_1}+\sqrt{h_2}\right)^2}{p\%-q\%} \tag{6.23}$$

6.4.3 $L < D$: Length of Summit Vertical Curve less than the Stopping Sight Distance

In Figure 6.6, the vehicle (eye height h_1) and object (object cut off height h_2) are situated on the gradients *p* and *q* respectively and are at a distance *D* apart. *D* is the stopping sight distance and exceeds the length of the parabolic vertical curve. T_1 and T_2 are tangent points and the lines T_1T_2 and *AB* are parallel. *AB* is tangential to the curve at *E*, which is the mid-point of the curve.

The following manipulations yield an equation for the length of the vertical curve.

$$\text{The gradient of the line } T_1T_2 = \frac{\text{rise}}{\text{run}} = \frac{p(L/2)+q(L/2)}{L} = \frac{p+q}{2} \tag{6.24}$$

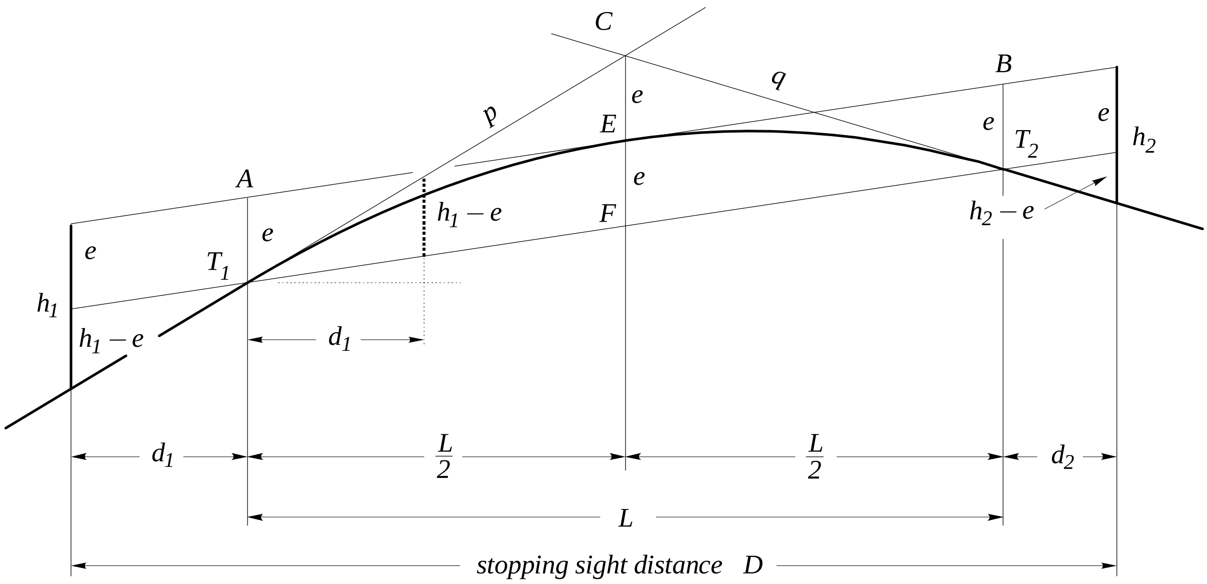


Figure 6.6

An expression for the vertical distance $h_1 - e$ can be obtained by considering the grade *p* and the gradient of the line T_1T_2. In Figure 6.6, the distance $h_1 - e$ is also shown as a dotted line at a distance d_1 from T_1.

$$\begin{aligned} h_1 - e &= p(d_1) - \left(\frac{p+q}{2}\right)d_1 \\ &= \frac{2p}{2}d_1 - \frac{p}{2}d_1 - \frac{q}{2}d_1 \end{aligned}$$

giving

$$h_1 - e = \left(\frac{p-q}{2}\right)d_1 \tag{6.25}$$

Similarly

$$h_2 - e = \left(\frac{p-q}{2}\right)d_2 \tag{6.26}$$

Adding equations (6.25) and (6.26) gives

$$h_1 + h_2 - 2e = \left(\frac{p-q}{2}\right)(d_1 + d_2)$$

but $d_1 + d_2 = D - L$ so we may write

$$h_1 + h_2 - 2e = \left(\frac{p-q}{2}\right)(D - L) \tag{6.27}$$

From the previous section, the vertical distance *e* is given by equation (6.16)

$$e = \left(\frac{q-p}{8}\right)L$$

However, for a summit curve, $q - p$ will be a negative quantity, making *e* negative, thus, for *e* positive we write

$$2e = \left(\frac{p-q}{4}\right)L \tag{6.28}$$

Substituting equation (6.28) into (6.27) gives

$$h_1 + h_2 - \left(\frac{p-q}{4}\right)L = \left(\frac{p-q}{2}\right)(D - L)$$

Multiplying both sides by $\left(\frac{4}{p-q}\right)$ and re-arranging gives and expression for *L*

$$L = 2D - \frac{4(h_1 + h_2)}{p-q} \tag{6.29}$$

or, if gradients are given in percentages

$$L = 2D - \frac{400(h_1 + h_2)}{p\% - q\%} \tag{6.30}$$

6.5 Length of Sag Vertical Curve for Stopping Sight Distance *D* and Clearance Height H_C

Stopping Sight Distance *D* (Non-Overtaking Sight Distance) is used to determine the minimum length of a vertical curve. *D* shall be equal to the Stopping Distance D_S of a vehicle travelling at design speed *V* when an unobstructed view is provided between a point h_1 (eye height) above road pavement and a stationary object of height h_2 (object cut off height) in the lane of travel. Overhead obstructions, such as road or railway overpasses, sign or tollway gantries may limit the sight distance available on sag vertical curves.

In Figure 6.7, the overhead obstruction (bridge) is at height H_C above the road pavement. H_C is the clearance height.

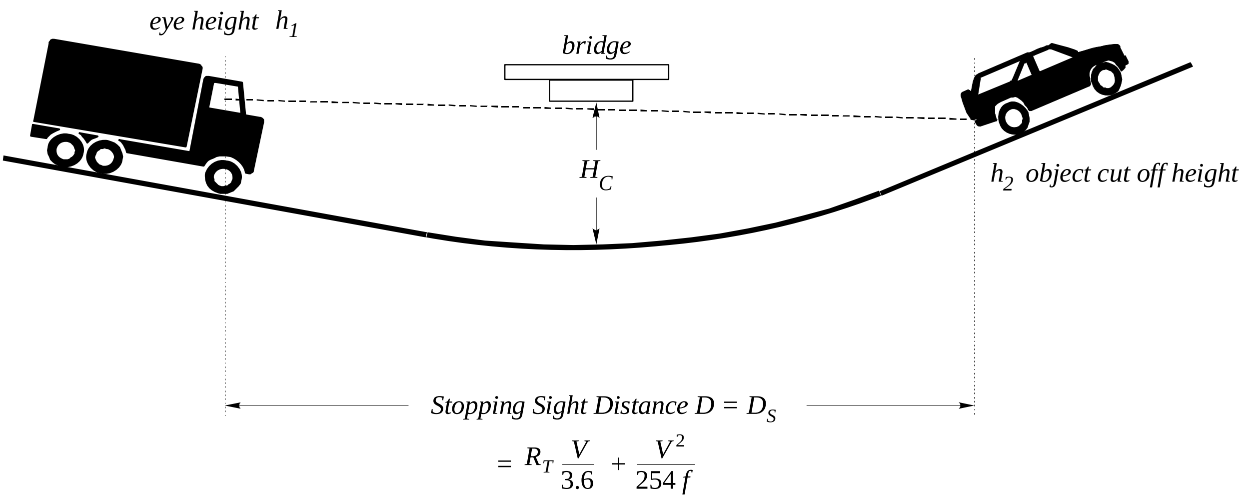


Figure 6.7

To determine a minimum vertical curve length *L*, using Stopping Sight Distance *D* and Clearance Height H_C three cases arise

(i) $L > D$: length of vertical curve greater than the stopping sight distance. In this case, the vehicle and the object are both on the vertical curve.
(ii) $L = D$: length of vertical curve equal to the stopping sight distance. In this case, the vehicle is at the beginning of the vertical curve and the object is at the end of the curve.
(iii) $L < D$: length of vertical curve shorter than the stopping sight distance. In this case, the vehicle and the object are both on the grades joined by the vertical curve.

6.5.1 $L > D$: Length of Sag Vertical Curve Greater than Stopping Sight Distance

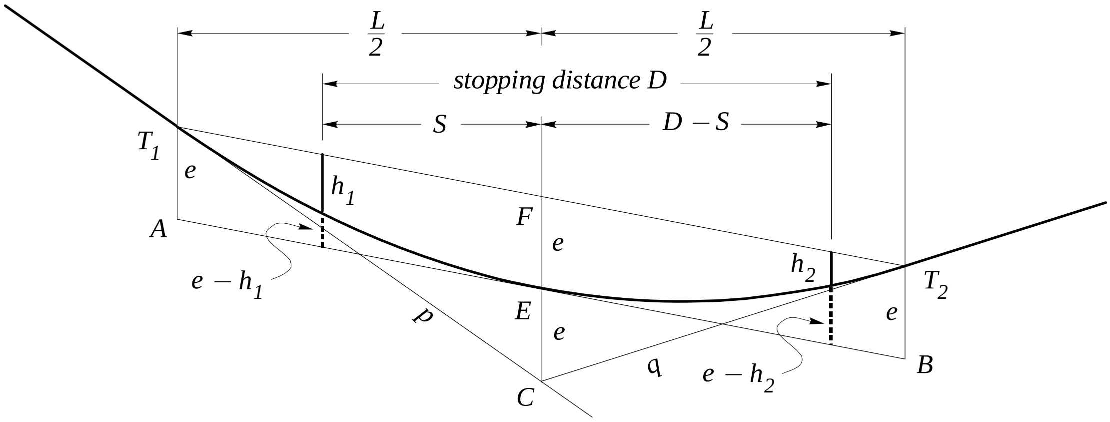


Figure 6.8

In a similar manner to Section 6.4.1 (summit vertical curves), two equations can be written and manipulated to yield expressions for the minimum length of a vertical curve.

$$e - h_1 = \left(\frac{q-p}{2L}\right)S^2 \tag{6.31}$$

$$e - h_1 = \left(\frac{q-p}{2L}\right)\left(D-S\right)^2 \tag{6.32}$$

By taking square roots of both sides of equations (6.31) and (6.32)we have

$$\sqrt{e-h_1} = S\sqrt{\frac{q-p}{2L}}$$

$$\sqrt{e-h_2} = \left(D-S\right)\sqrt{\frac{q-p}{2L}}$$

Adding the equations eliminates S

$$\sqrt{e-h_1} + \sqrt{e-h_2} = D\sqrt{\frac{q-p}{2L}}$$

Squaring both sides and re-arranging gives

$$L = \frac{D^2\left(q-p\right)}{2\left(\sqrt{e-h_1}+\sqrt{e-h_2}\right)^2} \tag{6.33}$$

In equation (6.33) e can be replaced by H_C the Clearance Height, the vertical distance between the point of tangency E and the obstruction to the line of sight between objects of height h_1 and h_2.

$$L = \frac{D^2(q-p)}{2\left(\sqrt{H_C - h_1} + \sqrt{H_C - h_2}\right)^2} \tag{6.34}$$

If the gradients are given in percentages then equation (6.34) becomes

$$L = \frac{D^2(q\% - p\%)}{200\left(\sqrt{H_C - h_1} + \sqrt{H_C - h_2}\right)^2} \tag{6.35}$$

Figures 6.9 and 6.10 show obstructions located vertically above the mid point of the curve and at a point to the left of the mid point. The line of sight between objects of height h_1 and h_2 is parallel to the tangent to the curve. The point of tangency is vertically below the obstruction.

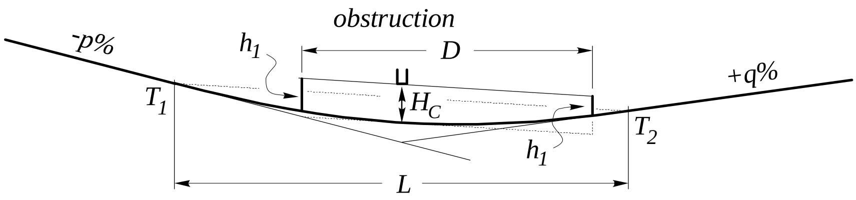


Figure 6.9
Obstruction at vertical height H_C above mid point of vertical curve

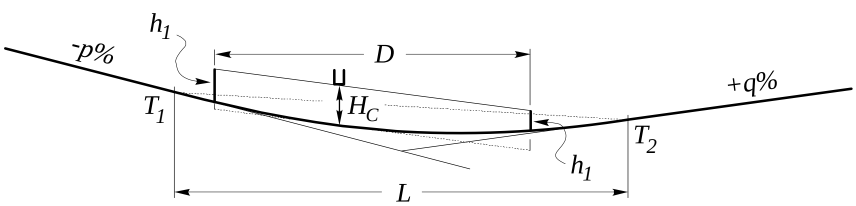


Figure 6.10
Obstruction at vertical height H_C above point to the left of mid point of vertical curve

6.5.2 $L = D$: Length of Sag Vertical Curve equal to the Stopping Sight Distance

In this case, letting $D = L$ in equations (6.34) and (6.35) gives

$$L = \frac{2\left(\sqrt{H_C - h_1} + \sqrt{H_C - h_2}\right)^2}{q - p} \tag{6.36}$$

or, if gradients are given in percentages

$$L = \frac{200\left(\sqrt{H_C - h_1} + \sqrt{H_C - h_2}\right)^2}{q\% - p\%} \tag{6.37}$$

6.5.3 $L < D$: Length of Sag Vertical Curve less than the Stopping Sight Distance

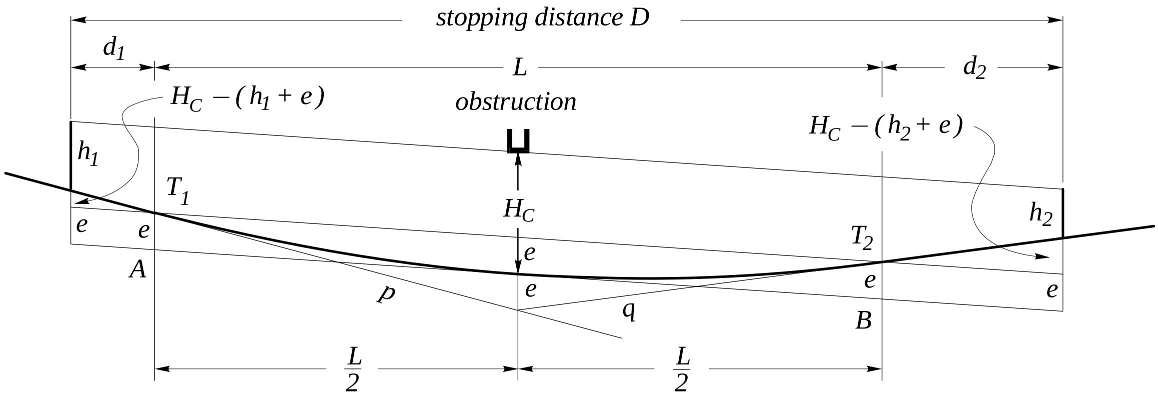


Figure 6.11

In Figure 6.11, the vehicle (eye height h_1) and object (object cut off height h_2) are situated on the gradients p and q respectively and are at a distance D apart. D is the stopping sight distance and exceeds the length of the parabolic vertical curve. T_1 and T_2 are tangent points and the lines T_1T_2 and AB are parallel. AB is tangential to the curve at the mid-point.

The following manipulations yield an equation for the length of the vertical curve. Equation (6.24) gives the gradient of the line T_1T_2

$$\text{gradient } T_1T_2 = \frac{p+q}{2}$$

An expression for the vertical distance $H_C - (h_1 + e)$ can be obtained by considering the grade p and the gradient of the line T_1T_2.

$$H_C - (h_1 + e) = \left(\frac{p+q}{2}\right)d_1 - p(d_1)$$
$$= \frac{p}{2}d_1 + \frac{q}{2}d_1 - \frac{2p}{2}d_1$$

giving

$$H_C - (h_1 + e) = \left(\frac{q-p}{2}\right)d_1 \tag{6.38}$$

Similarly

$$H_C - (h_2 + e) = \left(\frac{q-p}{2}\right)d_2 \tag{6.39}$$

Adding equations (6.38) and (6.39) gives

$$2H_C - (h_1 + h_2) - 2e = \left(\frac{q-p}{2}\right)(d_1 + d_2)$$

but $d_1 + d_2 = D - L$ and $2e = \left(\frac{q-p}{4}\right)L$ (see equation (6.28) noting that the for a sag vertical curve $q - p$ is a positive quantity). Substituting these expressions gives

$$2H_C - (h_1 + h_2) - \left(\frac{q-p}{4}\right)L = 2\left(\frac{q-p}{4}\right)(D - L)$$

Multiplying both sides by $\left(\frac{4}{p-q}\right)$ and re-arranging gives and expression for L

$$L = 2D - \left(\frac{4}{p-q}\right)\left[2H_C - (h_1 + h_2)\right] \tag{6.40}$$

or, if gradients are given in percentages

$$L = 2D - \left(\frac{400}{p\% - q\%}\right)\left[2H_C - (h_1 + h_2)\right] \tag{6.41}$$

6.6 Length of Sag Vertical Curve for Headlight Sight Distance *S*

Figure 6.12 shows a vehicle on a sag vertical curve whose headlights are at a vertical height *h* above the road pavement at the mid point of the curve. The longitudinal axis of the vehicle is parallel to the tangent to the vertical curve at the mid point *E*. The useful portion of the headlight beam diverges (upwards) at an angle θ and strikes the road pavement at a horizontal distance *S* (the Headlight Sight Distance) from the vehicle.

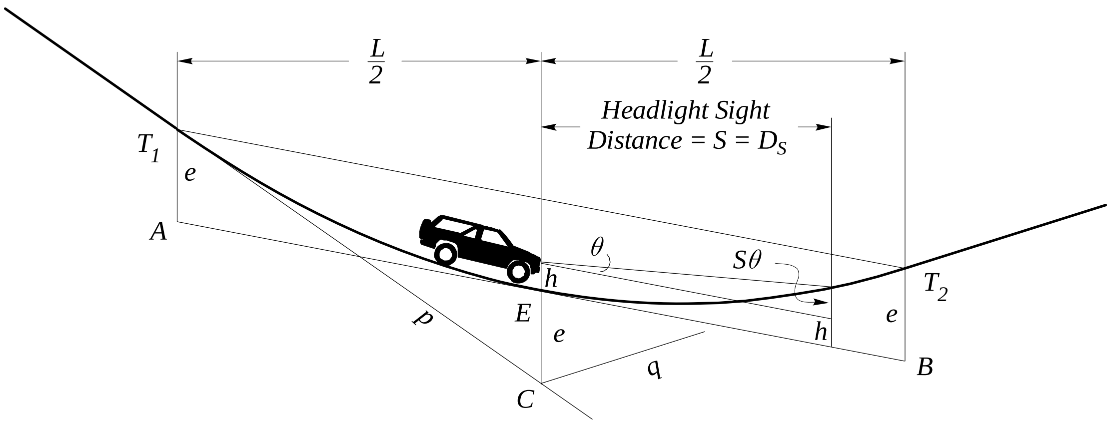


Figure 6.12

To determine a minimum vertical curve length *L*, using Headlight Sight Distance *S*, three cases arise

(i) $L > S$: length of vertical curve greater than the headlight sight distance. In this case, the vehicle and the limit of the light beam are both on the vertical curve.
(ii) $L = S$: length of vertical curve equal to the headlight sight distance. In this case, the vehicle is at the beginning of the vertical curve and the limit of the light beam is at the end of the curve.
(iii) $L < S$: length of vertical curve shorter than the headlight sight distance. In this case, the vehicle and the limit of the light beam are both on the grades joined by the vertical curve.

6.6.1 $L > S$: Length of Sag Vertical Curve Greater than Headlight Sight Distance

In a similar manner to previous sections, the property of the vertical distance between the tangent and the parabolic curve can be employed to give the following equation

$$h + S\theta = \left(\frac{q-p}{2L}\right)S^2 \tag{6.42}$$

In equation (6.42) it is assumed that the vertical distance, shown as $S\theta$, is equal to the small arc of a large circle of radius *S* subtending the angle θ at its centre (the headlight of the vehicle). This is a reasonable assumption, since in practice, the angle θ is small (usually 1°) and the grades are also small.

Rearranging equation (6.42) gives

$$L = \frac{S^2(q-p)}{2(h+S\theta)} \tag{6.43}$$

or, if gradients are given in percentages

$$L = \frac{S^2(q\% - p\%)}{200(h+S\theta)} \tag{6.44}$$

For a light beam with $\theta = 1^\circ$ (0.017453 radians) and $h = 0.750$ m, equation (6.44) becomes

$$L = \frac{S^2(q\% - p\%)}{150 + 3.5S} \tag{6.45}$$

6.6.2 $L \le S$: Length of Sag Vertical Curve less than or equal to the Headlight Sight Distance

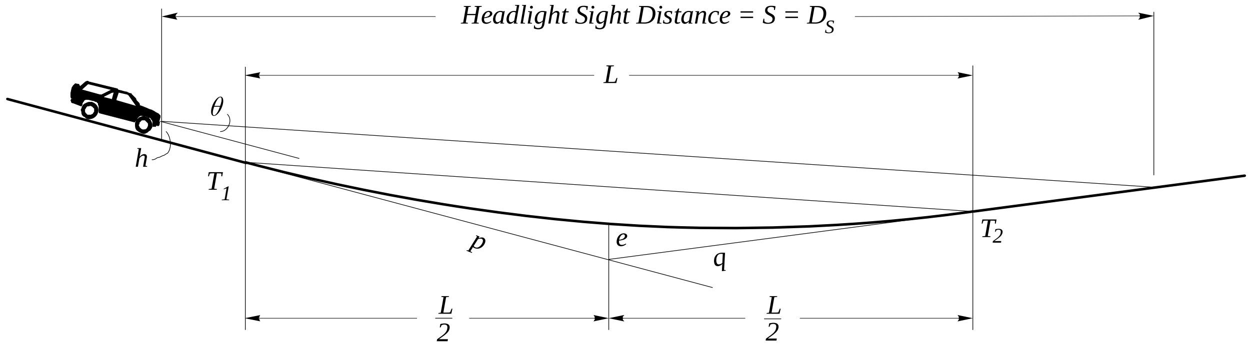


Figure 6.13

Figure 6.13 shows a vehicle on the grade *p* (before the tangent point) whose useful portion of the headlight beam intersects the grade *q* (after the tangent point) on the other side of the curve. In this case *L* < *S*. Since we are interested in determining <u>minimum</u> lengths of vertical curves, we may consider the case when the vehicle's headlamp is at T_1 and the headlight beam strikes the road pavement at T_2, i.e., the case when *L* = *S*. In such a case, substituting *L* for *S* in equation (6.43) gives the formula to be used for $L \le S$

$$L = \frac{2(h+S\theta)}{q-p} \tag{6.46}$$

or, if gradients are given in percentages

$$L = \frac{200(h+S\theta)}{q\% - p\%} \tag{6.47}$$

For a light beam with $\theta = 1^\circ$ (0.017453 radians) and $h = 0.750$ m, equation (6.47) becomes

$$L = \frac{150 + 3.5S}{q\% - p\%} \tag{6.48}$$